COMPLEX INTEGRABLE SYSTEMS OF MANY PARTICLES

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G. Sarkissian, V.S., J. Phys. A: Math. Theor. 55 (2022), 385203

N. Belousov, G. Sarkissian, V.S., work in progress

Euler: the gamma function

$$\Gamma(x) = \int_0^\infty t^{x-1} e^{-t} dt, \qquad \mathrm{Re}(x) > 0, \quad \Gamma(x+1) = x\Gamma(x)$$

and the beta integral

$$\int_0^1 t^{\alpha-1}(1-t)^{\beta-1} dt = \frac{\Gamma(\alpha)\Gamma(\beta)}{\Gamma(\alpha+\beta)}, \qquad \mathrm{Re}(\alpha), \mathrm{Re}(\beta) > 0.$$

Many applications: Feynman integrals, string amplitude, probability measure, etc. Complexification:

$$t \in \mathbb{R} \to z \in \mathbb{C}, \quad \int dt \to \int_{\mathbb{C}} d^2 z = \int_{\mathbb{R}^2} d\,\mathrm{Re}(z)\, d\,\mathrm{Im}(z)$$

with the single-valuedness condition

$$[z]^\alpha := z^\alpha \bar{z}^{\alpha'}, \quad \bar{z} = z^*, \quad \alpha' - \alpha = n \in \mathbb{Z}.$$

Complex beta integral (Gelfand, Graev, Vilenkin, 1962)

$$\int_{\mathbb{C}} [w - z_1]^{\alpha-1} [z_2 - w]^{\beta-1} \frac{d^2 w}{\pi} = \frac{\mathbf{\Gamma}(\alpha|\alpha')\mathbf{\Gamma}(\beta|\beta')}{\mathbf{\Gamma}(\alpha+\beta|\alpha'+\beta')} [z_2 - z_1]^{\alpha+\beta-1}.$$

Here and below $1' = 1$ and the "complex" gamma function

$$\mathbf{\Gamma}(\alpha|\alpha') := \frac{\Gamma(\alpha)}{\Gamma(1-\alpha')} = \frac{\Gamma(\frac{n+\mathrm{i}x}{2})}{\Gamma(1+\frac{n-\mathrm{i}x}{2})} =: \mathbf{\Gamma}(x,n),$$

where $x \in \mathbb{C}$ and $n \in \mathbb{Z}$. One has

$$\mathbf{\Gamma}(\alpha+1|\alpha') = \mathbf{\Gamma}(x-\mathrm{i}, n+1) = \alpha\mathbf{\Gamma}(\alpha|\alpha'),$$
$$\mathbf{\Gamma}(\alpha|\alpha'+1) = \mathbf{\Gamma}(x-\mathrm{i}, n-1) = -\alpha'\mathbf{\Gamma}(\alpha|\alpha'),$$

From $\Gamma(x)\Gamma(1-x) = \pi/\sin\pi x \Rightarrow$

$$\mathbf{\Gamma}(\alpha|\alpha') = (-1)^{\alpha-\alpha'}\mathbf{\Gamma}(\alpha'|\alpha), \qquad \mathbf{\Gamma}(x,-n) = (-1)^n\mathbf{\Gamma}(x,n),$$
$$\mathbf{\Gamma}(\alpha|\alpha')\mathbf{\Gamma}(1-\alpha|1-\alpha') = (-1)^{\alpha-\alpha'},$$

Linear fractional transformations of $w, z_1, z_2 \Rightarrow$

$$\int_{\mathbb{C}} [z_1 - w]^{\alpha-1} [z_2 - w]^{\beta-1} [z_3 - w]^{\gamma-1} \frac{d^2 w}{\pi}$$
$$= \frac{\mathbf{\Gamma}(\alpha, \beta, \gamma)}{[z_3 - z_2]^{\alpha} [z_1 - z_3]^{\beta} [z_2 - z_1]^{\gamma}},$$

where $\alpha + \beta + \gamma = \alpha' + \beta' + \gamma' = 1$ and

$$\mathbf{\Gamma}(\alpha, \ldots, \gamma) = \mathbf{\Gamma}(\alpha|\alpha') \ldots \mathbf{\Gamma}(\gamma|\gamma').$$

This is a star-triangle relation $\Rightarrow$ solutions of YBE $\Rightarrow$ noncompact XXX chain.

Multidimensional generalization: Selberg integral.

Complexification: $2d$ CFT (Dotsenko, Fateev, 1985), (Aomoto, 1987), the most general form (Sarkissian, V.S., 2021)

The standard Newton's binomial theorem

$$ {}_1F_0(a;x) = \sum_{n=0}^{\infty} \frac{(a)_n}{n!} x^n = (1-x)^{-a}, \quad (a)_n = a(a+1)\dots(a+n-1). $$

Complexification:

$$ \frac{1}{[x+y]^{\alpha}} = \frac{1}{4\pi \mathbf{\Gamma}(\alpha|\alpha')} \sum_{N\in\mathbb{Z}} \int_L d\nu \, \frac{\mathbf{\Gamma}(s|s')\mathbf{\Gamma}(\alpha - s|\alpha' - s')}{[x]^{\alpha-s}[y]^{s}}, $$

where

$$ s = \tfrac{1}{2}(N + \mathrm{i}\nu), \quad \alpha = \tfrac{1}{2}(m + \mathrm{i}a), \quad N, m \in \mathbb{Z}. $$

The contour L lies in the strip $\mathrm{Im}(a) < \mathrm{Im}(\nu) < 0$.

$\Rightarrow$ The Mellin-Barnes form of STR (Derkachov, Manashov, Valinevich, 2018)

$$ \frac{1}{4\pi} \sum_{n\in\mathbb{Z}} \int_{-\infty}^{\infty} \prod_{j=1}^{3} \mathbf{\Gamma}(b_j{+}y, n_j{+}n)\mathbf{\Gamma}(a_j{-}y, n{-}m_j) dy = \prod_{j,k=1}^{3} \mathbf{\Gamma}(b_j{+}a_k, n_j{+}m_k), $$

where $\mathrm{Im}(b_j)$, $\mathrm{Im}(a_j) < 0$ and the balancing condition

$$ \sum_{j=1}^{3}(b_j + a_j) = -2\mathrm{i}, \qquad \sum_{j=1}^{3}(n_j + m_j) = 0. $$

The elliptic gamma function

$$\Gamma(z;p,q) := \prod_{j,k=0}^{\infty} \frac{1 - z^{-1}p^{j+1}q^{k+1}}{1 - zp^j q^k}, \quad |q|, |p| < 1,$$

$$\Gamma(tz^{\pm 1};p,q) := \Gamma(tz;p,q)\Gamma(tz^{-1};p,q).$$

The functional equations

$$\Gamma(qz;p,q) = \theta(z;p)\Gamma(z;p,q), \quad \Gamma(pz;p,q) = \theta(z;q)\Gamma(z;p,q),$$

$$\theta(z;p) = (z;p)_\infty (pz^{-1};p)_\infty, \quad (z;p)_\infty = \prod_{k=1}^{\infty}(1 - zp^k).$$

For $|t_j| < 1$, $\prod_{j=1}^{6} t_j = pq$, the elliptic beta integral (V.S., 2000):

$$\frac{(p;p)_\infty (q;q)_\infty}{4\pi \mathrm{i}} \int_{\mathbb{T}} \frac{\prod_{j=1}^{6} \Gamma(t_j z^{\pm 1};p,q)}{\Gamma(z^{\pm 2};p,q)} \frac{dz}{z} = \prod_{1 \le j < k \le 6} \Gamma(t_j t_k;p,q).$$

The trigonometric degeneration: $p \to 0$ with fixed $z, q \Rightarrow$

$$\Gamma(z;0,q) = \frac{1}{(z;q)_\infty}.$$

Jackson's (trigonometric) q-gamma function

$$\Gamma_q(x) := \frac{(q;q)_\infty}{(q^x;q)_\infty}(1-q)^{1-x}, \quad |q|<1,\ x\in\mathbb{C},$$

$$\Gamma_q(qx) = \frac{1-q^x}{1-q}\Gamma_q(x), \quad \lim_{q\to 1^-}\Gamma_q(x) = \Gamma(x).$$

The hyperbolic degeneration $z, p, q \to 1$:

$$\Gamma(e^{-2\pi v u}; e^{-2\pi v\omega_1}, e^{-2\pi v\omega_2}) \underset{v\to 0^+}{=} e^{-\pi\frac{2u-\omega_1-\omega_2}{12v\omega_1\omega_2}}\gamma^{(2)}(u;\omega_1,\omega_2),$$

The hyperbolic q-gamma function (Ruijsenaars, 1997)

$$\gamma^{(2)}(u;\omega) := \exp\left(-\mathrm{PV}\int_{\mathbb{R}}\frac{e^{ux}}{(1-e^{\omega_1 x})(1-e^{\omega_2 x})}\frac{dx}{x}\right).$$

$$= e^{-\frac{\pi \mathrm{i}}{2}B_{2,2}(u;\omega)}\frac{(\tilde{q}e^{2\pi\mathrm{i}\frac{u}{\omega_1}};\tilde{q})_\infty}{(e^{2\pi\mathrm{i}\frac{u}{\omega_2}};q)_\infty}, \qquad q = e^{2\pi\mathrm{i}\frac{\omega_1}{\omega_2}}, \quad \tilde{q} = e^{-2\pi\mathrm{i}\frac{\omega_2}{\omega_1}},$$

$$B_{2,2}(u;\omega) = \frac{1}{\omega_1\omega_2}\left((u-\frac{\omega_1+\omega_2}{2})^2 - \frac{\omega_1^2+\omega_2^2}{12}\right).$$

Functional equations and the inversion relation:

$$\frac{\gamma^{(2)}(u+\omega_1;\omega)}{\gamma^{(2)}(u;\omega)} := 2\sin\frac{\pi u}{\omega_2}, \quad \frac{\gamma^{(2)}(u+\omega_2;\omega)}{\gamma^{(2)}(u;\omega)} := 2\sin\frac{\pi u}{\omega_1},$$
$$\gamma^{(2)}(x;\omega)\gamma^{(2)}(\omega_1+\omega_2-x;\omega) = 1.$$

- The inverse of Shintani's (1977) double-sine function $1/S(x;\omega)$.
- The Faddeev's (1994) modular (non-compact quantum) dilogarithm.

The usual rational degeneration

$$\gamma^{(2)}(\omega_1 x;\omega) \underset{\omega_1\to 0}{=} \frac{\Gamma(x)}{\sqrt{2\pi}}\left(\frac{\omega_2}{2\pi\omega_1}\right)^{\frac{1}{2}-x}.$$

Complex rational degeneration: Bazhanov, Mangazeev, Sergeev, 2007 (*ad hoc* ansatz); Sarkissian, V.S., 2019 (rigorous derivation)

$$b_{CFT} := \sqrt{\frac{\omega_1}{\omega_2}} = \mathrm{i} + \delta, \quad \delta \to 0^+,$$

$$\gamma^{(2)}(\mathrm{i}\sqrt{\omega_1\omega_2}(n + x\delta); \omega_1, \omega_2) \underset{\delta\to 0^+}{=} e^{\frac{\pi \mathrm{i}}{2}n^2}(4\pi\delta)^{\mathrm{i}x-1}\mathbf{\Gamma}(x, n),$$

where $n \in \mathbb{Z}$, $x \in \mathbb{C}$. Then

$$\frac{(\tilde{q}e^{2\pi \mathrm{i}\frac{u}{\omega_1}}; \tilde{q})_\infty}{(e^{2\pi \mathrm{i}\frac{u}{\omega_2}}; q)_\infty} = \frac{\Gamma_q\left(\frac{n+\mathrm{i}x}{2} + O(\delta)\right)}{\Gamma_{\tilde{q}}\left(1 + \frac{n-\mathrm{i}x}{2} + O(\delta)\right)} \frac{(\tilde{q}; \tilde{q})_\infty}{(q; q)_\infty} \frac{(1 - \tilde{q})^{\frac{-n+\mathrm{i}x}{2} + O(\delta)}}{(1 - q)^{1-\frac{n+\mathrm{i}x}{2}+O(\delta)}}.$$

The crucial point: despite of $\tilde{q} = q + O(\delta^2)$, one has

$$\frac{(\tilde{q}; \tilde{q})_\infty}{(q; q)_\infty} = e^{\frac{\pi \mathrm{i}}{12}\left(\frac{\omega_2}{\omega_1}+\frac{\omega_1}{\omega_2}\right)}\left(-\mathrm{i}\frac{\omega_1}{\omega_2}\right)^{\frac{1}{2}} \underset{\delta\to 0^+}{=} e^{\frac{\pi \mathrm{i}}{12}}.$$

Apply this degeneration to integrable many-body systems of the Ruijsenaars type, i.e. finite-difference hyperbolic Calogero-Sutherland type models (Sarkissian, V.S., 2021).

Integrable many-body systems

Elliptic Ruijsenaars model Hamiltonian

$$\mathcal{H}_R = \sum_{j=1}^{N} \prod_{k=1,\neq j}^{N} \frac{\theta(tz_j z_k^{-1};p)}{\theta(z_j z_k^{-1};p)} T_j,$$

where $z_j, t, q, p \in \mathbb{C}$, $|p| < 1$, and $T_j = q$-shift operators

$$T_j \psi(z_1, \ldots, z_j, \ldots, z_N) = \psi(z_1, \ldots, qz_j, \ldots, z_N).$$

Elliptic van Diejen system (generalized BC_n Ruijsenaars system)

$$\mathcal{H}_{vD} = \sum_{j=1}^{N} \left(b_j(\underline{u}) e^{2\gamma \partial_{u_j}} + b_j(-\underline{u}) e^{-2\gamma \partial_{u_j}} \right) + v(\underline{u}),$$

where $e^{2\gamma \partial_{u_j}} f(\ldots, u_j, \ldots) = f(\ldots, u_j + 2\gamma, \ldots)$ and

$$b_j(\underline{u}) = \prod_{k=1,\neq j}^{N} \frac{\theta_1(u_j \pm u_k + \mu)}{\theta_1(u_j \pm u_k)} \prod_{r=0}^{3} \frac{\theta_{r+1}(u_j + \mu_r)\theta_{r+1}(u_j + \gamma + \mu_r')}{\theta_{r+1}(u_j)\theta_{r+1}(u_j + \gamma)},$$

$v(\underline{u})$ is some elliptic potential, $\theta_j(u) =$ Jacobi theta functions. .

Denote

$$q = e^{4\pi \mathrm{i} \gamma}, \quad z_j = e^{2\pi \mathrm{i} u_j}, \quad t = e^{2\pi \mathrm{i} \mu},$$

$$t_1 = -e^{2\pi \mathrm{i} \mu_1},\ t_2 = -p^{1/2} e^{2\pi \mathrm{i} \mu_2},\ t_3 = p^{1/2} e^{2\pi \mathrm{i} \mu_3},\ t_4 = e^{2\pi \mathrm{i} \mu_0},$$

$$t_5 = q^{1/2} e^{2\pi \mathrm{i} \mu_0'},\ t_6 = -q^{1/2} e^{2\pi \mathrm{i} \mu_1'},\ t_7 = -q^{1/2} p^{1/2} e^{2\pi \mathrm{i} \mu_2'},\ t_8 = q^{1/2} p^{1/2} e^{2\pi \mathrm{i} \mu_3'}.$$

and impose the balancing condition

$$2(N-1)\mu + \sum_{r=0}^{3}(\mu_r + \mu_r') = 0, \quad \text{or} \quad t^{2N-2}\prod_{m=1}^{8} t_m = p^2q^2.$$

$\Rightarrow$ up to a multiplicative and additive constants

$$\mathcal{H}_{vD} = \sum_{j=1}^{N}\Big(A_j(\underline{z})(T_j - 1) + A_j(\underline{z}^{-1})(T_j^{-1} - 1)\Big),$$

$$A_j(\underline{z}) = \frac{\prod_{m=1}^{8}\theta(t_m z_j;p)}{\theta(z_j^2, qz_j^2;p)}\prod_{\substack{k=1\\k\neq j}}^{N}\frac{\theta(tz_jz_k^{\pm1};p)}{\theta(z_jz_k^{\pm1};p)},$$

and $A_j(\ldots, pz_l, \ldots) = A_j(\underline{z})$.

The eigenvalue problem $\mathcal{H}_{vD}\psi(\underline{z}) = \lambda\psi(\underline{z})$ for $N=1$ yields

$$\frac{\prod_{j=1}^8 \theta(t_j z;p)}{\theta(z^2, qz^2;p)}(\psi(qz)-\psi(z))+\frac{\prod_{j=1}^8 \theta(t_j z^{-1};p)}{\theta(z^{-2}, qz^{-2};p)}(\psi(q^{-1}z)-\psi(z)) = \lambda\psi(z),$$

which becomes the elliptic hypergeometric equation for

$$t_8 = qt_7, \qquad \lambda = -\prod_{k=1}^{6} \theta\left(\frac{t_k t_8}{q};p\right).$$

with a solution

$$\psi(z) = \frac{V(q/ct_1, \dots, q/ct_5, cz, c/z, c/t_8;p,q)}{\Gamma(t_8 z^{\pm 1}, c^2 z^{\pm 1}/t_8;p,q)}, \quad c^2 = \frac{t_6 t_8}{p^4},$$

where V is an elliptic analogue of ${}_2F_1$ function (V.S., 2003)

$$V(w_1, \dots, w_8;p,q) = \frac{(p;p)_\infty (q;q)_\infty}{4\pi \mathrm{i}} \int_{\mathbb{T}} \frac{\prod_{j=1}^8 \Gamma(w_j x^{\pm 1};p,q)}{\Gamma(x^{\pm 2};p,q)} \frac{dx}{x},$$

where $\prod_{j=1}^8 w_j = (pq)^2$, $|w_j| < 1$.

Two-step degeneration: elliptic → hyperbolic → complex hypergeometric ⇒ recurrence-difference equation:

$$t_a = e^{-2\pi v g_a},\ z = e^{-2\pi v y},\ p = e^{-2\pi v \omega_1},\ q = e^{-2\pi v \omega_2},\ v \to 0^+ \ \Rightarrow$$

$$\frac{\prod_{j=1}^{8} \sin \frac{\pi}{\omega_1}(g_j + y)}{\sin \frac{2y\pi}{\omega_1} \sin \frac{(2y+\omega_2)\pi}{\omega_1}}(\chi(y+\omega_2)-\chi(y))+\frac{\prod_{j=1}^{8} \sin \frac{\pi}{\omega_1}(g_j - y)}{\sin \frac{2y\pi}{\omega_1} \sin \frac{(2y-\omega_2)\pi}{\omega_1}}(\chi(y-\omega_2)-\chi(y))$$

$$+\prod_{k=1}^{6} \sin \frac{\pi}{\omega_1}\,(g_k + g_7)\,\chi(y) = 0,$$

$$\sum_{j=1}^{8} g_j = 2(\omega_1 + \omega_2), \qquad g_8 = g_7 + \omega_2.$$

where $\chi(y) = I_h(...)/(product\, of\, four\, \gamma^{(2)}(...))$,

$$I_h(\underline{a}; \omega_1, \omega_2) = \int_{-\mathrm{i}\infty}^{\mathrm{i}\infty} \frac{\prod_{j=1}^{8} \gamma^{(2)}(a_j \pm x; \omega_1, \omega_2)}{\gamma^{(2)}(\pm 2x; \omega_1, \omega_2)} \frac{dx}{2\mathrm{i}\sqrt{\omega_1 \omega_2}},$$

$$\sum_{j=1}^{8} a_j = 2(\omega_1 + \omega_2).$$

$$y = \mathrm{i}\sqrt{\omega_1\omega_2}(m+u\delta), \quad x = \mathrm{i}\sqrt{\omega_1\omega_2}(n+v\delta), \quad g_j = \mathrm{i}\sqrt{\omega_1\omega_2}(r_j+\gamma_j\delta),$$

where $u, v, \gamma_j \in \mathbb{C}$, $m, n, r_j \in \frac{1}{2}\mathbb{Z}$, the limit $\sqrt{\frac{\omega_1}{\omega_2}} = \mathrm{i} + \delta$, $\delta \to 0^+$.

Then

$$\frac{\prod_{j=1}^{8}(\beta_j + z)}{2z(2z+1)}(\Psi(u-\mathrm{i}, m-1) - \Psi(u, m))$$

$$+\frac{\prod_{j=1}^{8}(\beta_j - z)}{2z(2z-1)}(\Psi(u+\mathrm{i}, m+1) - \Psi(u, m)) + \prod_{k=1}^{6}(\beta_k + \beta_7)\,\Psi(u, m) = 0,$$

where

$$\beta_j = \frac{\mathrm{i}\gamma_j - r_j}{2}, \qquad z = \frac{\mathrm{i}u - m}{2}, \quad r_j, m \in \mathbb{Z} + \mu, \ \mu = 0, \frac{1}{2},$$

with the additional constraints

$$\sum_{j=1}^{8} \gamma_j = -4\mathrm{i}, \quad \sum_{j=1}^{8} r_j = 0, \quad r_8 = r_7 - 1, \quad \gamma_8 = \gamma_7 - \mathrm{i}.$$

A solution $\Psi(u, m) = S(...)/(product\, of\, four\, \mathbf{\Gamma}(...))$,

$$S(\underline{s},\underline{n}) = \frac{1}{8\pi} \sum_{n\in\mathbb{Z}+\nu} \int_{-\infty}^{\infty} (n^2+v^2) \prod_{j=1}^{8} \mathbf{\Gamma}(s_k \pm v, n_k \pm n) dv,$$

$n_k \in \mathbb{Z}+\nu$, $\nu = 0, \frac{1}{2}$, with

$$s_k = -3\mathrm{i} - \gamma_k - \frac{1}{2}(\gamma_6+\gamma_8), \quad n_k = 1 - r_k - \frac{1}{2}(r_6+r_8), \quad k = 1,\dots,5,$$

$$s_{6,7} = \frac{1}{2}(\gamma_6+\gamma_8) + 2\mathrm{i} \pm u, \quad n_{6,7} = \frac{1}{2}(r_6+r_8) - 2 \pm m,$$

$$s_8 = \frac{1}{2}(\gamma_6-\gamma_8) + 2\mathrm{i}, \quad n_8 = \frac{1}{2}(r_6-r_8) - 2.$$

Extension to the general balanced case

$$\mathcal{H}_{rat} = \sum_{j=1}^{N} \left(B_j(\underline{u}, \underline{m})(T_{u_j,m_j} - 1) + B_j(-\underline{u}, -\underline{m})(T_{u_j,m_j}^{-1} - 1) \right),$$

where the difference-recurrence operators act as follows

$$T_{u_j,m_j}^{\pm 1} f(\underline{u}, \underline{m}) = f(\ldots, u_j \mp \mathrm{i}, \ldots, m_j \mp 1, \ldots)$$

and the potentials have the form

$$B_j(\underline{u}, \underline{m}) = \prod_{\substack{k=1 \\ k \neq j}}^{N} \frac{(\beta + z_j + z_k)(\beta + z_j - z_k)}{(z_j + z_k)(z_j - z_k)} \frac{\prod_{l=1}^{8}(\beta_l + z_j)}{2z_j(2z_j + 1)},$$

$$z_j = \frac{\mathrm{i}u_j - m_j}{2}, \qquad \beta = \frac{\mathrm{i}\gamma - r}{2}, \qquad \beta_k = \frac{\mathrm{i}\gamma_k - r_k}{2},$$

where $u_j, \gamma, \gamma_k \in \mathbb{C}$, $r \in \mathbb{Z}$, $m_j, r_k \in \mathbb{Z} + \mu, \mu = 0, \frac{1}{2}$, with the balancing condition

$$(2N - 2)\gamma + \sum_{k=1}^{8} \gamma_k = -4\mathrm{i}, \qquad (2N - 2)r + \sum_{k=1}^{8} r_k = 0.$$

The scalar product

$$\langle \varphi, \psi \rangle_{rat} = \frac{1}{(8\pi)^N N!} \sum_{m_j \in \mathbb{Z}+\mu} \int_{u_j \in \mathbb{R}} \varphi(\underline{u}, \underline{m}) \psi(\underline{u}, \underline{m})$$
$$\times \prod_{1 \leq j < k \leq N} \frac{\mathbf{\Gamma}(\gamma \pm u_j \pm u_k, r \pm m_j \pm m_k)}{\mathbf{\Gamma}(\pm u_j \pm u_k, \pm m_j \pm m_k)}$$
$$\times \prod_{j=1}^{N} \left[\prod_{\ell=1}^{8} \mathbf{\Gamma}(\gamma_\ell \pm u_j, r_\ell \pm m_j) \right] (u_j^2 + m_j^2)\, du_j,$$

Hermicity:

$$\langle \varphi, \mathcal{H}_{rat} \psi \rangle_{rat} = \langle \mathcal{H}_{rat} \varphi, \psi \rangle_{rat}.$$

$\langle 1,1\rangle_{rat}$ for $\gamma_7+\gamma_8=-2\mathrm{i}$, $r_7+r_8=0$ yields the most general form of the complex Selberg integral (Sarkissian, V.S., 2021)

$$\frac{1}{(8\pi)^N N!}\sum_{m_j\in\mathbb{Z}+\mu}\int_{u_j\in\mathbb{R}}\prod_{1\le j<k\le N}\frac{\mathbf{\Gamma}(\gamma\pm u_j\pm u_k, r\pm m_j\pm m_k)}{\mathbf{\Gamma}(\pm u_j\pm u_k,\pm m_j\pm m_k)}$$
$$\times\prod_{j=1}^N\left[\prod_{\ell=1}^6\mathbf{\Gamma}(\gamma_\ell\pm u_j, r_\ell\pm m_j)\right](u_j^2+m_j^2)\,du_j=(-1)^{r\frac{N(N-1)}{2})}$$
$$\times\prod_{j=1}^N\frac{\mathbf{\Gamma}(j\gamma,jr)}{\mathbf{\Gamma}(\gamma,r)}\prod_{1\le\ell<s\le 6}\mathbf{\Gamma}((j-1)\gamma+\gamma_\ell+\gamma_s,(j-1)r+r_\ell+r_s),$$

where $r\in\mathbb{Z}$, $m_j,r_\ell\in\mathbb{Z}+\mu$, $\mu=0,\frac{1}{2}$, and

$$(2N-2)\gamma+\sum_{\ell=1}^6\gamma_\ell=-2\mathrm{i},\quad (2N-2)r+\sum_{\ell=1}^6 r_\ell=0.$$

Ordinary rational Ruijsenaars model Hamiltonian

$$\mathcal{H}_R = \sum_{j=1}^{N} \prod_{k=1,\neq j}^{N} \frac{z_j - z_k + g}{z_j - z_k}\, e^{\partial_{z_j}},$$

Two particle case in the center of mass, $z_1 + z_2 = 0$,

$$\mathcal{H}_R = \frac{z+g}{z}\, e^{\partial_z} + \frac{z-g}{z}\, e^{-\partial_z}, \quad z = z_1 - z_2.$$

Two particle hyperbolic Ruijsenaars Hamiltonian eigenvalue problem in the center of mass

$$\frac{\sin \frac{\pi}{\omega_1}(x+g)}{\sin \frac{\pi}{\omega_1} x} \chi(x+\omega_2) + \frac{\sin \frac{\pi}{\omega_1}(x-g)}{\sin \frac{\pi}{\omega_1} x} \chi(x-\omega_2) = 2\cos \frac{\pi\lambda}{\omega_1}\, \chi(x)$$

Its solution

$$\chi(x) = \int_{-\mathrm{i}\infty}^{\mathrm{i}\infty} \gamma^{(2)}(\tfrac{\omega_1+\omega_2-g}{2} \pm \tfrac{x}{2} \pm z) e^{2\pi \mathrm{i} \frac{\lambda z}{\omega_1 \omega_2}} \frac{dz}{\mathrm{i}\sqrt{\omega_1\omega_2}}.$$

Take parametrizations

$$g = \mathrm{i}\sqrt{\omega_1\omega_2}(r + \gamma\delta), \quad z = \mathrm{i}\sqrt{\omega_1\omega_2}(k + y\delta), \quad r, k \in \mathbb{Z},$$
$$\lambda = \mathrm{i}\sqrt{\omega_1\omega_2}(2N + \alpha), \quad x = \mathrm{i}\sqrt{\omega_1\omega_2}(n + u\delta), \quad N, n \in \mathbb{Z},$$

and consider the *new* limit (Belousov, Sarkissian, V.S, 2024)

$$\sqrt{\frac{\omega_1}{\omega_2}} = \mathrm{i} + \delta, \quad \delta \to 0^+, \quad N \to \infty, \quad 2N\delta = \beta = \ fixed,$$

$$\chi(x) \to (4\pi\delta)^{2\mathrm{i}\gamma} e^{\pi \mathrm{i}(n+r+nr)}\, \Psi(u, n),$$

$$\Psi(u, n) = \sum_{k=\mathbb{Z}+\nu} \int_{-\infty}^{\infty} \mathbf{\Gamma}(-\tfrac{1}{2}\gamma - \mathrm{i} \pm \tfrac{1}{2}u \pm y, -\tfrac{r}{2} \pm \tfrac{n}{2} \pm k) e^{-2\pi \mathrm{i}(\alpha k + \beta y)} dy,$$

where $\nu = 0$, if $r + n$ is even, and $\nu = \frac{1}{2}$, if $r + n$ is odd. The eigenvalue equation becomes

$$\left(\frac{z + \rho}{z} e^{\partial_z} + \frac{z - \rho}{z} e^{-\partial_z} \right) \Psi(u, n) = \left(e^{\lambda} + e^{-\lambda} \right) \Psi(u, n),$$

$$e^{\partial_z} = e^{-\partial_n} e^{-\mathrm{i}\partial_u}, \quad e^{\partial_z}\Psi(u, n) = \Psi(u - \mathrm{i}, n - 1),$$

$$\lambda = \pi(\mathrm{i}\alpha - \beta), \quad z = \frac{\mathrm{i}u - n}{2}, \quad \rho = \frac{\mathrm{i}\gamma - r}{2}$$

$\Rightarrow$ related to the model by Molchanov, Neretin, 2018.