Going beyond Vogel's universality for simple Lie algebras

A. P. Isaev[1,2]

[1]Bogoliubov Laboratory of Theoretical Physics, JINR, Dubna

[2]Chair of Theoretical Physics, Faculty of Physics,
M.V.Lomonosov Moscow State University

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Content

Vogel parameters, Vogel map and universal Lie algebra.

1. Consider tensor products of r adjoint representations of the simple LA $\mathfrak{g}$ and consider Clebsch-Gordan expansion of these products:

$$\mathrm{ad}^{\otimes r} := \underbrace{\mathrm{ad} \otimes \mathrm{ad} \otimes \cdots \otimes \mathrm{ad}}_{r} = \oplus_\lambda \, n_\lambda T_\lambda \; , \qquad (1)$$

where T_λ are irreps, λ – parameters which numerate irreps (e.g. highest weights) and $n_\lambda \in \mathbb{Z}_{>0}$ are multiplicities.

2. The elements of the vector space of rep $\mathrm{ad}^{\otimes r}$ are rank r tensors $t^{a_1 a_2 \dots a_r}$. Invariant subspaces in $V_{\mathrm{ad}}^{\otimes r}$ are spaces of $t^{a_1 a_2 \dots a_r}$ with special symmetrization of indices $(a_1, a_2, \dots a_r)$ (according to Young diagrams $\vdash r$): $t_{\pm}^{a_1 a_2} = \frac{1}{2}(t^{a_1 a_2} \pm t^{a_2 a_1})$. Thus, it is possible to group the representations in the r.h.s. of (1) so that the decomposition is converted into

$$\mathrm{ad}^{\otimes r} = \oplus_\Lambda \, T_\Lambda \; , \qquad T_\Lambda := \mathbb{P}_\Lambda(\mathrm{ad}^{\otimes r}) \, , \qquad (2)$$

where Λ are Young diagrams $\vdash r$ and T_Λ are (reducible) reps in the invariant subspaces which are extracted from $V_{\mathrm{ad}}^{\otimes r}$ by Young projectors $\mathbb{P}_\Lambda$ related to Λ. The decomposition (2) is universal for all Lie algebras $\mathfrak{g}$.

3. Amazing fact: it was noticed [P.Deligne (1996), P.Vogel (1999), J.M.Landsberg and L.Manivel (2002),...] that, for first $r = 2, 3, 4$, subreps T_Λ in the r.h.s. of $\mathrm{ad}^{\otimes r} = \oplus_\Lambda T_\Lambda$ can be decomposed further

$$T_\Lambda = \oplus_{c_\Lambda} \; T^{(\Lambda)}_{c_\Lambda} \quad \Rightarrow \quad \mathrm{ad}^{\otimes r} = \oplus_\Lambda \; \oplus_{c_\Lambda} \; T^{(\Lambda)}_{c_\Lambda} \;, \qquad (3)$$

such that decomposition (3) is **universal** for all simple Lie algebras $\mathfrak{g}$. Here c_Λ are parameters which numerate subreps $T^{(\Lambda)}_{c_\Lambda}$ in $T^{(\Lambda)}$; they are related to values of quadratic Casimir.

4. Moreover, there are **remarkable universal formulas** for $\dim(T^{(\Lambda)}_{c_\Lambda})$ for all simple LAs $\mathfrak{g}$. Formulas for $\dim(T^{(\Lambda)}_{c_\Lambda})$ are represented as **rational and homogeneous symmetric functions of 3 real parameters** (α, β, γ) called **Vogel parameters**, and all simple Lie algebras $\mathfrak{g}$ are special points in the space of (α, β, γ).

Example: $\mathrm{ad}^{\otimes 2}$ $(r = 2)$. For all simple LAs (with rank > 1) we have decomposition

$$\mathrm{ad}^{\otimes 2} = \mathbb{P}_{[1^2]}(\mathrm{ad}^{\otimes 2}) + \mathbb{P}_{[2]}(\mathrm{ad}^{\otimes 2}) \; = \; (\mathrm{ad} + \mathsf{X}_2) + \big(\mathbf{1} + \mathsf{Y}(\alpha) + \mathsf{Y}(\beta) + \mathsf{Y}(\gamma)\big) \; .$$

Dim. formulas for reps in the r.h.s. are homogeneous rational functions in (α, β, γ) (symmetry in (α, β, γ) is permutation of $\mathsf{Y}(\alpha), \mathsf{Y}(\beta), \mathsf{Y}(\gamma)$).
First we have famous P.Deligne formula:

$$\dim \mathfrak{g} \equiv \dim(\mathrm{ad}) \; = \; \frac{(\alpha - 2t)(\beta - 2t)(\gamma - 2t)}{\alpha\beta\gamma} = \frac{(\hat{\alpha} - 1)(\hat{\beta} - 1)(\hat{\gamma} - 1)}{\hat{\alpha}\hat{\beta}\hat{\gamma}} \, ,$$

$$\hat{\alpha} := \frac{\alpha}{2t}, \quad \hat{\beta} := \frac{\beta}{2t}, \quad \hat{\gamma} := \frac{\gamma}{2t} \, , \qquad \boxed{t := \alpha + \beta + \gamma} \, .$$

Also we have

$$\dim(\mathsf{X}_2) \; = \; \frac{1}{2} \dim \mathfrak{g} \, (\dim \mathfrak{g} - 3) = \frac{(1 - \hat{\alpha}^2)(1 - \hat{\beta}^2)(1 - \hat{\gamma}^2)}{(\hat{\alpha}\hat{\beta}\hat{\gamma})^2} \; .$$

For $\dim(Y(\alpha)), \ldots$ we have similar formulas.

Since all $\dim(T_{c_\wedge})$ are homogeneous symmetric functions of Vogel parameters (α, β, γ), it is possible to fix one of them, e.g. $\alpha = -2$. For this choice the sum $t := \alpha + \beta + \gamma$ coincides with dual Coxeter number $h^\vee$.

Table 1

Type	Lie algebra	α	β	γ	$t = h^\vee = \alpha + \beta + \gamma$	$\hat{\gamma} = \frac{\gamma}{2t}$
A_n	$s\ell(n+1)$	-2	2	$n+1$	$n+1$	$1/2$
B_n	$so(2n+1)$	-2	4	$2n-3$	$2n-1$	$\frac{2n-3}{2(2n-1)}$
C_n	$sp(2n)$	-2	1	$n+2$	$n+1$	$\frac{n+2}{2(n+1)}$
D_n	$so(2n)$	-2	4	$2n-4$	$2n-2$	$\frac{n-2}{2(n-1)}$
G_2	$\mathfrak{g}_2$	-2	$10/3$	$8/3$	4	$1/3$
F_4	$\mathfrak{f}_4$	-2	5	6	9	$1/3$
E_6	$\mathfrak{e}_6$	-2	6	8	12	$1/3$
E_7	$\mathfrak{e}_7$	-2	8	12	18	$1/3$
E_8	$\mathfrak{e}_8$	-2	12	20	30	$1/3$

Note that, for all exceptional Lie algebras we have $2t = 3\gamma \;\rightarrow\; \hat{\gamma} = 1/3$.

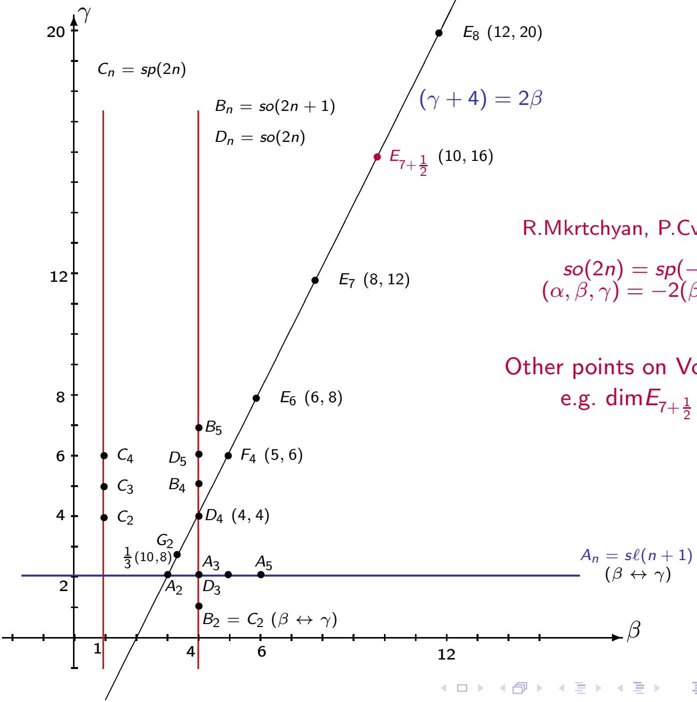
Since all $\dim(T_{C_\Lambda})$ are homogeneous symmetric functions of the Vogel parameters, one can consider all simple Lie algebras as points on the 2d plane $\mathcal{P}_{(\alpha=-2)}$ in 3d space of the Vogel parameters (α, β, γ). More precisely they are points in $\mathbb{RP}^2/S_3$ (the Vogel map).

Before we represent the Vogel map, we note that condition $2t = 3\gamma$, for exceptional LAs, defines the line $(\gamma + 4) = 2\beta$ on the plane $\mathcal{P}_{(\alpha=-2)} \in \mathbb{R}^3$. Remarkable fact: points of Lie algebras $s\ell(3)$ and $so(8)$ are also on this line.

Type	Lie algebra	α	β	γ	$t = h^\vee = \alpha + \beta + \gamma$	$\hat{\gamma} = \frac{\gamma}{2t}$
A_2	$s\ell(3)$	-2	2	3	3	$\frac{\beta}{2t} = 1/3$
D_4	$so(8)$	-2	4	4	6	$1/3$
G_2	$\mathfrak{g}_2$	-2	$10/3$	$8/3$	4	$1/3$
F_4	$\mathfrak{f}_4$	-2	5	6	9	$1/3$
E_6	$\mathfrak{e}_6$	-2	6	8	12	$1/3$
E_7	$\mathfrak{e}_7$	-2	8	12	18	$1/3$
E_8	$\mathfrak{e}_8$	-2	12	20	30	$1/3$

Unified description of all simple LA by means of 3 parameters (α, β, γ) leads to the conjecture of existing the **universal LA**.

Vogel map (1999)



R.Mkrtchyan, P.Cvitanović

$so(2n) = sp(-2n) \Leftrightarrow$
$(\alpha, \beta, \gamma) = -2(\beta, \alpha, \gamma)$

Other points on Vogel map
e.g. $\dim E_{7+\frac{1}{2}} = 190$

Example: $\mathrm{ad}^{\otimes 2}$, $(r=2)$. For all simple LAs (with rank >1) we have decomposition

$$\mathrm{ad}^{\otimes 2} = \mathbb{A}(\mathrm{ad}^{\otimes 2}) + \mathbb{S}(\mathrm{ad}^{\otimes 2}) \;=\; (\mathrm{ad} + \mathsf{X}_2) + \big(\mathbf{1} + \mathsf{Y}(\alpha) + \mathsf{Y}(\beta) + \mathsf{Y}(\gamma)\big)\;.$$

Dim. formulas for reps are homogeneous rational functions in (α,β,γ) (symmetry in (α,β,γ) is permutation of $\mathsf{Y}(\alpha),\mathsf{Y}(\beta),\mathsf{Y}(\gamma)$) [Vogel(1999)]:

$$\dim(\mathrm{ad}) \equiv \dim\mathfrak{g} = \frac{(\alpha-2t)(\beta-2t)(\gamma-2t)}{\alpha\beta\gamma}\,, \quad \boxed{t := \alpha+\beta+\gamma}\,, \tag{4}$$

$$\dim(\mathsf{X}_2) = \tfrac{1}{2}\dim\mathfrak{g}\,(\dim\mathfrak{g}-3)\,,\ \ 20|_{sl(3)},\ 350|_{so(8)}\,, \qquad \dim(\mathbf{1}) = 1\,,$$

$$\boxed{\dim\mathsf{Y}(\alpha) = \frac{(2t-3\alpha)(\beta-2t)(\gamma-2t)t(\beta+t)(\gamma+t)}{\alpha^2(\alpha-\beta)(\alpha-\gamma)\beta\gamma}}\,, \quad 27|_{sl(3)} \sim [4,2],\ 300|_{so(8)}$$

$$\dim\mathsf{Y}(\beta) = \dim\mathsf{Y}(\alpha)|_{\alpha\leftrightarrow\beta}\,, \quad 0|_{sl(3)},\ \tfrac{0}{0} = 35, 35', 35''|_{so(8)}$$

$$\dim\mathsf{Y}(\gamma) = \dim\mathsf{Y}(\alpha)|_{\alpha\leftrightarrow\gamma}\,, \quad 8|_{sl(3)},\ \tfrac{0}{0} = 0|_{so(8)}$$

In the rhs we give dims for two "exceptional" algebras $\mathfrak{sl}_3, \mathfrak{so}_8$.

Remark. For the exceptional line $2t = 3\gamma$, we have $\dim\mathsf{Y}(\gamma) = 0$. It means that, for axceptional LA, in the decomposition of $\mathrm{ad}^{\otimes 2}$, the representation $\mathsf{Y}(\gamma)$ is missing:

$$\mathrm{ad}^{\otimes 2} = \mathbb{A}(\mathrm{ad}^{\otimes 2}) + \mathbb{S}(\mathrm{ad}^{\otimes 2}) = (\mathrm{ad} + \mathsf{X}_2) + \big(\mathbf{1} + \mathsf{Y}(\alpha) + \mathsf{Y}(\beta)\big)\;. \tag{5}$$

Some achievements in the universal description of simple LA & LG.

1.) The generating function of universal eigenvalues $C_{\rm ad}^{(k)}$ of the higher Casimir operators in the ad-representation of $\mathfrak{g}$ [R.Mkrtchyan, A.Sergeev and A.Veselov (2012)]

$$\hat{C}(z) \; = \; \sum_{k=0}^{\infty} C_{\rm ad}^{(k)} z^k = \frac{1}{\dim(\mathfrak{g})} \left(\frac{1}{1+z} + \frac{\dim Y(\alpha)}{1+\frac{z\alpha}{2t}} + \frac{\dim Y(\beta)}{1+\frac{z\beta}{2t}} + \frac{\dim Y(\gamma)}{1+\frac{z\gamma}{2t}} \right) + \\ + \frac{1}{2}\dim(\mathfrak{g}) + \frac{1}{1+\frac{z}{2}} - \frac{3}{2} \; .$$

2.) Formula for volumes of compact simple Lie groups G [R.Mkrtchyan, A.Veselov]

$$\mathrm{Vol}(G) = (2^{3/2}\pi)^{\dim \mathfrak{g}} \; e^{-\Phi(\alpha,\beta,\gamma)} \; ,$$

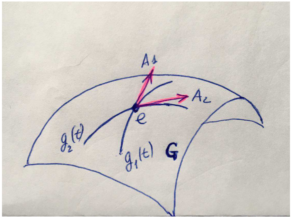
where $\Phi(\alpha,\beta,\gamma) = \int_0^\infty dz \frac{F(z/t)}{z(e^z-1)}$ and

$$F(z) = \frac{\mathrm{sh}\frac{z}{4}(\alpha - 2t) \, \mathrm{sh}\frac{z}{4}(\beta - 2t) \, \mathrm{sh}\frac{z}{4}(\gamma - 2t)}{\mathrm{sh}\frac{z}{4}\alpha \; \mathrm{sh}\frac{z}{4}\beta \; \mathrm{sh}\frac{z}{4}\gamma} - \dim \mathfrak{g} \; . \qquad (6)$$

Here the first term in rhs is the deformation of the universal formula (4) for $\dim \mathfrak{g} = \frac{(\alpha-2t)(\beta-2t)(\gamma-2t)}{\alpha\beta\gamma}$ (it is clear that $F(z)|_{z=0} = 0$).

A Lie group G is a smooth manifold. Consider the tangent vector space $T_e(G)$ to the Lie group G at the unit element $e \in G$.

Definition. *The tangent vector space $T_e(G)$, equipped with the multiplication $[A_1,\ A_2] \in T_e(G)$ $(\forall A_1, A_2 \in T_e(G))$ with axioms:*
1) Anticommutativity: $[A_1,\ A_2] = -[A_2,\ A_1]$,
2) Jacoby identity: $[[A_1,\ A_2],\ A_3] + [[A_3,\ A_1],\ A_2] + [[A_2,\ A_3],\ A_1] = 0$,
is called the Lie algebra $\mathfrak{g}$ of the Lie group G.



Let $X_a \,|_{a=1,\dots,\dim\mathfrak{g}} \in T_e(G)$ be basis elements of Lie algebra (LA) $\mathfrak{g}$:

$$[X_a,\ X_b] = C_{ab}^d\, X_d\ , \tag{7}$$

C_{ab}^d – are structure constants. Matrices $\mathrm{ad}(X_a)_b^d = C_{ab}^d$ define **the adjoint representation** of $\mathfrak{g}$. The **invariant Cartan-Killing metric** in $T_e(G)$ is

$$\mathrm{g}_{ab} \equiv \mathrm{Tr}(\mathrm{ad}(X_a)\cdot \mathrm{ad}(X_b)) = C_{ac}^d\, C_{bd}^c\ . \tag{8}$$

For **simple Lie algebras**, the metric g_{ab} is invertible:

$$\mathrm{g}_{ab}\,\mathrm{g}^{bc} = \delta_a^c \ ,$$

and unique up to a normalization factor: $\mathrm{g}_{ab} \to \lambda \mathrm{g}_{ab}$.
For compact Lie algebras $\mathfrak{g}$, one can chose the basis: $\mathrm{g}_{ab} = -\delta_{ab}$.

The **classification of simple Lie algebras** (E.Cartan-H.Weyl):
4 infinite series (accidental isomorphisms are not taken into account):

$$\mathbf{1}.\,A_n:\ \mathfrak{s}\ell(n+1);\quad \mathbf{2}.\,B_n:\ \mathfrak{so}(2n+1);\quad \mathbf{3}.\,C_n:\ \mathfrak{sp}(2n);\quad \mathbf{4}.\,D_n:\ \mathfrak{so}(2n);$$

$$\dim \mathfrak{s}\ell(N) = N^2-1\,,\quad \dim \mathfrak{so}(N) = \frac{N(N-1)}{2}\,,\quad \dim \mathfrak{sp}(N) = \frac{N(N+1)}{2}\,,$$

and 5 exceptional LA: $\mathfrak{g}_2$, $\mathfrak{f}_4$, $\mathfrak{e}_6$, $\mathfrak{e}_7$, $\mathfrak{e}_8$ with dims: 14, 52, 78, 133, 248.

The main object is **split (or polarized) Casimir operator** of LA $\mathfrak{g}$ is

$$\widehat{C} = \mathrm{g}^{ab} X_b \otimes X_a \equiv X^a \otimes X_a \in \mathfrak{g} \otimes \mathfrak{g} . \qquad (9)$$

The operator $\widehat{C}$ is independent of the choice of the basis X_a in $\mathfrak{g}$ and is related to the standard **quadratic Casimir operator** (central element in the enveloping algebra $\mathcal{U}(\mathfrak{g})$)

$$C^{(2)} = \mathrm{g}^{ab} X_b \cdot X_a \in \mathcal{U}(\mathfrak{g}) . \qquad (10)$$

Relation is via comultiplication $\Delta(X_a) = (X_a \otimes I + I \otimes X_a)$:

$$\begin{array}{c} \Delta(C^{(2)}) = \Delta(X^a) \cdot \Delta(X_a) = C^{(2)} \otimes I + I \otimes C^{(2)} + 2\,\widehat{C} \quad \Rightarrow \\ \widehat{C} = \frac{1}{2}\big(\Delta(C^{(2)}) - C^{(2)} \otimes I - I \otimes C^{(2)}\big) . \end{array} \qquad (11)$$

Remark. The split Casimir operator $\widehat{C}$ commutes with the action of $\mathfrak{g}$:

$$[\Delta(A), \widehat{C}] = [(A \otimes I + I \otimes A), \widehat{C}] = 0 , \quad \forall A \in \mathfrak{g} .$$

Split Casimir operator $\widehat{C}$ appears in many applications: in the RT, in the theory of integrable systems, as colour factors in the nonabelian gauge theories, ...

1) Higher Casimir operators $C^{(k)}$ (for $k > 2$) are constructed via split operator $\widehat{C}$ [S.Okubo, J. Math. Phys. 18(1977) 2382; A.P.I. and V.A. Rubakov, Theory of Groups and Symmetries, WS (2018)]. Indeed, define

$$(\widehat{C})^k = X_{a_1} \cdots X_{a_k} \otimes X^{a_1} \cdots X^{a_k} \in \mathcal{U}(\mathfrak{g}) \otimes \mathcal{U}(\mathfrak{g}) ,$$

then we take *ad*-representation in the second factor and then take trace

$$C^{(k)} = \mathrm{Tr}_2\big((I \otimes \mathrm{ad})\widehat{C}^k\big) = X_{a_1} \cdots X_{a_k} \underbrace{\mathrm{Tr}\big(\mathrm{ad}(X^{a_1} \cdots X^{a_k})\big)}_{d^{a_1 \cdots a_k}} \in \mathcal{U}(\mathfrak{g}) .$$

2) Kohno-Drinfeld Lie algebra and k-split Casimir operator

We define

$$\widehat{C}_{ij} := \mathrm{g}^{ab} \big(I^{\otimes(i-1)} \otimes X_a \otimes I^{\otimes(j-i-1)} \otimes X_b \otimes I^{\otimes(n-j)}\big) \in \mathcal{U}(\mathfrak{g})^{\otimes n} .$$

Defining relations for Kohno-Drinfeld Lie algebra

$$[\widehat{C}_{ij}, \, \widehat{C}_{ik} + \widehat{C}_{jk}] = 0 , \quad [\widehat{C}_{ij}, \, \widehat{C}_{k\ell}] = 0 .$$

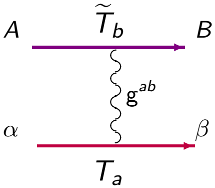
n-split Casimir operator: $\widehat{C}_{(n)} := \sum_{i<j}^{n} \widehat{C}_{ij}$ – Hamiltonians for nonlocal spin chains.

Let T and $\widetilde{T}$ be two representations of $\mathfrak{g}$. One can visualize split Casimir operator in the representation $T \otimes \widetilde{T}$:

$$(T^{\alpha}_{\beta} \otimes \widetilde{T}^{A}_{B})\,\widehat{C} \;=\; \mathrm{g}^{ab}\; T^{\alpha}_{\beta}(X_a)\; \widetilde{T}^{A}_{B}(X_b) \;\equiv\; \mathrm{g}^{ab}\,(T_a)^{\alpha}_{\beta}\,(\widetilde{T}_b)^{A}_{B}\,, \qquad (12)$$

where $\alpha, \beta = 1, ..., \dim T$ and $A, B = 1, ..., \dim \widetilde{T}$:

$$(T_a)^{\alpha}_{\beta}\;\; \mathrm{g}^{ab}\;\; (\widetilde{T}_b)^{A}_{B} \qquad =$$



Colour factor for the Feynman diagram describing scattering of two particles in the representations T and $\widetilde{T}$ by gauge field $A \in \mathfrak{g}$.

Let T_{λ_1} and T_{λ_2} be two irreps of $\mathfrak{g}$ with highest weights λ_1 and λ_2 acting in spaces $\mathcal{V}_{\lambda_1}$ and $\mathcal{V}_{\lambda_2}$. Consider the decomposition $T_{\lambda_1} \otimes T_{\lambda_2} = \sum_\lambda n_\lambda T_\lambda$, where T_λ are irreps with HW λ and n_λ are their multiplicity. Denote the space of T_λ as $\mathcal{V}_\lambda$. Then, from relation $\Delta(C^{(2)}) = C^{(2)} \otimes I + I \otimes C^{(2)} + 2\,\widehat{C}$ we obtain

$$\widehat{C}_{\lambda_1 \times \lambda_2} \cdot \mathcal{V}_\lambda = \tfrac{1}{2}(c_2^{(\lambda)} - c_2^{(\lambda_1)} - c_2^{(\lambda_2)})\mathcal{V}_\lambda \;.$$

Here $\widehat{C}_{\lambda_1 \times \lambda_2} := (T_{\lambda_1} \otimes T_{\lambda_2})(\widehat{C})$ and $c_2^{(\lambda)} = (\lambda, \lambda + 2\,\delta)$ is the value of $C^{(2)}$ on irrep T_λ; $\delta = \frac{1}{2}\sum_{\alpha>0}\alpha$. Note that $T(\widehat{C})$ is diagonalizable for simple LA in any T, but in general its spectrum is degenerate. It implies the characteristic identity

$$\prod_\lambda{}' \Big(\widehat{C}_{\lambda_1 \times \lambda_2} - \hat{c}^{\,\lambda}_{\lambda_1,\lambda_2}\Big) = 0\;, \quad \hat{c}^{\,\lambda}_{\lambda_1,\lambda_2} := \frac{1}{2}(c_2^{(\lambda)} - c_2^{(\lambda_1)} - c_2^{(\lambda_2)})\;,$$

where $\prod'_\lambda$ means that the product runs over only those λ that corresponds to unequal eigenvalues $\hat{c}^{\,\lambda}_{\lambda_1,\lambda_2}$. Then we find projectors in $T_{{}_{\lambda_1 \times \lambda_2}} := T_{\lambda_1} \otimes T_{\lambda_2}$ onto invariant subspaces of $\widehat{C}_{\lambda_1 \times \lambda_2}$ with eigenvalues $a_\lambda := \hat{c}^{\,\lambda}_{\lambda_1,\lambda_2}$:

$$\mathsf{P}_{(a_\lambda)} = \prod_{\mu \neq \lambda} \frac{(\widehat{C}_{\lambda_1 \times \lambda_2} - a_\mu)}{a_\lambda - a_\mu} \quad \Rightarrow \quad T_{{}_{\lambda_1 \times \lambda_2}} = \sum_j \mathsf{P}_{(a_\lambda)} \cdot \Big(T_{{}_{\lambda_1 \times \lambda_2}}\Big)\;.$$

The invariant subspaces of $\mathsf{P}_{(a_\lambda)} \cdot \Big(T_{{}_{\lambda_1 \times \lambda_2}}\Big)$ are called Casimir subspaces.

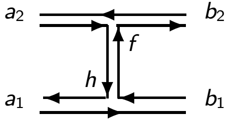
Our method of universal description of LA is based on the extracting of invariant subspaces in $V_{\rm ad}^{\otimes r} = {\rm ad}^{\otimes r}$ by means of the char. ident. for r-split CO $\widehat{C}_{\rm ad}$.

The split Casimir operators and universality in ad$^{\otimes 2}$:

$$(\widehat{C}_{\rm ad})^{a_1 a_2}_{b_1 b_2} \equiv ({\rm ad} \otimes {\rm ad})^{a_1 a_2}_{b_1 b_2} (X_h \otimes X^h) = (X_h)^{a_1}_{\ b_1} (X^h)^{a_2}_{\ b_2} = C^{a_1}_{hb_1} \, C^{a_2}_{fb_2} \, {\rm g}^{hf} ,$$

acts in the space $V_{\rm ad}^{\otimes 2}$ and $V_{\rm ad} \simeq {\rm ad} \simeq \mathfrak{g}$ is the space of ad-representation. Since ad-representation embedded in $T \otimes (T^{\rm T})^{-1}$, one can consider adj. indices $a, b, c, \ldots$ as pairs of fundamental and antifundamental indices $a = (i, \bar{j}), \; b = (k, \bar{\ell}), \ldots$. In view of this, matrices $(\widehat{C}_{\rm ad})^{a_1 a_2}_{b_1 b_2}$ can be represented as Feynman "colour" diagrams (oriented and not oriented lines correspond to $\mathfrak{sl}_N$ and $\mathfrak{so}_N$, $\mathfrak{sp}_{2n}$ cases)

$$(\widehat{C}_{\rm ad})^{a_1 a_2}_{b_1 b_2} \; = \; C^{a_1}_{hb_1} \, C^{a_2}_{fb_2} \, {\rm g}^{hf} \quad =$$



Our aim is to find char. identity $\prod_{i=1}^{k} (\widehat{C}_{\rm ad} - a_i) = 0$ for split CO $\widehat{C}_{\rm ad}$. Then we find k projectors in ${\rm ad}^{\otimes 2}$ onto invariant subspaces of $\widehat{C}_{\rm ad}$ with eigenvalues a_j:

$$\mathrm{P}_{(a_j)} = \prod_{i \neq j} \frac{(\widehat{C}_{\rm ad} - a_i)}{a_j - a_i} \qquad \Rightarrow \qquad {\rm ad}^{\otimes 2} = \sum_j \mathrm{P}_{(a_j)} \cdot \left({\rm ad}^{\otimes 2} \right) .$$

Introduce symmetrized and antisymmetrized parts of $\widehat{C}_{\rm ad}$

$$\widehat{C}_{\pm} = \mathbf{P}_{\pm}^{(ad)}\ \widehat{C}_{\rm ad}\ , \qquad (\widehat{C}_{\pm})^{a_1 a_2}_{b_1 b_2} = \frac{1}{2}((\widehat{C}_{\rm ad})^{a_1 a_2}_{b_1 b_2} \pm (\widehat{C}_{\rm ad})^{a_2 a_1}_{b_1 b_2})\ ,$$

where $\mathbf{P}_{+}^{(ad)} = \frac{1}{2}(\mathbf{I} + \mathbf{P})$ and $\mathbf{P}_{-}^{(ad)} = \frac{1}{2}(\mathbf{I} - \mathbf{P})$ are projectors on symmetric $\widehat{C}_{+}$ and antisymmetric $\widehat{C}_{-}$ parts of $\widehat{C}_{\rm ad}$ in $(V_{\rm ad})^{\otimes 2} \simeq {\rm ad}^{\otimes 2}$.

Proposition 1. *For all simple LA $\mathfrak{g}$ the SCO $\widehat{C}_{-}$ satisfy char. identity*

$$\boxed{\widehat{C}_{-}(\widehat{C}_{-} + \frac{1}{2}) = 0} \Leftrightarrow \boxed{\widehat{C}_{-}^2 = -\frac{1}{2}\widehat{C}_{-}}\ , \qquad (13)$$

Since identity (13) is quadratic, we have two projectors $\mathrm{P}_{(0)}, \mathrm{P}_{(-\frac{1}{2})}$ on two subrepresentations X_1, X_2

$$\boxed{\mathbf{P}_{-}^{(ad)}({\rm ad}\otimes{\rm ad}) = \mathrm{P}_{(0)}({\rm ad}\otimes{\rm ad}) + \mathrm{P}_{(-\frac{1}{2})}({\rm ad}\otimes{\rm ad}) = \mathrm{X}_1 + \mathrm{X}_2 = {\rm ad} + \mathrm{X}_2},$$

where $\dim \mathrm{X}_1 = \mathbf{Tr}(\mathrm{P}_{(0)}) = \dim\mathfrak{g}, \quad \dim \mathrm{X}_2 = \mathbf{Tr}(\mathrm{P}_{(-\frac{1}{2})}) = \frac{1}{2}\dim\mathfrak{g}(\dim\mathfrak{g} - 3)$.

Proposition 2. *For all LA of the classical series* $A_n = \mathfrak{sl}_{n+1}$, $B_n = \mathfrak{so}_{2n+1}$, $C_n = \mathfrak{sp}_{2n}$, $D_n = \mathfrak{so}_{2n}$ *(except* $\mathfrak{sl}_3$ *and* $\mathfrak{so}_8$*), in ad-representation,* $\widehat{C}_+$ *has the universal char identity*

$$\boxed{(\widehat{C}_+ + 1)(\widehat{C}_+ + \frac{\alpha}{2t})(\widehat{C}_+ + \frac{\beta}{2t})(\widehat{C}_+ + \frac{\gamma}{2t})\mathbf{P}_+^{(\mathrm{ad})} = 0}, \quad \#4 \qquad (14)$$

where $(\frac{\alpha}{2t} + \frac{\beta}{2t} + \frac{\gamma}{2t}) = 1/2 \;\; \Rightarrow \;\; (t = \alpha + \beta + \gamma)$. The values of the Vogel parameters α, β, γ for $s\ell(N), so(N), sp(N)$ are given in Table 1.

From char. identity (14), we deduce four universal projectors $\mathsf{P}^{(+)}_{(a_i)}$ on the invariant subspaces $V_{(a_i)} \subset \mathbf{P}^{(\mathrm{ad})}_{+}(V_{\mathrm{ad}}^{\otimes 2})$ (with eigenvalues a_i of $\widehat{C}_{+}$)

$$\mathbf{P}^{(\mathrm{ad})}_{+}(V_{\mathrm{ad}}^{\otimes 2}) = (\mathsf{P}^{(+)}_{(-1)} + \mathsf{P}^{(+)}_{(-\frac{\alpha}{2t})} + \mathsf{P}^{(+)}_{(-\frac{\beta}{2t})} + \mathsf{P}^{(+)}_{(-\frac{\gamma}{2t})})V_{\mathrm{ad}}^{\otimes 2} = \\ = V_{(-1)} + V_{(-\frac{\alpha}{2t})} + V_{(-\frac{\beta}{2t})} + V_{(-\frac{\gamma}{2t})}\,.$$

$$\mathsf{P}^{(+)}_{(-\frac{\alpha}{2t})} = \mathsf{P}^{(+)}(\alpha|\beta,\gamma)\,, \quad \mathsf{P}^{(+)}_{(-\frac{\beta}{2t})} = \mathsf{P}^{(+)}(\beta|\alpha,\gamma)\,, \quad \mathsf{P}^{(+)}_{(-\frac{\gamma}{2t})} = \mathsf{P}^{(+)}(\gamma|\alpha,\beta)\,.$$

The representations of $\mathfrak{g}$ in the subspaces $V_{(-1)}$, $V_{(-\frac{\alpha}{2t})}$, $V_{(-\frac{\beta}{2t})}$, $V_{(-\frac{\gamma}{2t})}$ were respectively denoted by Vogel as $\mathsf{X}_0 = \mathbf{1}$, $Y_2(\alpha)$, $Y_2(\beta)$, $Y_2(\gamma)$

$$\boxed{\mathbf{P}^{(\mathrm{ad})}_{+}(\mathrm{ad}^{\otimes 2}) = \mathsf{X}_0 + Y_2(\alpha) + Y_2(\beta) + Y_2(\gamma)}\,.$$

Thus, Prop. 1,2 justify the Vogel statement for LA of classical series.
Theorem. (P.Vogel, 1999)

$$\mathrm{ad}^{\otimes 2} = \mathbf{P}_{-}^{(\mathrm{ad})}(\mathrm{ad}^{\otimes 2}) + \mathbf{P}_{+}^{(\mathrm{ad})}(\mathrm{ad}^{\otimes 2}) \;=\; (\mathrm{ad} + \mathsf{X}_2) + \big(\mathbf{1} + \mathsf{Y}(\alpha) + \mathsf{Y}(\beta) + \mathsf{Y}(\gamma)\big).$$

Finally, we calculate (by means of trace formulas) the Vogel universal expressions for the dim of the invariant eigenspaces $V_{(a_i)}$:

$$\dim V_{(-1)} = \mathbf{Tr}\ \mathsf{P}^{(+)}_{(-1)} = 1\,,$$
$$\dim Y_2(\alpha) \equiv \dim V_{(-\frac{\alpha}{2t})} = \mathbf{Tr}\ \mathsf{P}^{(+)}_{(-\frac{\alpha}{2t})} = -\frac{(3\alpha-2t)(\beta-2t)(\gamma-2t)t(\beta+t)(\gamma+t)}{\alpha^2(\alpha-\beta)\beta(\alpha-\gamma)\gamma}\,,$$
$$\dim Y_2(\beta) \equiv \dim V_{(-\frac{\beta}{2t})} = \mathbf{Tr}\ \mathsf{P}^{(+)}_{(-\frac{\beta}{2t})} = -\frac{(3\beta-2t)(\alpha-2t)(\gamma-2t)t(\alpha+t)(\gamma+t)}{\beta^2(\beta-\alpha)\alpha(\beta-\gamma)\gamma}\,,$$
$$\dim Y_2(\gamma) \equiv \dim V_{(-\frac{\gamma}{2t})} = \mathbf{Tr}\ \mathsf{P}^{(+)}_{(-\frac{\gamma}{2t})} = -\frac{(3\gamma-2t)(\beta-2t)(\alpha-2t)t(\beta+t)(\alpha+t)}{\gamma^2(\gamma-\beta)\beta(\gamma-\alpha)\alpha}\,.$$

Remark 1. The cases of algebras $\mathfrak{sl}_3$ and $\mathfrak{so}_8$ are exceptional – their char. identity (for symmetric part of $\widehat{C}$) has the order 3.

Remark 2. For exceptional LA cases $\frac{\gamma}{2t} = \frac{1}{3}$ and we have $\dim Y_2(\gamma) = 0$.

Universal char identities for $\widehat{C}$ for exceptional Lie algebras in $\mathrm{ad}^{\otimes 2}$.

The antisymmetric $\widehat{C}_-$ and symmetric $\widehat{C}_+$ parts of the split Casimir operators in the ad-representation for all exceptional Lie algebras $\mathfrak{g}$ obey the universal identities

$$\boxed{\widehat{C}_- \left(\widehat{C}_- + \frac{1}{2}\right) = 0}, \quad \boxed{(\widehat{C}_+ + 1)\big(\widehat{C}_+^2 + \frac{1}{6}\widehat{C}_+ - 2\mu\big)\ \mathbf{P}_+^{(\mathrm{ad})} = 0}, \quad \#3\,, \tag{15}$$

where the universal parameter μ is fixed as follows

$$\mu = \frac{5}{6(2 + \dim(\mathfrak{g}))}\,, \tag{16}$$

and for algebras $\mathfrak{g}_2, \mathfrak{f}_4, \mathfrak{e}_6, \mathfrak{e}_7, \mathfrak{e}_8$ with dimensions $14, 52, 78, 133, 248$ we have respectively $\mu = \frac{5}{96}, \frac{5}{324}, \frac{1}{96}, \frac{1}{162}, \frac{1}{300}$.

Moreover the char identities for $\widehat{C}_+$ for algebras $\mathfrak{s}\ell_3$ and $\mathfrak{so}_8$ have the same structure (15) with $\mu = \frac{1}{12}$ and $\mu = \frac{1}{36}$.

From (15) we obtain the factorized form of the universal char. identity for $\widehat{C}_+$

$$(\widehat{C}_+ + 1)(\widehat{C}_+^2 + \frac{1}{6}\widehat{C}_+ - 2\mu)\mathbf{P}_+^{(\mathrm{ad})} \equiv (\widehat{C}_+ + 1)(\widehat{C}_+ + \frac{\alpha}{2t})(\widehat{C}_+ + \frac{\beta}{2t})\mathbf{P}_+^{(\mathrm{ad})} = 0\,, \quad (17)$$

where we introduced the notation for two roots of eq. $\widehat{C}_+^2 + \frac{1}{6}\widehat{C}_+ - 2\mu = 0$:

$$\frac{\alpha}{2t} = \frac{1-\mu'}{12}\,, \quad \frac{\beta}{2t} = \frac{1+\mu'}{12}\,, \qquad \mu' := \sqrt{1+288\mu} = \sqrt{\frac{\dim\mathfrak{g}+242}{\dim\mathfrak{g}+2}}\,. \quad (18)$$

These roots are related by $3(\alpha+\beta) = t$, and for $\alpha = -2$ this relation defines the line of the exceptional LA on the Vogel (β, γ) plane (as we discussed above). We note that μ' is a rational number (since $\frac{\alpha}{2t}$ and $\frac{\beta}{2t}$ are rational) only for certain finite sequence of $\dim\mathfrak{g}$:

$$\dim\mathfrak{g} = 3, 8, 14, 28, 47, 52, 78, 96, 119, 133, 190, 248, 287, 336, 484, 603, 782, 1081, 1680, 3479\,, \quad (19)$$

which includes the dimensions $14, 52, 78, 133, 248$ of the exceptional Lie algebras $\mathfrak{g}_2, \mathfrak{f}_4, \mathfrak{e}_6, \mathfrak{e}_7, \mathfrak{e}_8$, and the dimensions 8 and 28 of $s\ell(3)$ and $so(8)$, which are sometimes also referred to as exceptional. Dim. 190 corresponds to $\mathfrak{e}_{7+\frac{1}{2}}$.

Remark. The sequence (19) contains $\dim \mathfrak{g}^* = (10m - 122 + 360/m)$, $(m \in \mathbb{N})$ referring to the adjoint representations of the so-called E_8 family of algebras $\mathfrak{g}^*$; see the Cvitanović book. For such dimensions we have relation $\mu' = |(m+6)/(m-6)|$. Two numbers 47 and 119 from sequence (19) do not belong to the sequence $\dim \mathfrak{g}^*$. Thus, the interpretation of these two numbers as the dimensions of some algebras is missing. Moreover, for values $\dim \mathfrak{g}$ given in (19), using (18), one can calculate dimensions of the corresponding representations $Y(\alpha)$:

$$\dim V_{(-\frac{\alpha}{2t})} = \Big\{5, 27, 77, 300, \tfrac{14553}{17}, 1053, 2430, \tfrac{48608}{13}, \tfrac{111078}{19}, 7371, 15504, 27000, \tfrac{841279}{23}, \tfrac{862407}{17}, 107892, \tfrac{2205225}{13}, \tfrac{578151}{2}, 559911, \tfrac{42507504}{31}, \tfrac{363823677}{61}\Big\}$$

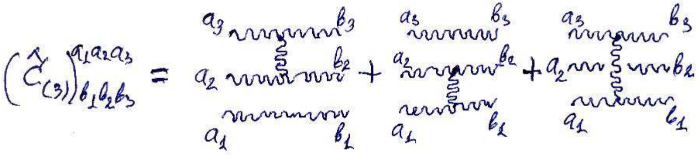
Since $\dim V_{(-\frac{\alpha}{2t})}$ should be integer, we conclude that there not exist Lie algebras with dimensions $47, 96, 119, 287, 336, 603, 782, 1680, 3479$, for which we assume characteristic identity (17) and the trace formulas.

Universal formulas for 3-split Casimir operator in $\mathrm{ad}^{\otimes 3}$

The matrix $\widehat{C}^{a_1 a_2 a_3}_{b_1 b_2 b_3} := (\widehat{C}_{(3)})^{a_1 a_2 a_3}_{b_1 b_2 b_3}$ of the 3-split Casimir operator is

$$(\widehat{C}_{(3)})^{a_1 a_2 a_3}_{b_1 b_2 b_3} = (\widehat{C}_{12} + \widehat{C}_{13} + \widehat{C}_{23})^{a_1 a_2 a_3}_{b_1 b_2 b_3} \,, \qquad (20)$$

and acts in the space $V_{\mathrm{ad}}^{\otimes 3}$ of the representation $\mathrm{ad}^{\otimes 3}$.



According to

$$\mathrm{ad}^{\otimes 3} = (\mathrm{P}_{[3]} + \mathrm{P}_{[2,1]} + \mathrm{P}_{[1^3]})\, \mathrm{ad}^{\otimes 3} \,,$$

we have decomposition

$$\widehat{C}_{(3)} = (\mathrm{P}_{[3]} + \mathrm{P}_{[2,1]} + \mathrm{P}_{[1^3]})\widehat{C}_{(3)} = \widehat{C}_{[3]} + \widehat{C}_{[2,1]} + \widehat{C}_{[1^3]} \,.$$

A.P. Isaev, S.O. Krivonos, A.A. Provorov,

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All calculations were done with "Mathematica".

Proposition 3. *For 3-split Casimirs $\widehat{C}_{[1^3]}$, $\widehat{C}_{[3]}$ and $\widehat{C}_{[2,1]}$ we have the universal char. identities*

$$\widehat{C}_{[1^3]}\left(\widehat{C}_{[1^3]}+\tfrac{1}{2}\right)\left(\widehat{C}_{[1^3]}+\tfrac{3}{2}\right)\left(\widehat{C}_{[1^3]}+\tfrac{1}{2}+\hat{\alpha}\right)\left(\widehat{C}_{[1^3]}+\tfrac{1}{2}+\hat{\beta}\right)\left(\widehat{C}_{[1^3]}+\tfrac{1}{2}+\hat{\gamma}\right)=0, \quad \#6$$

$$\begin{gathered}\left(\widehat{C}_{[3]}+\tfrac{1}{2}\right)\left(\widehat{C}_{[3]}+1\right)\left(\widehat{C}_{[3]}+\tfrac{1}{2}-\hat{\alpha}\right)\left(\widehat{C}_{[3]}+\tfrac{1}{2}-\hat{\beta}\right)\left(\widehat{C}_{[3]}+\tfrac{1}{2}-\hat{\gamma}\right)\times \\ \left(\widehat{C}_{[3]}+3\hat{\alpha}\right)\left(\widehat{C}_{[3]}+3\hat{\beta}\right)\left(\widehat{C}_{[3]}+3\hat{\gamma}\right)\mathrm{P}_{[3]}=0, \quad \#8 \\ \\ \left(\widehat{C}_{[2,1]}+\tfrac{1}{2}\right)\left(\widehat{C}_{[2,1]}+1\right)\left(\widehat{C}_{[2,1]}+\tfrac{1}{2}-\hat{\alpha}\right)\left(\widehat{C}_{[2,1]}+\tfrac{1}{2}-\hat{\beta}\right)\left(\widehat{C}_{[2,1]}+\tfrac{1}{2}-\hat{\gamma}\right)\times \\ \left(\widehat{C}_{[2,1]}+\tfrac{1}{2}+\hat{\alpha}\right)\left(\widehat{C}_{[2,1]}+\tfrac{1}{2}+\hat{\beta}\right)\left(\widehat{C}_{[2,1]}+\tfrac{1}{2}+\hat{\gamma}\right)\times \\ \left(\widehat{C}_{[2,1]}+\tfrac{3}{2}\hat{\alpha}\right)\left(\widehat{C}_{[2,1]}+\tfrac{3}{2}\hat{\beta}\right)\left(\widehat{C}_{[2,1]}+\tfrac{3}{2}\hat{\gamma}\right)\mathrm{P}_{[2,1]'}=0, \quad \#11\,.\end{gathered} \tag{21}$$

where $\hat{\alpha}=\frac{\alpha}{2t}$, $\hat{\beta}=\frac{\beta}{2t}$, $\hat{\gamma}=\frac{\gamma}{2t}$. All formulas in (21) are homogeneous and symmetric under permutations (α,β,γ). Our results are in agreement with [P.Vogel (1999), A.M. Cohen and R. de Man (1996)].

The dimensions of irreps corresponding to the eigenvalues of $\widehat{C}_{[2,1]}$

[A.P. Isaev, S.O. Krivonos, A.A. Provorov,
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$$
\begin{aligned}
\dim_{-\frac{1}{2}} &= 2X_2 = && 2 \times \tfrac{1}{2} \dim(g)\,(\dim(g) - 3)\,, \\
\dim_{-1} &= 2X_1 = && 2 \times \dim(g), \\
\dim_{\hat\alpha-\frac{1}{2}} &= B = && \frac{(\hat\alpha-1)(\hat\beta-1)(\hat\gamma-1)(2\,\hat\alpha+\hat\beta)(2\,\hat\alpha+\hat\gamma)(2\,\hat\beta+1)(2\,\hat\gamma+1)(3\,\hat\beta-1)(3\,\hat\gamma-1)}{8\,\hat\alpha^2\,(\hat\alpha-\hat\beta)(\hat\alpha-\hat\gamma)(2\,\hat\beta-\hat\gamma)(2\hat\gamma-\hat\beta)\,\hat\beta^2\,\hat\gamma^2}, \\
\dim_{\hat\beta-\frac{1}{2}} &= B' = && \frac{(\hat\beta-1)(\hat\gamma-1)(\hat\alpha-1)(2\,\hat\beta+\hat\gamma)(2\,\hat\beta+\hat\alpha)(2\,\hat\gamma+1)(2\,\hat\alpha+1)(3\,\hat\gamma-1)(3\,\hat\alpha-1)}{8\,\hat\beta^2\,(\hat\beta-\hat\gamma)(\hat\beta-\hat\alpha)(2\,\hat\gamma-\hat\alpha)(2\hat\alpha-\hat\gamma)\,\hat\gamma^2\,\hat\alpha^2}, \\
\dim_{\hat\gamma-\frac{1}{2}} &= B'' = && \frac{(\hat\gamma-1)(\hat\alpha-1)(\hat\beta-1)(2\,\hat\gamma+\hat\alpha)(2\,\hat\gamma+\hat\beta)(2\,\hat\alpha+1)(2\,\hat\beta+1)(3\,\hat\alpha-1)(3\,\hat\beta-1)}{8\,\hat\gamma^2\,(\hat\gamma-\hat\alpha)(\hat\gamma-\hat\beta)(2\,\hat\alpha-\hat\beta)(2\hat\beta-\hat\alpha)\,\hat\alpha^2\,\hat\beta^2}, \\
\dim_{-\hat\alpha-\frac{1}{2}} &= Y_2 = && -\frac{(3\hat\alpha-1)(\hat\beta-1)(\hat\gamma-1)(2\hat\beta+1)(2\hat\gamma+1)}{8\hat\alpha^2(\hat\alpha-\hat\beta)(\hat\alpha-\hat\gamma)\hat\beta\hat\gamma}, \\
\dim_{-\hat\beta-\frac{1}{2}} &= Y_2' = && -\frac{(3\hat\beta-1)(\hat\gamma-1)(\hat\alpha-1)(2\hat\gamma+1)(2\hat\alpha+1)}{8\hat\beta^2(\hat\beta-\hat\gamma)(\hat\beta-\hat\alpha)\hat\gamma\hat\alpha}, \\
\dim_{-\hat\gamma-\frac{1}{2}} &= Y_2'' = && -\frac{(3\hat\gamma-1)(\hat\alpha-1)(\hat\beta-1)(2\hat\alpha+1)(2\hat\beta+1)}{8\hat\gamma^2(\hat\gamma-\hat\alpha)(\hat\gamma-\hat\beta)\hat\alpha\hat\beta}, \\
\dim_{-\frac{3}{2}\hat\alpha} &= C = && -\frac{2}{3}\,\frac{(1+2\hat\alpha)(1+2\hat\beta)(1+2\hat\gamma)(1-\hat\beta)(1-\hat\gamma)(\hat\beta+\hat\gamma)(2\hat\beta+\hat\gamma)(2\hat\gamma+\hat\beta)}{\hat\alpha^3\hat\beta\hat\gamma(\hat\alpha-2\hat\beta)(\hat\alpha-2\hat\gamma)(\hat\alpha-\hat\beta)(\hat\alpha-\hat\gamma)} \\
\dim_{-\frac{3}{2}\hat\beta} &= C' = && -\frac{2}{3}\,\frac{(1+2\hat\beta)(1+2\hat\gamma)(1+2\hat\alpha)(1-\hat\gamma)(1-\hat\alpha)(\hat\gamma+\hat\alpha)(2\hat\gamma+\hat\alpha)(2\hat\alpha+\hat\gamma)}{\hat\beta^3\hat\gamma\hat\alpha(\hat\beta-2\hat\gamma)(\hat\beta-2\hat\alpha)(\hat\beta-\hat\gamma)(\hat\beta-\hat\alpha)} \\
\dim_{-\frac{3}{2}\hat\gamma} &= C'' = && -\frac{2}{3}\,\frac{(1+2\hat\gamma)(1+2\hat\alpha)(1+2\hat\beta)(1-\hat\alpha)(1-\hat\beta)(\hat\alpha+\hat\beta)(2\hat\alpha+\hat\beta)(2\hat\beta+\hat\alpha)}{\hat\gamma^3\hat\alpha\hat\beta(\hat\gamma-2\hat\alpha)(\hat\gamma-2\hat\beta)(\hat\gamma-\hat\alpha)(\hat\gamma-\hat\beta)}.
\end{aligned}
$$

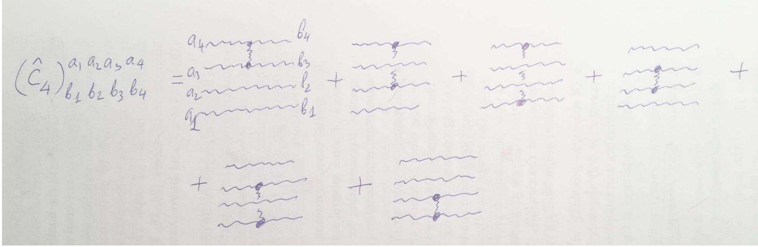
All calculations were done with "Mathematica".

Universal formulas for 4-split Casimir operator in $\mathrm{ad}^{\otimes 4}$

The matrix of the 4-split Casimir operator is

$$(\widehat{C}_{(4)})^{a_1 a_2 a_3 a_4}_{b_1 b_2 b_3 b_4} = (\widehat{C}_{12} + \widehat{C}_{13} + \widehat{C}_{14} + \widehat{C}_{23} + \widehat{C}_{24} + \widehat{C}_{34})^{a_1 a_2 a_3 a_4}_{b_1 b_2 b_3 b_4} \, , \qquad (22)$$

and acts in the space $V_{\mathrm{ad}}^{\otimes 3}$ of the representation $\mathrm{ad}^{\otimes 3}$.



According to

$$\mathrm{ad}^{\otimes 4} = (\mathrm{P}_{[4]} + \mathrm{P}_{[3,1]} + \mathrm{P}_{[2^2]} + \mathrm{P}_{[2,1^2]} + \mathrm{P}_{[1^4]})\, \mathrm{ad}^{\otimes 4} \, ,$$

we have decomposition

$$\widehat{C}_{(4)} = \widehat{C}_{[4]} + \widehat{C}_{[3,1]} + \widehat{C}_{[2^2]} + \widehat{C}_{[2,1^2]} + \widehat{C}_{[1^4]} \, .$$

Symmetric module $\mathrm{P}_{[4]}(\mathrm{ad}^{\otimes 4})$ includes the following representations:
[M.Avetisyan, A.P.I., S.Krivonos, R.Mkrtchyan, The uniform structure of $\mathfrak{g}^{\otimes 4}$, ArXive:2311.05358]

$$\mathrm{P}_{[4]}(\mathrm{ad}^{\otimes 4}) = 2 \oplus J \oplus J' \oplus J'' \oplus X_2 \oplus \mathbb{Z}_3 \oplus 3Y_2 \oplus 3Y_2' \oplus 3Y_2'' \oplus C \oplus C' \oplus C'' \oplus Y_4 \oplus Y_4' \oplus Y_4'' \oplus D \oplus D' \oplus D'' \oplus D''' \oplus D'''' \oplus D''''', \quad \#21\,.$$

The universal dimension formulae of some of these representations are:

$$\begin{aligned}
dim J &= \frac{(\hat\alpha+\hat\beta)(\hat\alpha+\hat\gamma)(2\hat\alpha+\hat\beta-\hat\gamma)(2\hat\alpha+2\hat\beta-\hat\gamma)(2\hat\alpha-\hat\beta+\hat\gamma)(\hat\alpha+2\hat\beta+\hat\gamma)}{4\,\hat\alpha^2\hat\beta^2\hat\gamma^2(\hat\alpha-\hat\beta)(\hat\alpha-\hat\gamma)(\hat\beta-2\hat\gamma)(\hat\beta-\hat\gamma)^2(2\hat\beta-\hat\gamma)(\hat\alpha-\hat\beta-\hat\gamma)} \times \\
&\quad (2\hat\alpha+2\hat\beta+\hat\gamma)(2\hat\alpha-\hat\beta+2\hat\gamma)(\hat\alpha+\hat\beta+2\hat\gamma)(2\hat\alpha+\hat\beta+2\hat\gamma)(\hat\alpha+2\hat\beta+2\hat\gamma), \\
dim J' &= dim J_{\hat\alpha\leftrightarrow\hat\beta}, \quad dim J'' = dim J_{\hat\alpha\leftrightarrow\hat\gamma}, \\
dim\,\mathbb{Z}_3 &= 2 dim \widehat{X}_3 = \frac{2}{9}(N^2-1)^2(N^2-9) \quad \text{for } sl(N), \\
&= dim\,\mathbb{X}_3 = \frac{1}{6} dim\mathfrak{g}(dim\mathfrak{g}-1)(dim\mathfrak{g}-8), \quad \text{for } so(N) \text{ and exceptional algebras}, \\
dim Y_4 &= -\frac{(\hat\alpha-1)(2\hat\alpha-1)(7\hat\alpha-1)(\hat\beta-1)(\hat\alpha+\hat\beta-1)(2\hat\alpha+\hat\beta-1)(3\hat\alpha+\hat\beta-1)(\hat\gamma-1)}{24\hat\alpha^4(\hat\alpha-\hat\beta)(2\hat\alpha-\hat\beta)(3\hat\alpha-\hat\beta)\hat\beta(\hat\alpha-\hat\gamma)(2\hat\alpha-\hat\gamma)(3\hat\alpha-\hat\gamma)\hat\gamma} \times \\
&\quad (\hat\alpha+\hat\gamma-1)(2\hat\alpha+\hat\gamma-1)(3\hat\alpha+\hat\gamma-1), \\
dim Y_4' &= (dim Y_4)_{\hat\alpha\leftrightarrow\hat\beta}, \quad dim Y_4'' = (dim Y_4)_{\hat\alpha\leftrightarrow\hat\gamma}, \\
dim D &= \frac{(3\hat\alpha-2\hat\beta-2\hat\gamma)(\hat\alpha-\hat\beta-2\hat\gamma)(\hat\beta+\hat\gamma)(\hat\alpha+\hat\beta+\hat\gamma)(2\hat\alpha+\hat\beta+\hat\gamma)(2\hat\beta+\hat\gamma)(\hat\alpha+2\hat\beta+\hat\gamma)}{\hat\alpha^3(\hat\alpha-\hat\beta)^2(3\hat\alpha-\hat\beta)\hat\beta^2(\hat\alpha-2\hat\gamma)(\hat\alpha-\hat\gamma)(2\hat\alpha-\hat\gamma)(\hat\beta-\hat\gamma)\hat\gamma} \times \\
&\quad (2\hat\alpha+2\hat\beta+\hat\gamma)(\hat\alpha+2\hat\gamma)(2\hat\alpha-\hat\beta+2\hat\gamma)(\hat\alpha+\hat\beta+2\hat\gamma)(2\hat\alpha+\hat\beta+2\hat\gamma)(\hat\alpha+2\hat\beta+2\hat\gamma), \qquad (23) \\
dim D' &= (dim D)_{\hat\alpha\leftrightarrow\hat\beta}, \quad dim D'' = (dim D)_{\hat\alpha\leftrightarrow\hat\gamma}, \quad dim D''' = (dim D)_{\hat\beta\leftrightarrow\hat\gamma}, \\
dim D'''' &= (dim D)_{\hat\alpha\to\hat\beta\to\hat\gamma\to\hat\alpha}, \quad dim D''''' = (dim D)_{\hat\alpha\to\hat\gamma\to\hat\beta\to\hat\alpha}. \qquad (24)
\end{aligned}$$

Beyond Vogel Universality

It turns out that universality of LA is observed not only in the decomp. of $\mathrm{ad}^{\otimes r}$.

1. Consider tensor product of defining $\Box = T$ and adjoint ad representations. For all simple LA (except $\mathfrak{e}_8$), we have $\Box \otimes \mathrm{ad} = \Box + W_1 + W_2$ and the 2-split Casimir operator

$$(\widehat{C}_{\Box\otimes\mathrm{ad}})^{i_1 a_1}_{i_2 a_2} = \mathrm{g}^{ab}\,(T_a)^{i_1}_{i_2}(X_b)^{a_1}_{a_2}\;,$$

satisfies universal char. identity

$$(\hat{C}_{\Box\otimes\mathrm{ad}} + \frac{1}{2})(\hat{C}_{\Box\otimes\mathrm{ad}} + \frac{\hat{\alpha}}{2})(\hat{C}_{\Box\otimes\mathrm{ad}} + \frac{\hat{\beta}}{2}) = 0\,, \quad \#3 \qquad (25)$$

where $\hat{\alpha}$ and $\hat{\beta}$ are Vogel parameters (for exceptional LA $\gamma \to \beta$); see Table 2

Table 2.

	$\mathfrak{sl}(N)$	$\mathfrak{so}(N)$	$\mathfrak{sp}(N=2r)$	$\mathfrak{g}_2$	$\mathfrak{f}_4$	$\mathfrak{e}_6$	$\mathfrak{e}_7$
$\hat{\alpha}$	$-1/N$	$-1/(N-2)$	$1/(N+2)$	$-1/4$	$-1/9$	$-1/12$	$-1/18$
$\hat{\beta}$	$1/N$	$2/(N-2)$	$-2/(N+2)$	$1/3$	$1/3$	$1/3$	$1/3$
$\dim\Box$	N	N	N	7	26	27	56

all eigenvalues in (25) were found by means of relation $\hat{c}^{\lambda}_{(2)} = \frac{1}{2}(c^{\lambda}_{(2)} - c^{\Box}_{(2)} - c^{\mathrm{ad}}_{(2)})$ and by means of Mathematica (S.O.Krivonos).

Dimensions of the irreps $\square, W_1, W_2$ with eigenvalues $-\frac{1}{2}, -\frac{\hat{\alpha}}{2}, -\frac{\hat{\beta}}{2}$

$$\dim_{-\frac{1}{2}} = \dim \square\,, \quad \dim_{-\frac{\hat{\alpha}}{2}} = \frac{(1-\hat{\beta})(1+2\hat{\beta})}{2\hat{\alpha}\hat{\gamma}(\hat{\alpha}-\hat{\beta})} \dim \square\,, \quad \dim_{-\frac{\hat{\beta}}{2}} = \dim_{-\frac{\hat{\alpha}}{2}}\Big|_{\hat{\alpha}\leftrightarrow\hat{\beta}}\,,$$

are found by the standard method with the help of

$$\mathrm{Tr}\,\widehat{C}_{\square\otimes \mathrm{ad}} = 0, \quad \mathrm{Tr}\,\widehat{C}^3_{\square\otimes \mathrm{ad}} = -\frac{1}{4}\mathrm{Tr}\,\widehat{C}^2_{\square\otimes \mathrm{ad}} = -\frac{1}{4}c^{\square}_{(2)} \cdot \dim \square$$

where $\boxed{c^{\square}_{(2)} = \frac{(\hat{\alpha}-1)(\hat{\beta}-1)}{4\hat{\gamma}}}$, $\hat{\alpha}+\hat{\beta}+\hat{\gamma} = \frac{1}{2}$.

2. Tensor product of defining rep. $\square = T$ and rep. $Y_2(\alpha)$ which appears in ad^2. For all simple LA (except $\mathfrak{e}_8$), we have $\square \otimes Y_2(\alpha) = W_1' + W_2' + W_3'$ and

$$(\widehat{C}_{\square\otimes Y_2} + \frac{\hat{\beta}}{2})(\widehat{C}_{\square\otimes Y_2} + \hat{\alpha})\big(\widehat{C}_{\square\otimes Y_2} + \frac{1}{2}(1-\hat{\alpha})\big) = 0\,, \quad \#3 \qquad (26)$$

where $\hat{\alpha}, \hat{\beta}$ are Vogel parameters (for exceptional LA $\gamma \to \beta$) given in Table 2. For this case we also have $\mathrm{Tr}\,\widehat{C}_{\square\otimes Y_2} = 0$

$$-4\mathrm{Tr}\,\widehat{C}^3_{\square\otimes Y_2} = \mathrm{Tr}\,\widehat{C}^2_{\square\otimes Y_2} = \frac{(\hat{\alpha}+\hat{\beta}-1)(1-\hat{\alpha})(1-\hat{\beta})(3\hat{\alpha}-1)(2\hat{\beta}+1)}{-8\hat{\gamma}(\hat{\alpha}-\hat{\gamma})(\hat{\alpha}-\hat{\beta})\hat{\alpha}} \dim \square$$

and one finds universal formulas for dimensions of W_1', W_2', W_3'.

$$\dim_{-\hat{\alpha}} := \frac{(2\hat{\alpha}-1-2\hat{\beta})(\hat{\alpha}-1)(\hat{\beta}-1)(1+2\hat{\beta})(\hat{\alpha}+\hat{\beta}-1)}{2(2\hat{\alpha}+2\hat{\beta}-1)(4\hat{\alpha}-1+2\hat{\beta})(\hat{\alpha}-\hat{\beta})\hat{\alpha}^2(2\hat{\alpha}-\hat{\beta})} \dim \Box$$

Discussion.

- Why does new universality arise in tensor products $\Box \otimes \mathrm{ad}$ and $\Box \otimes Y_2(\alpha)$?
- All values of higher Casimirs $c_{(k)}^{\Box}$ and $c_{(k)}^{Y_2(\alpha)}$ have universal representation.
- Char. identities for the split CO allow us to calculate colour factors for the amplitudes in "glue-dynamics" (including fundamental fields of $\Box$) and write them in the universal form via Vogel parameters.
- The universal description of the subrepresentations in $\mathrm{ad}^{\otimes n}$ for $n \geq 5$ is open problem. We have problems with analytical calculations on "Mathematica".
- n-Split Casimir operators are Hamiltonians for Heisenberg-type spin-chains with interactions between all nodes (not just the closest ones).