Интегральные уравнения в физике частиц: математическая постановка и методы решения

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Integral equations

A systematic field-theoretic approach to the description of hadronic matter on the basis of an effective-action functional in the quark sector of QCD can be formulated within the path-integral approach. The quark-action functional can be represented in the form

$$S_{\text{eff}}[q,\bar{q}] = \int \mathrm{d}x_1\,\mathrm{d}x_2\,\bar{q}_A(x_1)\left[G_0^{-1}(x_1,x_2)\right]_{AB} q_B(x_2)$$

$$-\frac{1}{2}\int \mathrm{d}x_1\,\mathrm{d}x_2\,\mathrm{d}y_1\,\mathrm{d}y_2\,\left[\bar{q}_A(x_1)q_B(y_1)\right]$$

$$\times\left[\mathscr{K}^{\eta}(x_1,y_1;x_2,y_2)\right]_{AB;CD}\left[\bar{q}_C(x_2)q_D(y_2)\right].$$

Here, the first term is the free-quark action, with

$$\left[G_0^{-1}(x_1,x_2)\right]_{AB} = (i\partial\!\!\!/ - \hat{m}_0)\delta_{AB}\delta(x_1-x_2),$$

and $\hat{\mathrm{m}}_0 = \mathrm{diag}(\mathrm{m}_0^{\mathrm{a}})$, $\mathrm{a} = 1,\ldots,\mathrm{N}$, is the current quark mass matrix. The indices A, B, C, D are a compact notation for Dirac as well as flavour and colour indices.

The channel decomposition of the four-point interaction kernel $\mathcal{K}^\eta$

$$[\mathscr{K}^\eta(x_1, y_1; x_2, y_2)]_{AB;CD} = (\Lambda^{\mathrm{H}}_{AD} \cdot \Lambda^{\mathrm{H}}_{CB})\mathscr{K}^\eta(x, y \mid X, Y),$$

where

$$\Lambda^{\mathrm{H}}_{AD} = (\xi^k \cdot \rho^b \cdot \zeta^j)_{AD}$$

$$\xi^k = \left(1,\ i\gamma_5,\ \frac{i}{\sqrt{2}}\gamma_\mu,\ \frac{i}{\sqrt{2}}\gamma_\mu\gamma_5\right),$$

$$\rho^b = \left(\sqrt{\tfrac{3}{2}}\,1_{\mathrm{f}},\ \frac{i}{\sqrt{2}}\tau^a\right), \quad a = 1, 2, 3 \quad \text{for } \mathrm{SU}(2)_{\mathrm{f}},$$

$$\zeta^j = \left(\tfrac{4}{3}1_{\mathrm{c}},\ \frac{i}{\sqrt{3}}\lambda^i\right), \quad i = 1, 2, \ldots, 8 \quad \text{for } \mathrm{SU}(3)_{\mathrm{c}}.$$

The orbital part of the interaction,

$$\mathscr{K}^{\eta}(x, y \mid X, Y) = W(x^{\perp}, y^{\perp})\delta^4(X-Y)\delta(x\cdot\eta)\delta(y\cdot\eta),$$

has a relativistic covariant form, with $\mathrm{x} = \mathrm{x}_1 - \mathrm{x}_2$, $\mathrm{X} = (\mathrm{x}_1 + \mathrm{x}_2)/2$ being the four-vectors of the relative and the center-of-mass coordinates of the incoming quark-antiquark pair;
$\mathrm{y} = \mathrm{y}_1 - \mathrm{y}_2$, $\mathrm{Y} = (\mathrm{y}_1 + \mathrm{y}_2)/2$ are those of the outgoing one, respectively. The four-dimensional δ-function guarantees the condition of the center-of-mass conservation, which is a consequence of the homogeneity of the space-time continuum.

Integral equations

The instantaneous interaction kernel is further assumed to neglect retardation effects in the s-channel so that it depends via $W(x^{\perp}, y^{\perp})$ only on the transverse components of the four-distances x_{μ} and y_{μ}, with respect to the (conserved) four-vector P_{μ}, of the center-of-mass momentum:

$$x_{\mu}^{\perp} = x_{\mu} - x_{\mu}^{\parallel}, \quad x_{\mu}^{\parallel} = \eta_{\mu}(x \cdot \eta), \quad \eta_{\mu} = \frac{P_{\mu}}{\sqrt{P^2}}.$$

$y^{\parallel}$ and $y^{\perp}$ are introduced accordingly.
The neglect of retardation effects in the s-channel is motivated by the analogy with quantum electrodynamics where the assumption of the dominance of instantaneous interactions in the formation of bound states can be justified. The projection on the subspace of equal-time processes is represented by $\delta(x \cdot \eta)$.
The potential of the effective quark interaction can be constructed in a gauge-invariant way within the reduced phase-space- quantization scheme.

For the application of the general nonlocal theory, there are two important classes of potentials which are contained as special cases in the general form of $W(x^{\perp}, y^{\perp})$
bilocal potentials:

$$W(x^{\perp}, y^{\perp}) = \delta^3(x^{\perp} - y^{\perp}) V(x^{\perp}),$$

separable potentials:

$$W(x^{\perp}, y^{\perp}) = V_0 g(x^{\perp}) g(y^{\perp}).$$

Within the standard functional-integral approach to bilocal field theory the action can be transformed by introducing bilocal bosonic fields $\mathcal{M}_{\mathrm{AB}}(\mathrm{x}_1, \mathrm{x}_2)$.
After integration over the quark fields, the effective bosonized action takes the form

$$S_{\mathrm{eff}}[\mathscr{M}] = -N_c\left[\frac{1}{2}\mathscr{M}(\mathscr{K}^{\eta})^{-1}\mathscr{M} + i\ \mathrm{Tr}\ \ln(-G_0^{-1} + \mathscr{M})\right].$$

Due to the particular choice of the instantaneous interaction kernel, the fields $\mathcal{M}_{\mathrm{AB}}(\mathrm{x}_1, \mathrm{x}_2) = \Lambda^{\mathrm{H}}_{\mathrm{AB}}\mathcal{M}_{\mathrm{AB}}(\mathrm{x}, \mathrm{X})$ can be introduced in such a way that they satisfy the Markov-Yukawa condition for instantaneous bilocal meson fields:

$$x_{\mu}\frac{\partial}{\partial X_{\mu}}\mathscr{M}^{\mathrm{H}}(x\,|\,X) = 0.$$

These fields are irreducible representations of the Poincare group with definite mass, $P^2 = M_H^2$, spin and orbital parts which can be expressed as

$$\mathscr{M}^{H}(x \mid X) = \int \frac{d\boldsymbol{P}}{(2\pi)^{3/2}\sqrt{2\omega_H(\boldsymbol{P})}} \left[e^{-i(P \cdot X)} \Gamma_H(x^{\perp} \mid P) a_H^{+}(\boldsymbol{P}) + e^{i(P \cdot X)} \bar{\Gamma}_H(x^{\perp} \mid -P) a_H^{-}(\boldsymbol{P}) \right] \delta(x \cdot \eta^{H}),$$

where $a_H^+(P)$, $a_H^-(P)$ are creation and annihilation operators of a bound state. The index H denotes the set of hadron quantum numbers and P_0 is fixed as $P_0 = \omega_H = \sqrt{P^2 + M_H^2}$ and $\Gamma_H(p^{\perp}|P)$, $\Gamma_H(p^{\perp}|-P)$ are the quark-meson-vertex amplitudes which are functions of the transverse component of the four-distance $x_\mu^{\perp}$ with respect to the four-vector P of the center-of-mass momentum.

Integral equations

$$\Sigma(x-y) = m^0 \delta^{(4)}(x-y) + i\mathscr{K}(x,y) G_\Sigma(x-y),$$

$$\Gamma = i\mathscr{K}(x,y) \int d^4 z_1\, d^4 z_2\, G_\Sigma(x-z_1)\Gamma(z_1,z_2)G_\Sigma(z_2-y),$$

In momentum space we obtain with

$$\underline{\Sigma}(k) = \int d^4x\, \Sigma(x) e^{ikx},$$

$$\underline{\Gamma}(q|\mathscr{P}) = \int d^4x\, d^4y\, \exp\left[i\frac{x+y}{2}\mathscr{P}\right] \exp[i(x-y)q]\, \Gamma(x,y)$$

$$\underline{\Sigma}(k) = m^0 + i\int \frac{d^4q}{(2\pi)^4} \underline{V}(k^\perp - q^\perp) \not\eta\, \underline{G}_\Sigma(q) \not\eta,$$

$$\underline{\Gamma}(k,\mathscr{P}) = i\int \frac{d^4q}{(2\pi)^4} \underline{V}(k^\perp - q^\perp) \not\eta \left[\underline{G}_\Sigma\left(q+\frac{\mathscr{P}}{2}\right)\Gamma(q|\mathscr{P})\underline{G}_\Sigma\left(q-\frac{\mathscr{P}}{2}\right)\right]\not\eta,$$

Integral equations

The quantities $\underline{\Sigma}$ and $\underline{\Gamma}$ depend only on the transversal momentum

$$\underline{\Sigma}(k) = \underline{\Sigma}(k^{\perp}), \qquad \underline{\Gamma}(k|\mathscr{P}) = \underline{\Gamma}(k^{\perp}|\mathscr{P}),$$

because of the instantaneous form of the potential $\underline{V}(k^{\perp})$ in any frame.

$$\underline{\Sigma}_a(q) = \not{q}^{\perp} + E_a(q^{\perp}) S_a^{-2}(q^{\perp})$$

for the self-energy with

$$S_a^{-2}(q^{\perp}) = \exp\{-\hat{\not{q}}^{\perp} 2v_a(q^{\perp})\}, \qquad \hat{q}_{\mu}^{\perp} = q_{\mu}^{\perp}/|q^{\perp}|,$$

where S_a is the Foldy-Wouthuysen type transformation matrix with the parameter v_a.

Integral equations

Then, one has

$$\underline{G}_{\Sigma_a} = [q_0 \not{\eta} - E_a(q^\perp) S_a^{-2}(q^\perp)]^{-1}$$
$$= \left[\frac{\Lambda^{(\eta)}_{(+)a}(q^\perp)}{q_0 - E_a(q^\perp) + i\varepsilon} + \frac{\Lambda^{(\eta)}_{(-)a}(q^\perp)}{q_0 + E_a(q^\perp) + i\varepsilon} \right] \not{\eta},$$

where

$$\Lambda^{(\eta)}_{(\pm)a}(q^\perp) = S_a(q^\perp) \Lambda^{(\eta)}_{(\pm)}(0) S_a^{-1}(q^\perp), \qquad \Lambda^{(\eta)}_{(\pm)}(0) = (1 \pm \not{\eta})/2$$

are the operators separating the states with positive $(+E_a)$ and negative $(-E_a)$ energies.

As a result, we obtain the following equations for the single-particle energy E and the angle v,

$$E_a(k^\perp) \cos 2v(k^\perp) = m_a^0 + \frac{1}{2} \int \frac{d^3 q^\perp}{(2\pi)^3} \underline{V}(k^\perp - q^\perp) \cos 2v(q^\perp),$$

$$E_a(k^\perp) \sin 2v(k^\perp) = |k^\perp| + \frac{1}{2} \int \frac{d^3 q^\perp}{(2\pi)^3} \underline{V}(k^\perp - q^\perp) |k^\perp \cdot q^\perp| \sin 2v(q^\perp).$$

The vertex function is given by

$$\Gamma_{ab}(k^{\perp}|\mathscr{P}) = \int \frac{d^3 q^{\perp}}{(2\pi)^3} V(k^{\perp} - q^{\perp}) \not{n} \psi_{ab}(q^{\perp}) \not{n},$$

where the bound-state wave function ψ_{ab} is given by

$$\psi_{ab}(q^{\perp}) = \not{n} \left[\frac{\bar{\Lambda}_{(+)a}(q^{\perp}) \Gamma_{ab}(q^{\perp}|\mathscr{P}) \Lambda_{(-)b}(q^{\perp})}{E_T - \sqrt{\mathscr{P}^2} + i\varepsilon} + \frac{\bar{\Lambda}_{(-)a}(q^{\perp}) \Gamma_{ab}(q^{\perp}|\mathscr{P}) \Lambda_{(+)b}(q^{\perp})}{E_T + \sqrt{\mathscr{P}^2} - i\varepsilon} \right] \not{n}.$$

$E_T = E_a + E_b$ means the sum of one-particle energies of the two particles (a) and (b)

$$\bar{\Lambda}_{(\pm)}(q^{\perp}) = S^{-1}(q^{\perp}) \Lambda_{(\pm)}(0) S(q^{\perp}) = \Lambda_{(\pm)}(-q^{\perp})$$

has been introduced.

Integral equations

$$(E_T(k^\perp) \mp \sqrt{\mathscr{P}^2})\Lambda^{(\eta)}_{(\pm)a}(k^\perp)\psi_{ab}(k^\perp)\Lambda^{(\eta)}_{(\mp)b}(-k^\perp)$$
$$= \Lambda^{(\eta)}_{(\mp)a}(k^\perp)\left[\int \frac{d^3q^\perp}{(2\pi)^3}\underline{V}(k^\perp - q^\perp)\psi_{ab}(q^\perp)\right]\Lambda^{(\eta)}_{(\mp)b}(-k^\perp).$$

Integral equations

$$E_a(\mathbf{k}) = \sqrt{(m_a^0)^2 + \mathbf{k}^2} \simeq m_a^0 + \frac{1}{2}\frac{\mathbf{k}^2}{m_a^0},$$

$$\tan 2v = \frac{k}{m^0} \to 0, \qquad S(\mathbf{k}) \simeq 1, \qquad \Lambda_{(\pm)} \simeq \frac{1 \pm \gamma_0}{2}.$$

$$\Lambda_{(+)}\psi\Lambda_{(-)} \simeq \psi_{\mathrm{Sch}}, \qquad \Lambda_{(-)}\psi\Lambda_{(+)} \simeq 0,$$

and finally the Schrödinger equation results in

$$\left[\frac{1}{2\mu}\mathbf{k}^{-2} + (m_a^0 + m_b^0 - M_A)\right]\psi_{\mathrm{Sch}}(\mathbf{k}) = \int \frac{d\mathbf{q}}{(2\pi)^3}\underline{V}(\mathbf{k}-\mathbf{q})\psi_{\mathrm{Sch}}(\mathbf{q}),$$

where $\mu = m_a \cdot m_b/(m_a + m_b)$. For an arbitrary total momentum $\mathscr{P}_\mu$,

$$\left[-\frac{1}{2\mu}(k_v^{\perp})^{-2} + (m_a^0 + m_b^0 - \sqrt{\mathscr{P}^2})\right]\psi_{\mathrm{Sch}}(k^{\perp}) = \int \frac{d^3q^{\perp}}{(2\pi)^3}\underline{V}(k^{\perp} - q^{\perp})\psi_{\mathrm{Sch}}(q^{\perp}),$$

Partial wave decomposition

This is achieved by expanding the functions $L(\mathbf{p})$, $\mathbf{N}(\mathbf{p})$ in spherical harmonics of total angular momentum, J. For the vector component it is convenient to use instead of the usual vector spherical harmonics the combinations

$$\begin{aligned}
\mathbf{Y}^1_{JM} &= \hat{\mathbf{p}} Y_{JM} &&= \alpha \mathbf{Y}_{JJ-1M} - \beta \mathbf{Y}_{JJ+1M},\\
\mathbf{Y}^2_{JM} &= (J(J+1))^{-1/2}\, \nabla_{\hat{\mathbf{p}}} Y_{JM} &&= \beta \mathbf{Y}_{JJ-1M} + \alpha \mathbf{Y}_{JJ+1M},\\
\mathbf{Y}^3_{JM} &= -i(J(J+1))^{-1/2}\, \hat{\mathbf{p}} \times \nabla_{\hat{\mathbf{p}}} Y_{JM} &&= \mathbf{Y}_{JJM},
\end{aligned}$$

with

$$\alpha = \sqrt{\frac{J}{2J+1}}, \qquad \beta = \sqrt{\frac{J+1}{2J+1}}, \qquad \alpha^2 + \beta^2 = 1.$$

The new functions $\mathbf{Y}^\lambda_{JM}$ have simple transformation properties under the operations $\hat{\mathbf{p}} \times \mathbf{Y}^\lambda_{JM}$ and $\hat{\mathbf{p}} \cdot \mathbf{Y}^\lambda_{JM}$. They are orthogonal and normalized as the usual vector spherical harmonics.

$$L^{(k)}(\mathbf{p}) = \frac{\ell^{(k)}_J(p)}{p} Y_{JM}(\hat{\mathbf{p}}), \qquad \mathbf{N}^{(k)}(\mathbf{p}) = \sum_{\lambda=1}^{3} \frac{n^{(k)}_{\lambda J}(p)}{p} \mathbf{Y}^\lambda_{JM}(\hat{\mathbf{p}}) \qquad (k = 1, 2).$$

Integral equations

For the angular matrix element of the potential kernel we use the definition

$$\frac{pq}{(2\pi)^3}\int d\Omega_p \int d\Omega_q Y^*_{L'M'}(\hat{\mathbf{p}})V(\mathbf{p}-\mathbf{q})Y_{LM}(\hat{\mathbf{q}}) \;=\; v_L(p,q)\delta_{LL'}\delta_{MM'}.$$

Note that the matrix element is independent of M, and that $v_L(p,q)$ is a symmetric functions of the radial variables p, q.

$$\frac{pq}{(2\pi)^3}\int d\Omega_p \int d\Omega_q \mathbf{Y}^*_{J'L'M'}(\hat{\mathbf{p}})V(\mathbf{p}-\mathbf{q})\mathbf{Y}_{JLM}(\hat{\mathbf{q}}) \;=\; v_L(p,q)\,\delta_{JJ'}\delta_{LL'}\delta_{MM'}$$

$$(L \;=\; J, J\pm 1)$$

we obtain the matrix elements

$$\frac{pq}{(2\pi)^3}\int d\Omega_p \int d\Omega_q \mathbf{Y}^{\lambda *}_{JM}(\hat{\mathbf{p}})V(\mathbf{p}-\mathbf{q})\mathbf{Y}^{\rho}_{JM}(\hat{\mathbf{q}}) \;=$$

$$\begin{cases} \alpha^2 v_{J-1}+\beta^2 v_{J+1} \;\equiv\; \overline{v}_J(p,q) & (\lambda,\rho)=(1,1) \\ \beta^2 v_{J-1}+\alpha^2 v_{J+1} \;\equiv\; \overline{\overline{v}}_J(p,q) & (\lambda,\rho)=(2,2) \\ v_J & (\lambda,\rho)=(3,3) \\ \alpha\beta(v_{J-1}-v_{J+1}) \;\equiv\; \tilde{v}_J(p,q) & (\lambda,\rho)=(1,2),(2,1) \end{cases}$$

Integral equations

$$
\begin{aligned}
M\ell_J^{\binom{2}{1}}(p) - E_p\ell_J^{\binom{1}{2}}(p) &= \int_0^\infty dq\left\{[c_p^{\mp}c_q^{\mp}v_J + s_p^{\mp}s_q^{\mp}\overline{v}_J]\ell_J^{\binom{1}{2}}(q) + is_p^{\mp}s_q^{\pm}\widetilde{v}_J n_{3J}^{\binom{2}{1}}(q)\right\} \\
Mn_{3J}^{\binom{2}{1}}(p) - E_p n_{3J}^{\binom{1}{2}}(p) &= \int_0^\infty dq\left\{[c_p^{\mp}c_q^{\mp}v_J + s_p^{\mp}s_q^{\mp}\overline{\overline{v}}_J]n_{3J}^{\binom{1}{2}}(q) - is_p^{\mp}s_q^{\pm}\widetilde{v}_J\ell_J^{\binom{2}{1}}(q)\right\}
\end{aligned}
$$

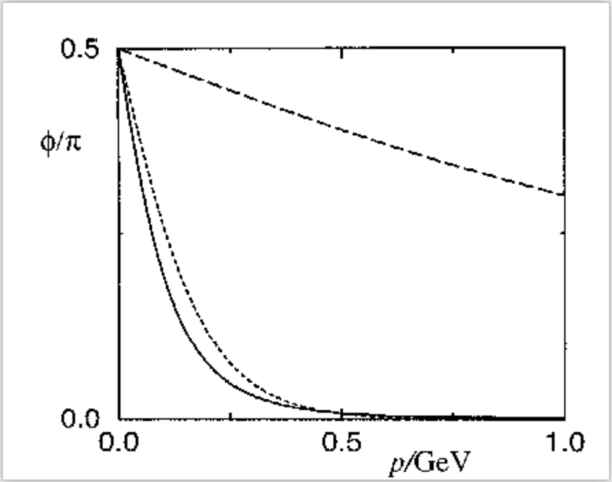
and

$$
\begin{aligned}
Mn_{1J}^{\binom{2}{1}}(p) - E_p n_{1J}^{\binom{1}{2}}(p) &= \int_0^\infty dq\left\{[c_p^{\pm}c_q^{\pm}\overline{v}_J + s_p^{\pm}s_q^{\pm}v_J]n_{1J}^{\binom{1}{2}}(q) + c_p^{\pm}c_q^{\mp}\widetilde{v}_J n_{2J}^{\binom{1}{2}}(q)\right\} \\
Mn_{2J}^{\binom{2}{1}}(p) - E_p n_{2J}^{\binom{1}{2}}(p) &= \int_0^\infty dq\left\{[c_p^{\mp}c_q^{\mp}\overline{\overline{v}}_J + s_p^{\mp}s_q^{\mp}v_J]n_{2J}^{\binom{1}{2}}(q) + c_p^{\mp}c_q^{\pm}\widetilde{v}_J n_{1J}^{\binom{1}{2}}(q)\right\}
\end{aligned}
$$

Integral equations

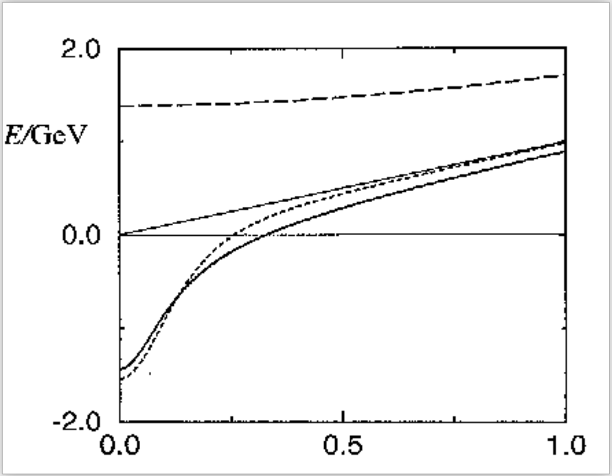
$V(r)$	$V(\mathbf{p}-\mathbf{q})$	$v_L(p,q)$
$1/r$	$\dfrac{4\pi}{\lvert\mathbf{p}-\mathbf{q}\rvert^2}$	$\dfrac{1}{\pi}Q_L(x)$
r	$-\dfrac{8\pi}{\lvert\mathbf{p}-\mathbf{q}\rvert^4}+C\delta^{(3)}(\mathbf{p}-\mathbf{q})$	$\dfrac{1}{\pi}\dfrac{1}{pq}\dfrac{dQ_L}{dx}(x)$
r^2	$(2\pi)^3\nabla^2_{\mathbf{q}}\delta^{(3)}(\mathbf{p}-\mathbf{q})$	$\left(\dfrac{d^2}{dq^2}+\dfrac{L(L+1)}{q^2}\right)\delta(p-q)$

Integral equations



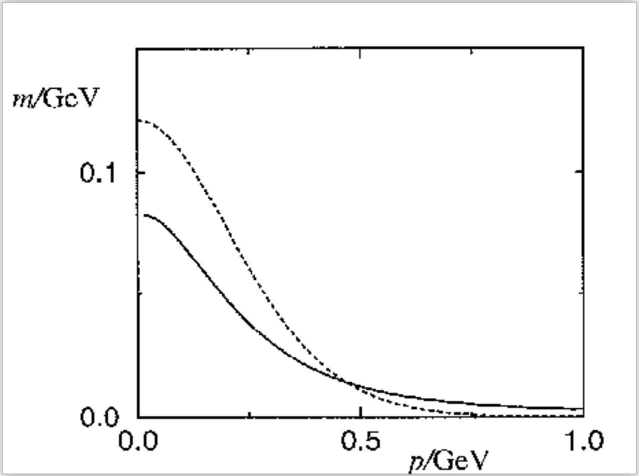
The chiral angle, $\phi(|\mathbf{p}|)$, from the Schwinger–Dyson equation for a light flavor ($m^{\upsilon} = 1$ MeV). Solid line: linear plus Coulomb potential ($\sigma = 0.41\,\text{GeV}$, $\alpha_s = 0.39$), dotted line: pure oscillator potential ($V_0 = 0.247\,\text{GeV}$). The dashed line shows the value for a constant heavy quark mass, $m_Q = m_c = 1.39\,\text{GeV}$.

Integral equations



The quark energy, $E(|\mathbf{p}|)$, from the Schwinger–Dyson equation for a light flavor ($m^0 = 1$ MeV). The potentials are those of fig.1. The dashed line shows the energy for a constant heavy quark mass, $m_Q = m_c = 1.39\,\text{GeV}$. The straight solid line indicates the asymptotic behavior, $E = |\mathbf{p}|$.

Integral equations



Задачи

1. Моделирование и численный анализ поведения псевдоскалярных мезонов в зависимости от вида потенциала взаимодействия.
2. Моделирование и численный анализ поведения векторных мезонов в зависимости от вида потенциала взаимодействия.
3. Свойства J/ψ и Υ.
4. Моделирование и численный анализ релятивистских уравнений для глюонных систем.
5. Моделирование и численный анализ релятивистских уравнении для многокварковых систем.
6. Моделирование распадов тяжелых кварконием в нелокальной эффективной теории.

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