Lectures 12-14: High Energy Cooling

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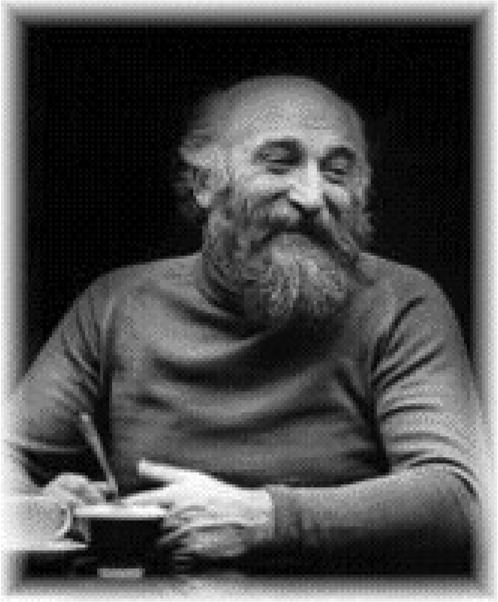
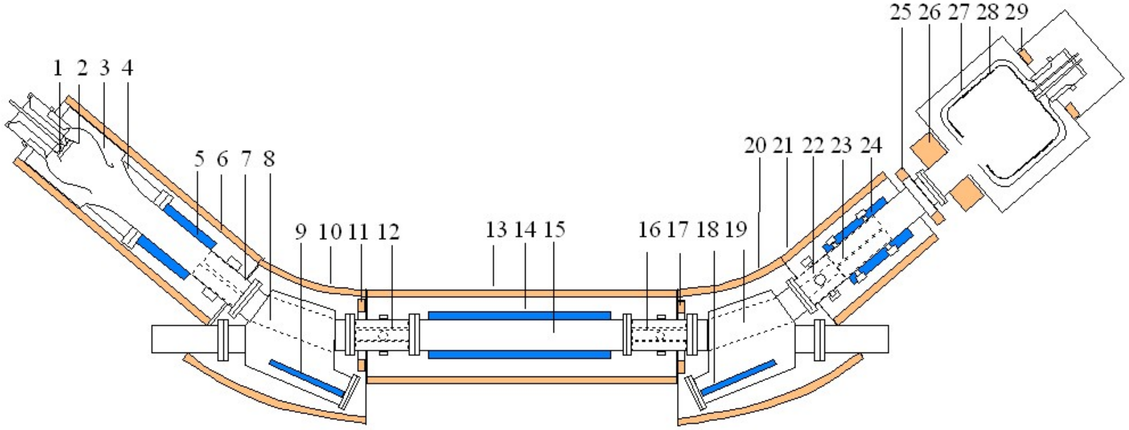
Objectives

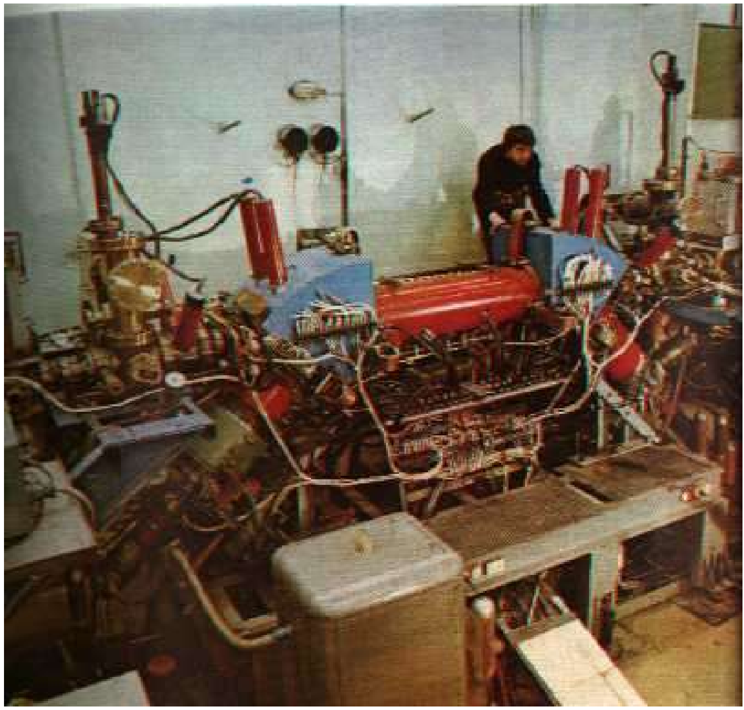
- Presently, there are two major methods of the cooling: the electron cooling and stochastic cooling.
- The stochastic cooling can be additionally separated on (1) the microwave stochastic cooling, (1) the optical stochastic cooling (OSC) and (3) the coherent electron cooling (CEC).
  - OSC and CEC are essentially extensions of microwave stochastic cooling operating in 1-10 GHz frequency range to the optical frequencies corresponding to 30-300 THz frequency range.
    - The OSC uses undulators as a pickup and a kicker, and an optical amplifier for signal amplification,
    - while the CEC uses an electron beam for all these functions.
- In these 3 lectures we consider electron and stochastic cooling mostly concentrating on cooling of high energy heavy particles (protons or ions) in the high energy colliders. Further in all equations we assume protons – the most challenging case.
- Later in the lectures 15 and 16 we consider the stochastic cooling at optical wavelengths

Electron cooling

- Invented in 1966 by A. M. Budker
  - In the beam frame - heavy particles come into equilibrium with electron gas
- Tested experimentally in BINP, Novosibirsk, in 1974-79 at NAP-M
  - 35 keV electron beam (65 MeV protons)
  - Magnetized electron cooling





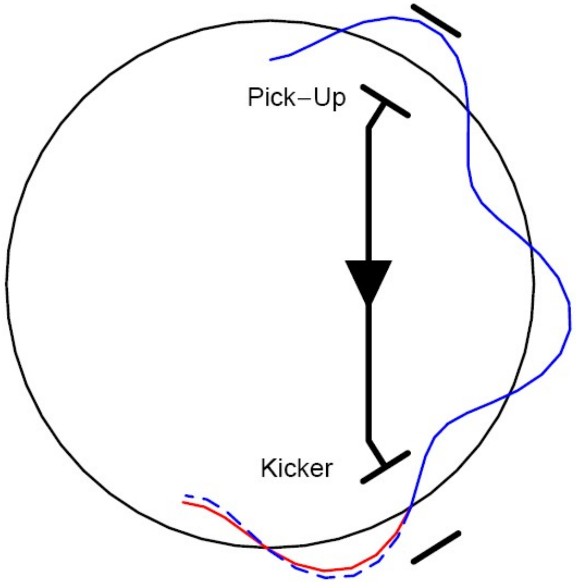


- Many installations since then, up to 300 kV electron beam (GSI, Darmstadt)
- FNAL 4.3 MeV cooler – next step in technology

Stochastic Cooling

- Invented in 1969 by Simon van der Meer
- Naïve transverse cooling model
  - 90 deg. between pickup and kicker
    $$\delta\theta = -g\theta$$
  - Averaging over betatron oscillations yields
    $$\delta\overline{\theta^2} = -\frac{1}{2}2g\overline{\theta^2} \equiv -g\overline{\theta^2}$$
  - Adding noise of other particles yields
    $$\delta\overline{\theta^2} = -g\overline{\theta^2} + N_{sample}g^2\overline{\theta^2} \equiv -\left(g - N_{sample}g^2\right)\overline{\theta^2}$$
  - That yields

$$\boxed{\delta\overline{\theta^2} = -\frac{1}{2}g_{opt}\overline{\theta^2} \quad , \quad g_{opt} = \frac{1}{2N_{sample}} \quad , \quad N_{sample} \approx N\frac{f_0}{W}}$$



- In accurate analytical theory the cooling process is described by Fokker-Planck equation
  - The theory is built on the same principle as plasma theory – which is a perturbation theory (large number of particles in the Debye sphere versus large number of particles in the sample)

Requirements for Cooling in Collision Mode

- Cooling time is typically set by IBS.
  - 20-40 minutes for ep collider for 275 GeV protons
- Cooling acceptances
  - Good beam lifetime in the presence of beam-beam effects requires cooling range to be > 5 - 6 σ.
- Overcooling in the bunch center has to be avoided
  - Overcooling greatly amplifies beam-beam effects
  - Ideally the cooling force should be proportional to particle amplitudes

Historical Remarks

- Maximum beam energy achieved in electron cooling with 8 GeV protons was demonstrated in Fermilab in the course of Tevatron Run II (2001 -2011).
  - This energy is well below required for most of modern proton colliders. There are few ideas how energy increase can be accomplished but no definite plans to demonstrate it in experiment
- The stochastic cooling was absolutely essential for stacking and cooling antiprotons in SPPS (CERN) and Tevatron (Fermilab).
  - Up to 2021, the stochastic cooling has been only operating at the microwave frequencies (f < 8 GHz).
  - BNL demonstrated SC of bunched heavy ions at RHIC
- First cooling at optical frequencies – the OSC - was demonstrated in Fermilab with electrons in 2021.
  - Passive OSC for now
  - *A* usage of electrons greatly decreased the cost of the experiment but still enabled us to study the physics in detail

Electron versus Stochastic Cooling

- The electron and stochastic cooling are based on completely different principles.
- The electron cooling is dissipative in its principle of operation and therefore the Liouville theorem is not applicable. That enables direct reduction of the beam phase space.
- The stochastic cooling is a "Hamiltonian" process which formally does not violate the Liouville theorem and cooling happens due to the phase space mapping so that phase space volumes containing particles are moved to the beam center while the rest mostly moves out. That makes stochastic colling rates strongly dependent on the beam particle density.
- Each method has its own domain where it achieves a superior efficiency. The electron cooling is preferred at a smaller energy and momentum spread, and its efficiency weakly depends on the particle density in the cooled beam. While the stochastic cooling is preferred at a higher energy, but its efficiency reduces fast with increase of particle phase density.

Electron Cooling

Simple Cooling Force Derivation

- Energy transfer from proton to the electron in the rest in the small scattering angle approximation

$$\Delta p \approx e\int E(t)dt = e\int E(z)\frac{dz}{\mathrm{v}} \xrightarrow[\int E(z)dz=\frac{2Ze}{\rho}]{Gauss\ theorem} \frac{2Ze^2}{\rho \mathrm{v}}$$

- To find the energy transfer to entire electron plasma we need to sum contributions of all electrons

$$F \equiv \frac{d\mathrm{E}}{dx} = \int \frac{\Delta p^2}{2m_e}\left(n_e 2\pi\rho d\rho\right) \approx \frac{\pi n_e}{m_e}\int\left(\frac{2Ze^2}{\rho \mathrm{v}}\right)^2 \rho d\rho = \frac{4\pi Z^2 n_e e^4}{m_e \mathrm{v}^2}\int\frac{d\rho}{\rho}$$

  - Logarithmic divergence at both limits needs to be removed
    ρ_{min} is set by conditions of applicability of small angular distribution
    ρ_{max} is set by plasma screening

⇨ $$F = \frac{4\pi Z^2 n_e e^4}{m_e \mathrm{v}^2}\ln\left(\frac{\rho_{\max}}{\rho_{\min}}\right) \xrightarrow[electron\ motion]{accounting} \mathbf{F} = \frac{4\pi Z^2 n_e e^4}{m_e \mathrm{v}^2}\ln\left(\frac{\rho_{\max}}{\rho_{\min}}\right)\int\frac{\mathbf{v}-\mathbf{v}'}{\left|\mathbf{v}-\mathbf{v}'\right|^3} f(\mathrm{v}')d\mathrm{v}'^3$$

Cooling Force in non-magnetized Cooling

$$\mathbf{F}(\mathbf{v}) = \frac{4\pi n_e e^4 L_c}{m_e} \int f(\mathbf{v}') \frac{\mathbf{v}-\mathbf{v}'}{|\mathbf{v}-\mathbf{v}'|^3} d\mathbf{v}'^3 = \frac{4\pi n_e e^4 L_c}{m_e} \nabla_{\mathrm{v}} \left(\int \frac{f(\mathbf{v}')}{|\mathbf{v}-\mathbf{v}'|} d\mathbf{v}'^3 \right)$$

Coulomb logarithm

$$L_c = \ln\left(\frac{r_{\max}}{r_{\min}}\right), \quad r_{\min} \approx \frac{2e^2}{m_e \mathrm{v}_{eff}{}^2}, \quad r_{\max} \approx \frac{\mathrm{v}_{eff}}{\omega_p}, \quad \mathrm{v}_{eff} = \max\left(|\mathbf{v}|, \overline{\mathrm{v}}_e\right),$$

The term with $dD_{ij}/d\mathrm{v}_j$ is im m_p/m_e times smaller and can be neglected

- By-Gaussian distribution of electrons in velocity

$$f(\mathbf{v}_\perp, \mathrm{v}_{||}) = \frac{1}{(2\pi)^{3/2} \sigma_{\mathrm{v}||}\sigma_{\mathrm{v}\perp}{}^2} \exp\left(-\frac{\mathrm{v}_{||}{}^2}{2\sigma_{\mathrm{v}||}{}^2} - \frac{\mathbf{v}_\perp{}^2}{2\sigma_{\mathrm{v}\perp}{}^2}\right), \quad \mathrm{v}_{||} \ll \mathrm{v}_\perp$$

- Similar to the IBS, the following formula $\displaystyle \frac{1}{\theta} = \frac{1}{\sqrt{\pi}} \int_0^\infty e^{-\lambda^2\theta^2} d\lambda$

enables to reduce the cooling force to single dimensional integral

$$\mathbf{F}(\mathbf{v}) = \frac{4\pi n_e e^4 L_c}{m_e} \nabla_{\mathrm{v}} \left(\frac{2}{\sqrt{\pi}} \int_0^\infty \frac{\exp\left(-\frac{\mathrm{v}_{||}{}^2 t^2}{1+2\sigma_{\mathrm{v}||}{}^2 t^2} - \frac{\mathbf{v}_\perp{}^2 t^2}{1+2\sigma_{\mathrm{v}\perp}{}^2 t^2}\right)}{\sqrt{\left(1+2\sigma_{\mathrm{v}||}{}^2 t^2\right)\left(1+2\sigma_{\mathrm{v}\perp}{}^2 t^2\right)^2}} dt \right)$$

Cooling Force

- For T_{beam}=0 the cooling force is

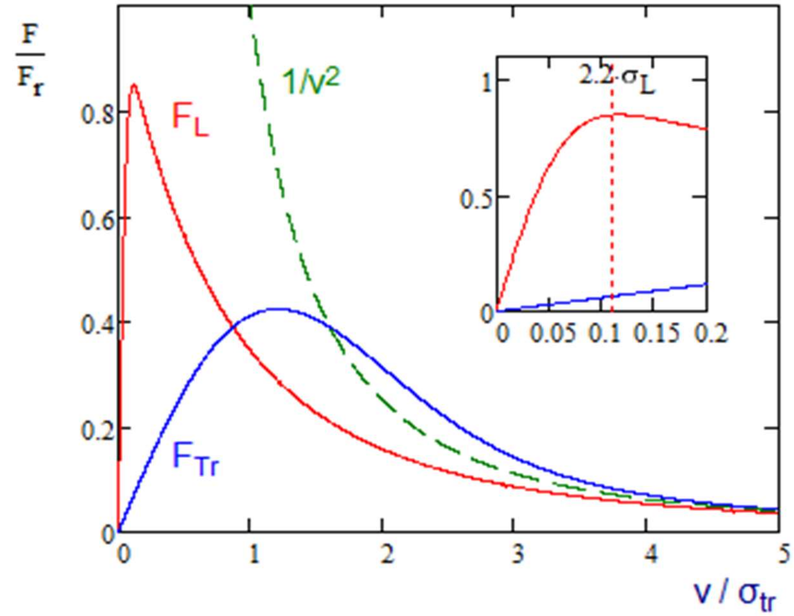
$$\mathbf{F}(\mathbf{v}) = \frac{4\pi n_e e^4 L_c}{m_e} \frac{\mathbf{v}}{\mathbf{v}^3}$$

- The force saturates at velocity where the plasma perturbation theory stops to work:

$$\rho_{\min} \approx \rho_{\max} \quad \text{or} \quad e^2 n^{1/3} \approx m\text{v}^2 / 2$$

$$\Rightarrow F_{\max} \approx e^2 n^{2/3}$$

The velocity, where the maximum is achieved, is orders of magnitude smaller than rms velocity in the proton beam



|| ($F_{||}(v_{||}, v_\perp=0)$) and ⊥ ($F_\perp(v_{||}=0, v_\perp)$) cooling forces on particle velocity; $F_r = 4\pi n_e e^4 L_c / (m_e \sigma_{v\perp}{}^2)$, $\sigma_{v||} = \sigma_{v\perp} / 20$.

- For electrostatic acceleration temperatures are:

$$T_\perp \approx T_{cathotde} \;, \quad T_{||} \approx T_{cathotde}{}^2 / W + 2e^2 n^{1/3} \ll T_\perp$$

- Strong accompanying magnetic field freezes out $T_\perp$ (magnetized cooling)
  - That greatly increases cooling force at small velocities. However, it is not helpful for collider cooling where T_{proton_beam} is much larger
  - It only makes overcooling in the distribution center

Diffusion Estimate

- A rough estimate for a single collision

$$\Delta p_p = \Delta p_e \approx \frac{2Ze^2}{\rho\left|\mathbf{v}_p - \mathbf{v}_e\right|} \quad \Rightarrow \quad \overline{\frac{d}{dt}\Delta p_p{}^2} \approx \int \Delta p_p{}^2 f(\mathrm{v}')2\pi\rho\left|\mathbf{v}_p - \mathbf{v}_e\right| d\rho d\mathrm{v}'^3$$

- Accurate accounting yields the Landau collision integral for two component plasma

$$\Rightarrow \quad \frac{df}{dt} = -\frac{\partial}{\partial p_i}\left(F_i f\right) + \frac{1}{2}\frac{\partial}{\partial p_i}\left(D_{ij}\frac{\partial f}{\partial p_j}\right), \quad \begin{cases} F_i(\mathrm{v}) = -\dfrac{4\pi n Z^2 e^4 L_c}{m_e}\displaystyle\int f(\mathrm{v}_e)\frac{u_i}{\left|\mathbf{u}\right|^3} d^3\mathrm{v}_e\,, \\ D_{ij}(\mathrm{v}) = 4\pi n Z^2 e^4 L_c \displaystyle\int f(\mathrm{v}_e)\frac{u^2\delta_{ij} - u_i u_j}{\left|\mathbf{u}\right|^3} d^3\mathrm{v}_e\,, \end{cases} \quad \mathbf{u} = \mathbf{v} - \mathbf{v}_e$$

  - In difference to single component plasma ion plasma considered in the IBS lecture the cooling force is larger in the mass ratio (Am_p/m_e)
  - Consequently, the contribution related to the gradient of diffusion

$$\frac{d}{dt}\overline{\delta p_i} = F_i(\mathbf{p}_0) + \left.\frac{\partial D_{ki}}{\partial \mathrm{v}_k}\right|_{\mathbf{p}=\mathbf{p}_0}$$

  which doubled cooling force in the IBS can be neglected

  - Obviously in equilibrium $m_e\overline{\mathrm{v}_e{}^2} = Am_p\overline{\mathrm{v}_p{}^2}$. That makes $\overline{\mathrm{v}_p{}^2} \ll \overline{\mathrm{v}_e{}^2}$

Single Pass Electron Cooling

- "Solenoid model" installation, BINP, 1979 - 1989
  - Next step in understanding of magnetized cooling
  - Low energy single pass device
  - Magnetic field non-uniformity has negligible effect on cooling due to small velocities of electrons

- Cooling section length – 2.4 m
- Electron energy – 460 eV
- Magnetic field – 1-4 kGs
- Measured longitudinal cooling length - ~20 m

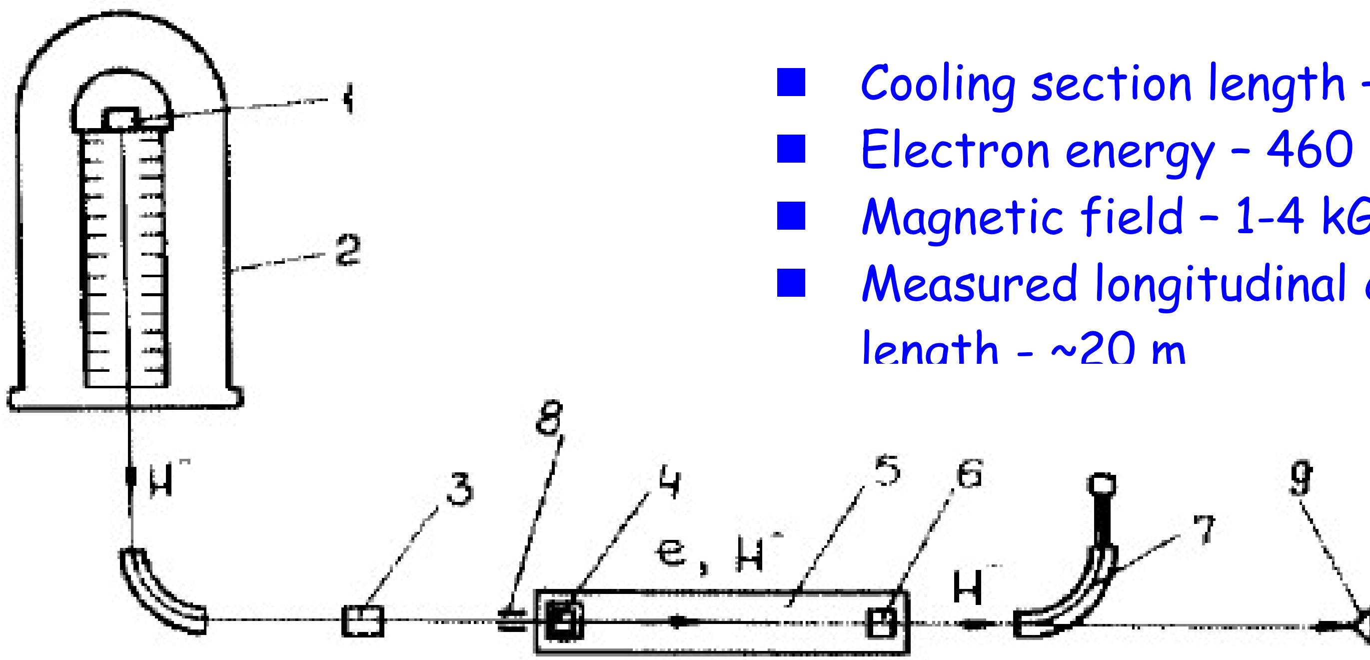
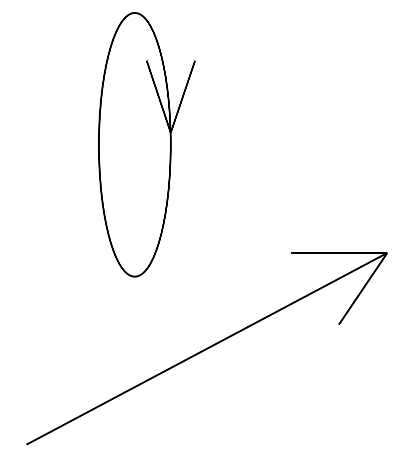


Fig. 1. Layout of the device:
1 — source of H^- ions; *2* — electrostatic accelerator; *3* — magnesium vapor target; *4* — electron gun; *5* — solenoid; *6* — collector of electrons; *7* — spectrometer; *8* — plates of transverse ion spread; *9* — two-coordinate position-sensitive detector.

Magnetized Electron Cooling



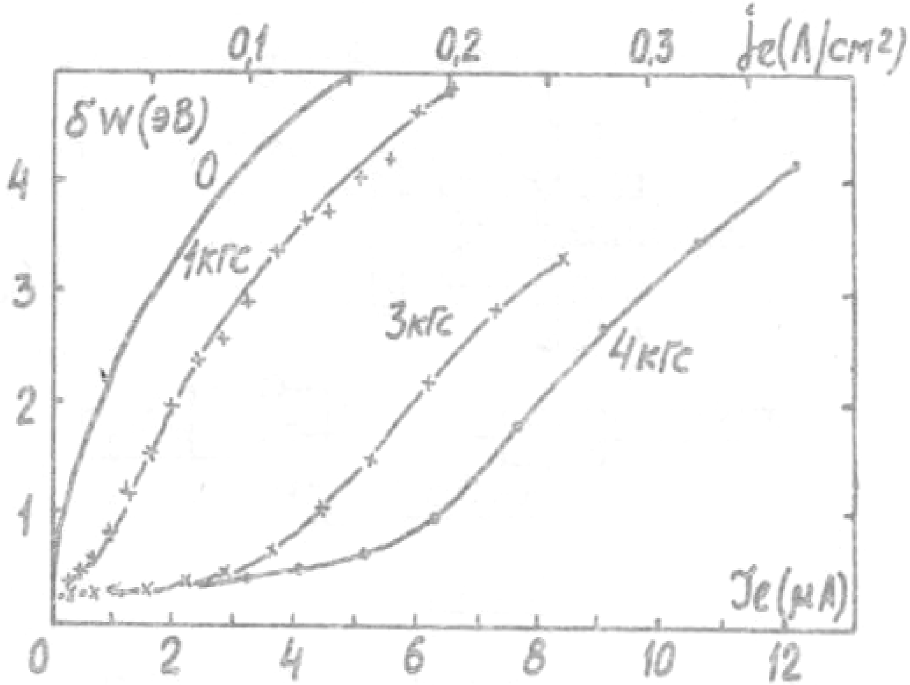
- Magnetic field switches off $T_{\perp}$
  - For $r_L < (r_{max} = v/\omega_p)$ the transverse temperature is magnetized out
    - That results in an increase of force for small velocities, $v_i < v_{\perp}$
- Longitudinal temperature of electrons is set by longitudinal-longitudinal relaxation to

$$T_{\parallel} \approx \frac{T_c^{\ 2}}{2W_e} + 1.9 e^2 n_e^{\ 1/3}$$

Growth of $T_{||}$ due to transverse-longitudinal relaxation (IBS or Boersch effect) is suppressed by magnetic field: $\rho_L < n_e^{-1/3}$

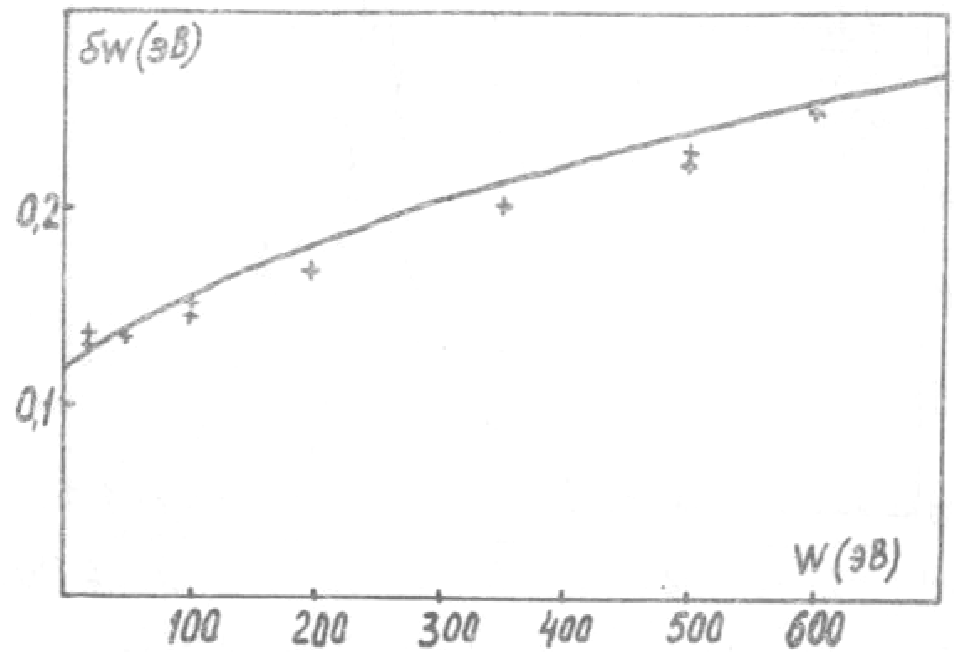
- **There is no accurate theory capable to describe cooling at small velocities!!!**

Dependence of electron temperature on beam parameters



Dependence of energy spread for electrons at the cooling section end on electron beam current and magnetic field. Electron energy is 470 eV; 0 – is a result of calculations for B=0 in according with

$$\frac{dT_{\parallel}}{dz} = k\frac{\pi e^3 jL_c}{W_e}\sqrt{\frac{m}{T_{\perp}}} \quad , \quad k \approx 0.87$$



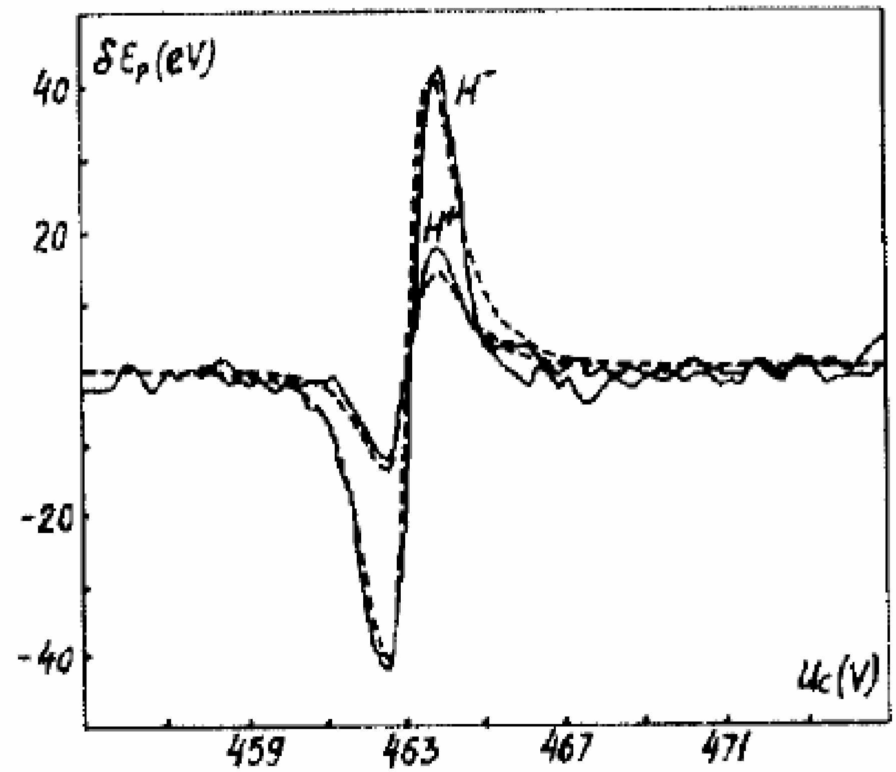
Dependence of energy spread for electrons at the end of 40 cm drift section on electron beam energy for low beam current (I_e=30 μA), Magnetic field is 1.3 kG. Solid line is calculated with

$$T_{\parallel} = \frac{T_c^{\ 2}}{2W_e} + 1.9e^2 n_e^{\ 1/3}$$

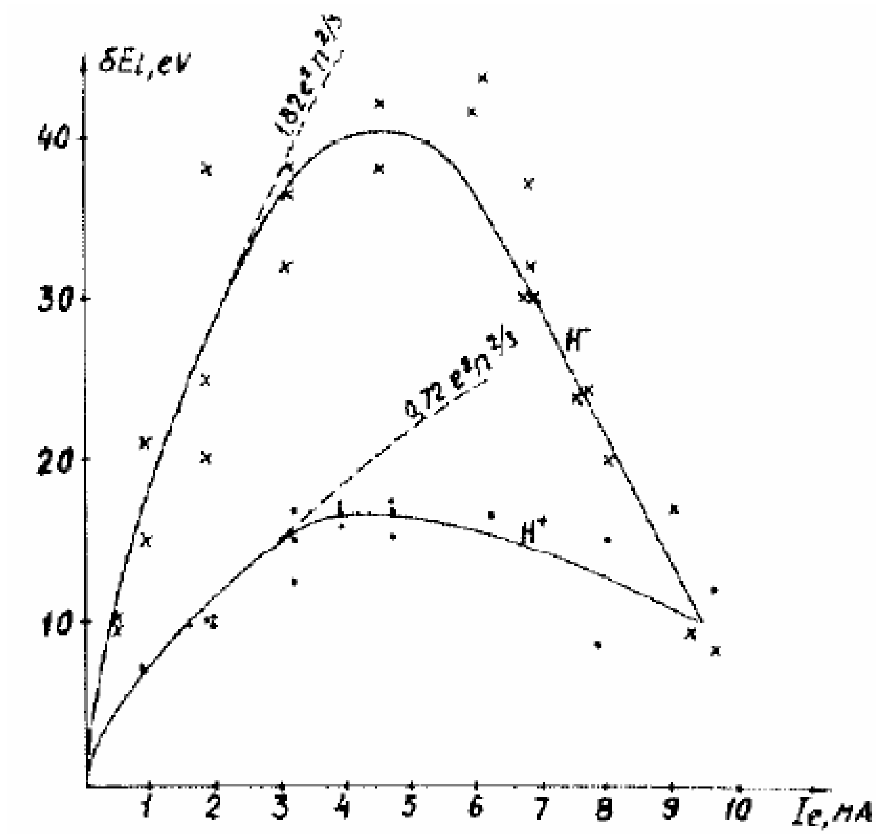
Maximum Cooling Force

- Maximum cooling force is achieved at velocities corresponding to

$\mathrm{v_i} \approx \mathrm{v_e} \approx \sqrt{e^2 n_e^{1/3} / m}$ and is equal to $F \approx e^2 n_e^{2/3}$



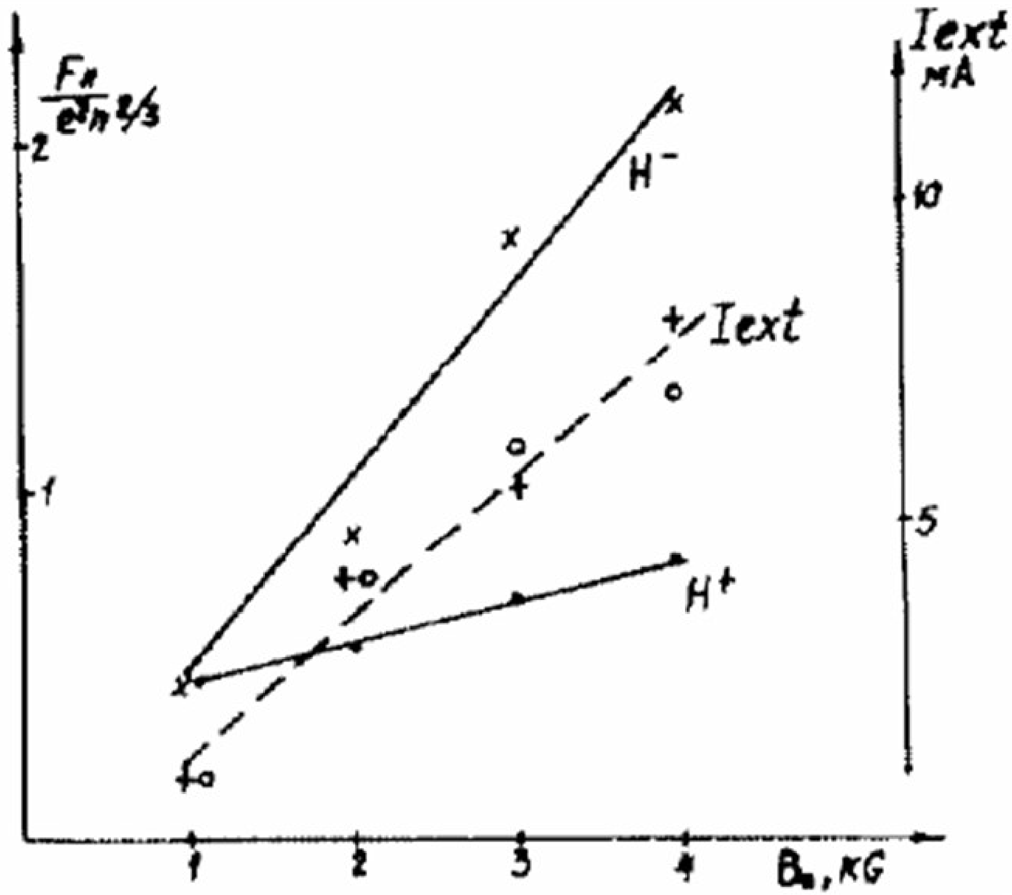
Energy variations of the ions of different-sign charge vs the electron energy and fitting the expression (5) to this dependence: the electron beam current 3 mA, magnetic field 3 kG. Numerical processing yields the following parameters:
H^-: $F_{max} l = 44.1$ eV, $\Delta E_0 = 1.21$ V, $U_0 = 463.1$ V;
H^+: $F_{max} l = 15.4$ eV, $\Delta E_0 = 1.30$ V, $U_0 = 463.0$ V;



The maximum energy variation for ions $\delta\varepsilon_i = lF_{\parallel max}$ vs the electron current; $B_0 = 3$ kG, × — H^-, ● — H^+, cooling section length $l = 2.4$ m. The dotted curves correspond to following expressions: $F_{max}^{-} = 1.82 e^2 n^{2/3}$, $F_{max}^{+} = 0.72 e^2 n^{2/3}$.

Maximum Cooling Force (continue)

- Larger magnetic field results in the larger cooling forces and the larger difference
- Saturation of maximum force with magnetic field was not observed in the experiments
- The difference between cooling positive and negative particles is related with additional pushing forward of electrons by negative ions (antiprotons)



The ratio $F_{max}/e^2n^{2/3}$ for low currents of the electron beam ($\times$ — H^-, $\bullet$ — H^+) and the electron beam current I_{ext} at which the longitudinal friction force F_{max} is maximum ($+$ — H^-, $\circ$ — H^+) vs the magnetic field.

Cooling Rates for Highly Relativistic El. Cooling

- For practical applications [1]

$$\lambda_{\parallel} \approx \frac{4\sqrt{2\pi} n_e r_e r_p L_c}{\gamma^4 \beta^4 \left(\Theta_{\perp} + 1.083\Theta_{\parallel}/\gamma\right)^{3/2} \sqrt{\Theta_{\perp}\Theta_{\parallel}}} L_{cs} f_0 \ ,$$

$$\lambda_{\perp} \approx \frac{\pi\sqrt{2\pi} n_e r_e r_p L_c}{\gamma^5 \beta^4 \Theta_{\perp}{}^2 \left(\Theta_{\perp} + \sqrt{2}\Theta_{\parallel}/\gamma\right)} L_{cs} f_0 \ ,$$

$$\frac{\Theta_{\parallel}}{\gamma\Theta_{\perp}} \le 2 \, .$$

$$\Theta_{\parallel} = \sqrt{\theta_{\parallel e}{}^2 + \theta_{\parallel p}{}^2} \ ,$$

$$\Theta_{\perp} = \sqrt{\theta_{\perp e}{}^2 + \theta_{\perp p}{}^2} \ ,$$

- Beam power grows fast with beam energy. For fixed $L_{cs} f_0$ one has

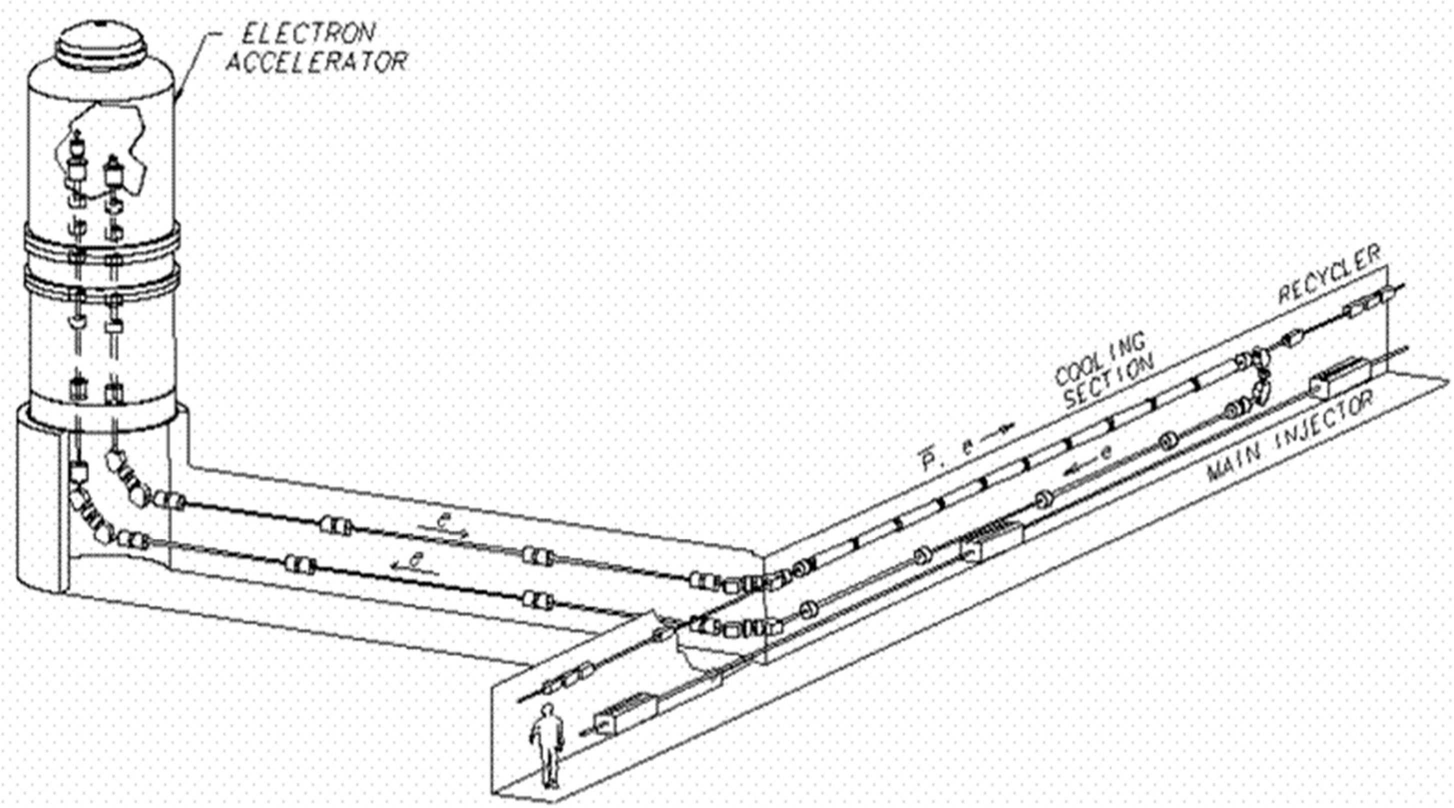
$$P = mc^2(\gamma - 1)\cdot \pi r_{eb}{}^2 e n_e c\beta \xrightarrow[r_{eb}{}^2 \propto 1/\gamma]{\varepsilon_n = const} \propto \gamma\gamma^5 \frac{1}{\gamma} = \gamma^5$$

  - For the 275 GeV proton beam one needs ~100 A electron beam current. It corresponds ~10 GW reactive beam power
- Typical rms angles in proton beam is ~10 - 20 μrad for 275 GeV
  - The straightness of magnetic field should be better – the extremely challenging problem

[1] V. Lebedev et al 2021 JINST **16** T01003

Electron Cooling at FNAL

- Fermilab made next step in the electron cooling technology
- Main Parameters
  - 4.34 MeV pelletron
  - 0.5 A DC electron beam with radius of about 4 mm
  - Magnetic field in the cooling section - 100 G
  - Interaction length – 20 m (out of 3319 m of Recycler circumference)



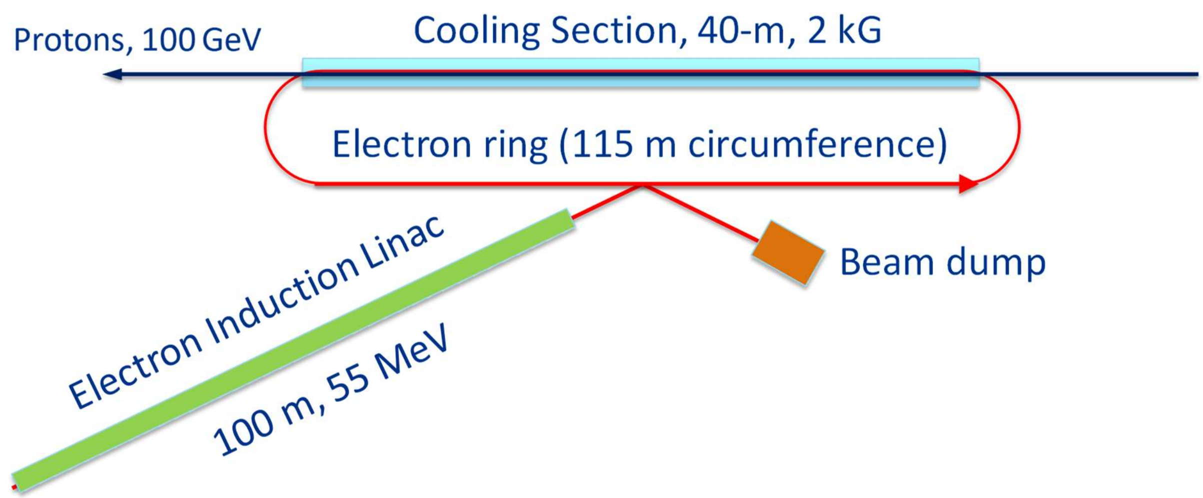
Electron Cooling at FNAL

What makes Fermilab electron cooler unique?

- No strong longitudinal magnetic field accompanying electron beam all the way from gun to collector
  - Angular-momentum-dominated beam transport line
  - Phase advance Q~6
  - Fully coupled motion
  - Length of beam transport~70 m
- Cooling with low-magnetic field – something that had never been tested, B=100 G
- Much higher energy than any cooler before (GSI ~0.3 MV)

Pelletron

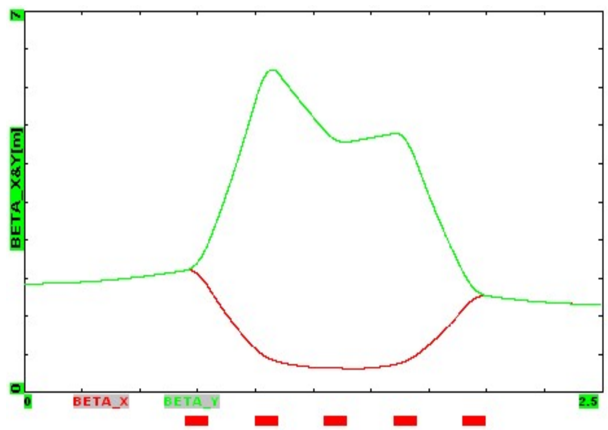
Possible Implementation of HE Electron Cooling

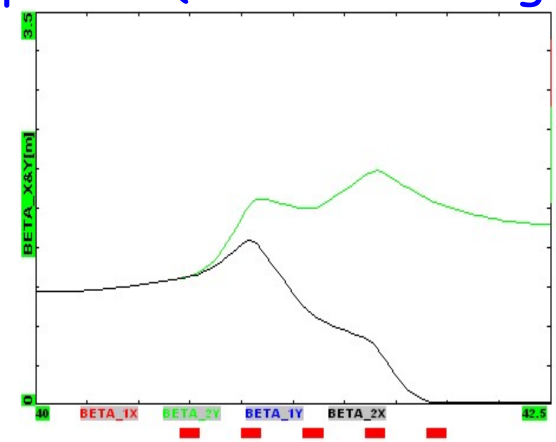


- Acceleration beam in induction linac with subsequent beam recirculation for ~10,000 turns (limited by IBS in e-beam). P~1 MW
  - The number of turns is limited by IBS in the electron beam
- Derbenev's transform is used to optimally match proton and electron velocities in the cooling section
  - Fully coupled ring optics
  - Electron gun cathode emersed in long. magnetic field to create rotational modes
- Major challenges: (1) space charge in electron beam, (2) beam stability (CSR impedances), (3) emittance growth due to interaction with proton bunches (suppressed by integer number of electron rotations in the cooling section)

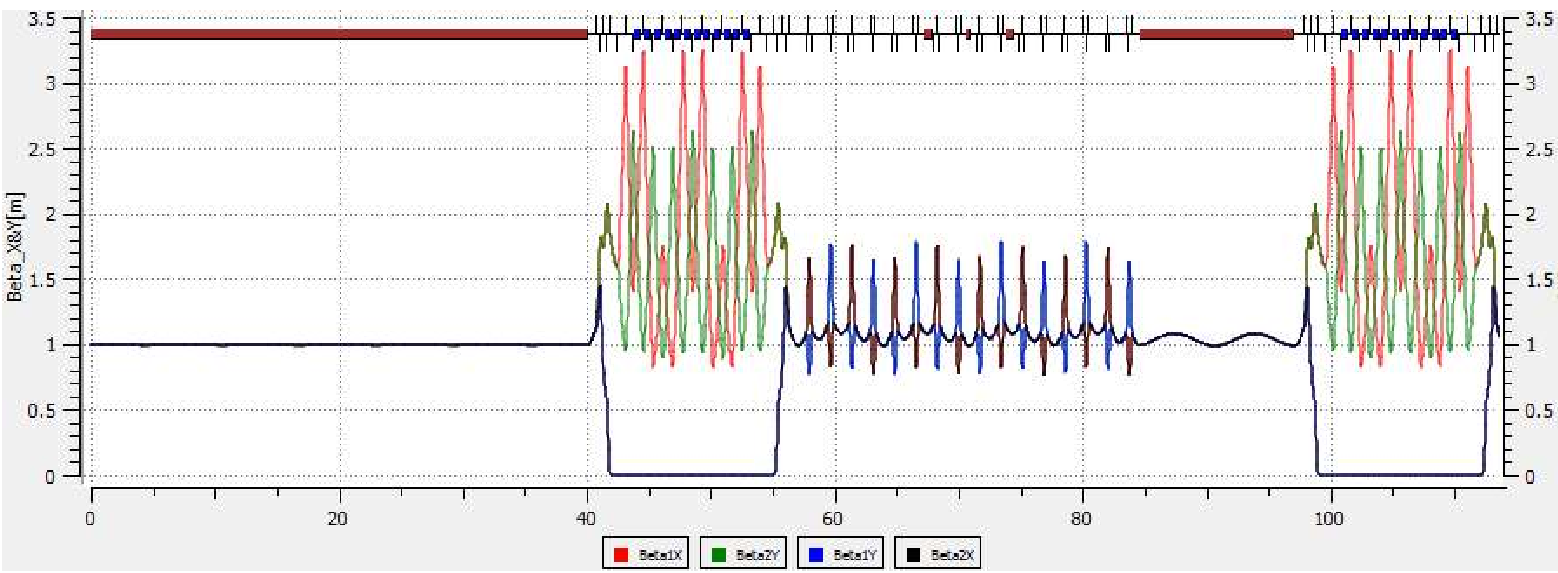
Derbenev's Adapter (Transform)

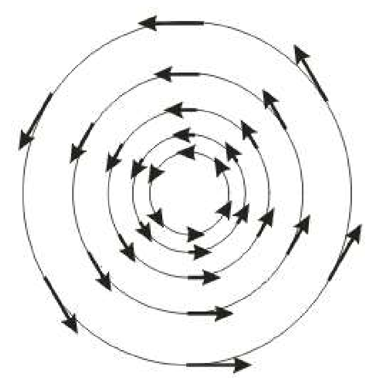
- Transformation of rotational betatron modes to flat uncoupled beam
  - Achieved by system of skew-quads making 90° difference in betatron phase advances for 2 planes (directed along quad planes)

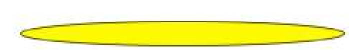




$$\begin{bmatrix} \sqrt{\beta} \\ -\dfrac{i/2+\alpha}{\sqrt{\beta}} \\ \sqrt{\beta_{1y}}\,e^{i\pi/2} \\ -\dfrac{i/2+\alpha}{\sqrt{\beta}}\,e^{i\pi/2} \end{bmatrix} \rightarrow \begin{bmatrix} \sqrt{\beta} \\ -\dfrac{i/2+\alpha}{\sqrt{\beta}} \\ \sqrt{\beta_{1y}} \\ -\dfrac{i/2+\alpha}{\sqrt{\beta}} \end{bmatrix}$$







Why Do We Need Derbenev's Adapter

- In the absence of magnetic field x and y norm. emittances are equal and are conserved in further beam motion/manipulations; $\varepsilon_{4n}=\varepsilon_{xn}\varepsilon_{yn}$
- Introduction of magnetic field enables controlled redistribution of mode emittances; $\varepsilon_{4n}=\varepsilon_{xn}\varepsilon_{yn}$

$$\varepsilon_{1n,2n} = \frac{\sqrt{\varepsilon_{4n}}}{\sqrt{1+\Phi_r^{\ 2}\beta_0^{\ 2}} \pm \Phi_r \beta_0}$$

  where $\beta_0 = a_e^{\ 2} / (\varepsilon_n / \beta\gamma)$ is the effective beta-function, a_e is the electron beam radius in the cooling section, β and γ are the relativistic factors, B_0 is the magnetic field in the cooling solenoid, and $\Phi_r = eB_0 / (2\gamma\beta m_e c^2)$ is its focusing strength.

  ⇨ independent control of the beam size and transverse angles

  ⇨ It enables to avoid large β-functions which makes beam optics more stable

- There was recently published a paper suggesting electron cooling for ep-collider without magnetic field in the cooling section
  - For the suggested parameters the interaction with proton beam space charge destroys the electron beam emittance at a fraction of cooling section length

Discussion on High Energy Electron Cooling

- Cooling at proton energies above ~20 GeV cannot be done as a classical electron cooling with electrostatic acceleration
- That leaves the following possibilities (or their combination):
  - Acceleration in the energy recovery SC linac
    - Bunching of electrons reduces current in comparison with DC beam
    - Small number of turns in a ring was also considered to additionally reduce linac current (problem with frequent injection & extraction)
  - Acceleration in energy recovery linac with beam storage in a ring for long time. Fast cooling of electrons to prevent IBS (Possibilities: SR cooling with wigglers, OSC)
  - Acceleration in induction linac with beam circulation in a ring for many turns
- Only last proposal was elaborated in some details
    - Still there is a not-answered question (chromaticity of Derbenev adapter)
- All choices are extremely challenging and require both theoretical and experimental studies

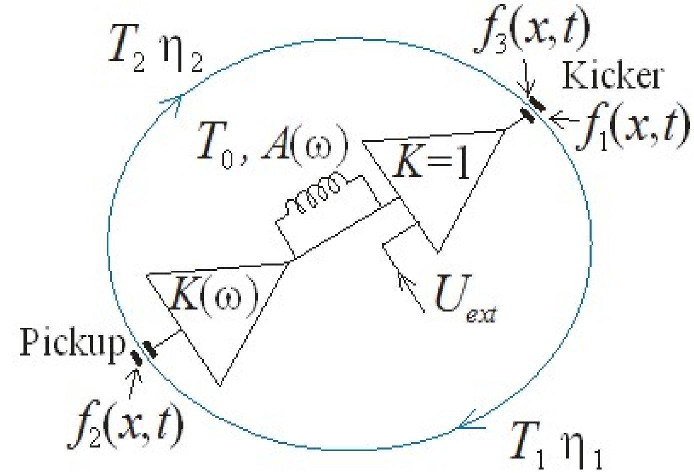
Stochastic Cooling

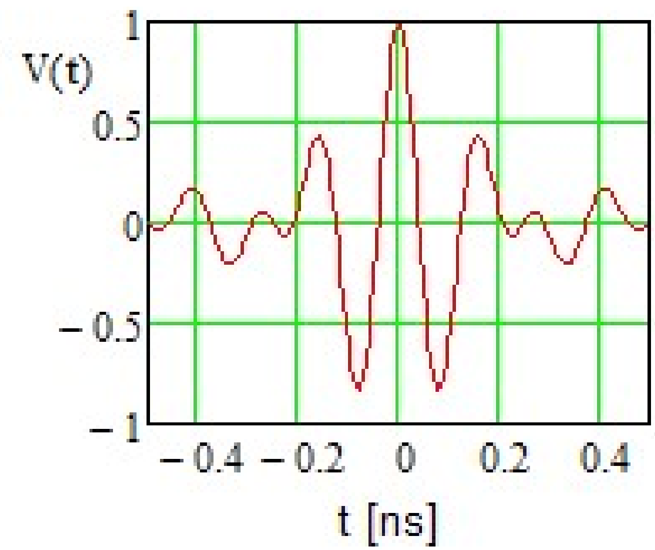
Methods of Longitudinal (Microwave) Stochastic Cooling

- Palmer cooling
  - Diff. pickup signal is proportional to particle momentum. It is measured by pickup at high dispersion location
  - Example: FNAL Accumulator
- Filter cooling
  - Signal proportional to particle momentum is obtained as difference of particle signals for two successive turns (notch filter)

$$U(t) = u(t) - u\left(t - T_0\left(1 + \eta\frac{\Delta p}{p}\right) + T_0\right) \approx \frac{du}{dt}T_0\eta\frac{\Delta p}{p}$$

  - Examples: FNAL Debuncher and Recycler
- Transient time cooling
  - No signal treatment
  - The same expression for kick as for FC
  - Larger diffusion => less effective than FC
  - Examples: OSC, CEC





particle in a system with constant gain in 4-8 GHz band

Before we start: Basics of Stochastic Cooling Theory

- SC theory is closely related to the plasma perturbation theory
  - Similar to the Vlasov equation with Landau collision term

$$\frac{\partial \psi}{\partial t} + \mathrm{v}_i \frac{\partial \psi}{\partial x_i} + \frac{\partial}{\partial p_i}\left(eE_i\psi\right) = \frac{1}{2}\frac{\partial}{\partial p_i}\left(F_i(x)\psi + D_{ij}(x)\frac{\partial \psi}{\partial p_j}\right)$$

    - No coherent motion (otherwise too large power)

      $\Rightarrow$ only $\partial \psi / \partial t$ is left in the left-hand side
  - Friction -> Cooling force
  - Diffusion due to collisions -> Diffusion due to particle interaction through cooling system
    - Diffusion coefficient is proportional to the spectral density of Schottky noise (slow process => non-resonant terms are negligible)
      - At betatron sidebands for $\perp$ cooling
      - At revolution harmonics for || cooling
  - Signal suppression due to particle interaction through cooling system
    - Reduces both Cooling Force and Diffusion
  - In most practical cases one can neglect cross-plane diffusion (D_{ij}=0, $i \neq j$)

Schottky Noise

Fourier transform: applicable if $f(t)\xrightarrow{t\to\pm\infty}0$

$$f_\omega=\frac{1}{2\pi}\int_{-\infty}^{\infty}f(t)e^{-i\omega t}dt \Leftrightarrow f(t)=\int_{-\infty}^{\infty}f_\omega e^{i\omega t}d\omega\ ,$$

$$\int_{-\infty}^{\infty}e^{i\omega t}d\omega=2\pi\delta(t) \quad => \quad \int_{-\infty}^{\infty}\left|f(t)\right|^2dt=\int_{-\infty}^{\infty}dt\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}d\omega d\omega' f_\omega f_{\omega'}{}^{*}e^{i(\omega-\omega')t}=2\pi\int_{-\infty}^{\infty}\left|f_\omega\right|^2d\omega$$

Spectral density of random noise: $f(t)$ is not zero at $t\to\pm\infty$ => divergence of FT

$$\overline{f_\omega f_{\omega'}{}^{*}}=P(\omega)\delta(\omega-\omega') \quad =>$$

$$K(\tau)\equiv\overline{f(t)f^{*}(t-\tau)}=\frac{1}{T}\int_{-T/2}^{T/2}f(t)f^{*}(t-\tau)dt=\frac{1}{T}\int_{-T/2}^{T/2}\left(\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}d\omega d\omega' f_\omega f_{\omega'}{}^{*}e^{i\omega(t-(t-\tau))}\right)dt$$

$$=\frac{1}{T}\int_{-T/2}^{T/2}dt\left(\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}d\omega d\omega' P(\omega)\delta(\omega-\omega')e^{i\omega\tau}\right)=\frac{1}{T}\int_{-T/2}^{T/2}\left(\int_{-\infty}^{\infty}e^{i\omega\tau}P(\omega)d\omega\right)dt=\int_{-\infty}^{\infty}P(\omega)e^{i\omega\tau}d\omega$$

Schottky noise of random pulses: $U(t)=\sum_n u(t-t_n)$

$$\overline{U^2}=\frac{1}{T}\int_{-T/2}^{T/2}dt\sum_{n,m}u(t-t_n)u^{*}(t-t_m)=\frac{1}{T}\int_{-T/2}^{T/2}dt\left(\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}d\omega d\omega'\sum_{n,m}u_\omega u_{\omega'}\overline{e^{i(\omega(t-t_n)-\omega'(t-t_n))}}\right)\xrightarrow{\overline{e^{i(\omega(t-t_n)-\omega'(t-t_m))}}=\delta_{nm}e^{i(\omega-\omega')(t-t_n)}}$$

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}u_\omega u_{\omega'}d\omega d\omega'\frac{1}{T}\int_{-T/2}^{T/2}dt\sum_n e^{i(\omega-\omega')(t-t_n)}=\dot{n}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}u_\omega u_{\omega'}d\omega d\omega'\int_{-\infty}^{\infty}dte^{i(\omega-\omega')t}=2\pi\dot{n}\int_{-\infty}^{\infty}\left|u_\omega\right|^2d\omega$$

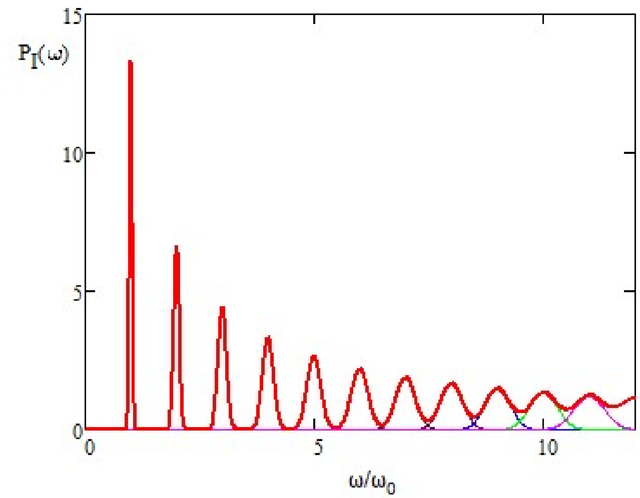
$\boxed{P(\omega)=2\pi\dot{n}\left|u_\omega\right|^2}$. For el. current $u(t)=e\delta(t)$ => $P_I(\omega)=\dfrac{e^2\dot{n}}{2\pi}=\dfrac{eI}{2\pi}$ => $\boxed{\overline{\Delta I^2}=2eI\Delta f}$

Schottky Noise in Circulating Beam

$$u(t) = e\delta(t) \Rightarrow u(t) = e\sum_{k=-\infty}^{\infty}\delta(t-kT) \quad \& \quad u_\omega = \frac{1}{T}e\sum_{k=-\infty}^{n\infty}\delta\left(\omega - \frac{2\pi}{T}k\right)$$

$$P_I(\omega) = \frac{eI}{2\pi} \Rightarrow \boxed{P(\omega) = \frac{eI}{2\pi}\sum_{n=-\infty}^{\infty}\frac{\psi_0\left(x(\omega/n)\right)}{\left|n\eta\left(x(\omega/n)\right)\right|}} \qquad \eta(x) = -\frac{1}{\omega}\frac{d\omega}{dx}, \quad x = \frac{\Delta p}{p}, \quad \int \psi_0(x)\,dx = 1$$

- This equation correctly describes noise even if Schottky bands overlap



- In vicinity of *n*-th harmonic for constant η one obtains:

$$\boxed{P_I(\Delta\omega_n) \equiv P_I(\omega - n\omega_0) = \frac{eI}{2\pi}\frac{1}{|k\eta|}\psi_0\left(\frac{\Delta\omega_n}{n\omega_0\eta}\right)}$$

- Compute integral around one harmonic: (accounting negative frequency will double it)

$$\overline{\Delta I_n^{\,2}} = \overline{\Delta I_{-n}^{\,2}} = \int P_I(\Delta\omega_n)\,d\Delta\omega_n = \int \frac{eI}{2\pi}\frac{1}{|n\eta|}\psi_0\left(\frac{\Delta\omega_n}{n\omega_0\eta}\right)d\Delta\omega_n = \frac{eI\omega_0}{2\pi}$$ compare to $\overline{\Delta I^2} = 2eI\Delta f$

- *i.e.* integral around each harmonic does not depend on *n* and the relative current fluctuations at n-th harmonic are: $\sqrt{\frac{\overline{\Delta I_n^{\,2}}}{I_0^{\,2}}} = \sqrt{\frac{e\omega_0}{2\pi I}} = \frac{1}{\sqrt{N}}$

- Schottky bands overlap when: $\omega_0 \approx n_{th}\omega_0\eta\frac{\Delta p}{p} \Rightarrow n_{th} \approx \frac{1}{\eta\,\Delta p/p}$

Effect of Random Noise on Oscillator

Noise spectral density and correlation function

$$\overline{f(t)f(t+\tau)} \equiv K(\tau) = \int_{-\infty}^{\infty} P(\omega)e^{i\omega\tau}d\omega\,, \quad K(\tau) = K(-\tau) \quad \Rightarrow \quad P(\omega) = P(-\omega) \geq 0$$

Growth of particle amplitude due to noise

- Equation of motion $\boxed{\ddot{x} + \omega_0{}^2 x = f(t)}$
- Solution: $x(t) = \frac{1}{\omega_0}\int_0^t f(t')\sin\left(\omega_0(t-t')\right)dt'$
- Growth of RMS amplitude with time

$$\overline{x(t)^2} = \frac{1}{\omega_0{}^2}\int_0^t dt' \int_0^t dt'' \overline{f(t')f(t'')}\sin\left(\omega_0(t-t')\right)\sin\left(\omega_0(t-t'')\right) \Rightarrow \quad \boxed{\overline{x(t)^2} = \frac{2\pi t}{\omega_0{}^2}P(\omega_0)}$$

Growth of particle amplitude due to kicker noise in a ring

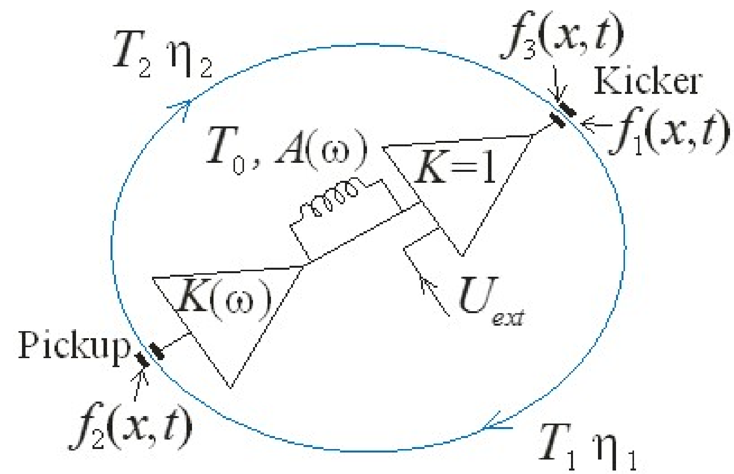
- Only resonance harmonics contributes to $d\varepsilon/dt$
  - 1/2 in $d\varepsilon_\perp/dt$ due to oscillatory motion

$$\boxed{\begin{aligned} &\frac{d}{dt}\overline{\Delta E^2} = \frac{\omega_0{}^2}{2\pi}\sum_{n=-\infty}^{\infty} P_E(\omega_0 n)\,, && \overline{\delta E^2} = \int_{-\infty}^{\infty} P_E(\omega)d\omega \\ &\frac{d\varepsilon_\perp}{dt} \equiv \frac{1}{\beta}\frac{d}{dt}\overline{x(t)^2} = \frac{\beta\omega_0{}^2}{4\pi}\sum_{n=-\infty}^{\infty} P_\theta(\omega_0(\nu+n))\,, && \overline{\theta^2} = \int_{-\infty}^{\infty} P_\theta(\omega)d\omega \end{aligned}}$$

Signal Suppression in Longitudinal Cooling

- Denote: $x \equiv \Delta p / p$
- No particle interaction

  => evolution of particle distribution:



$$\begin{cases} \psi f_2(x,t) = \psi_1(x, t - T_1(x)) \\ \psi_3(x,t) = \psi_2(x, t - T_2(x)) \\ \psi_1(x,t) = \psi_3(x - \delta p(t)/p_0,\ t) \end{cases}$$

where $\begin{cases} T_1(x) = T_{10} + T_0 \eta_1 x + ... \\ T_2(x) = T_{20} + T_0 \eta_2 x + ... \\ T(x) = T_0(1 + \eta x + ...) \end{cases}$

$\eta = \alpha - 1/\gamma^2$ is the slip-factor, and we call η_1 and η_2 the partial slip-factors

- Expending the last Eq. => $\tilde{\psi}_1(x,t) = \tilde{\psi}_3(x,t) - \dfrac{\delta p(t)}{p_0} \dfrac{d\psi_0(x)}{dx}$

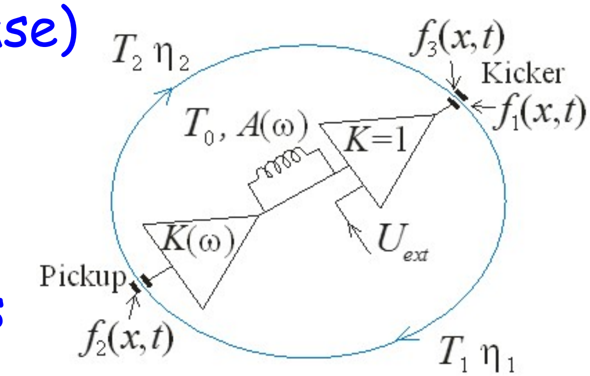
  and performing Fourier transform

$$\begin{cases} \tilde{\psi}_{2\omega}(x) = \tilde{\psi}_{1\omega}(x) \exp(-i\omega T_1(x)) \\ \tilde{\psi}_{3\omega}(x) = \tilde{\psi}_{2\omega}(x) \exp(-i\omega T_2(x)) \\ \tilde{\psi}_{1\omega}(x) = \tilde{\psi}_{3\omega}(x) - (df_0(x)/dx)(\delta p_\omega / p_0) \end{cases}$$

$$\Rightarrow \tilde{\psi}_{2\omega}(x) e^{i\omega T_1(x)} = \tilde{\psi}_{2\omega}(x) e^{-i\omega T_2(x)} - \frac{d\psi_0(x)}{dx} \frac{\delta p_\omega}{p_0}$$

Signal Suppression in Longitudinal Cooling (2)

- Introduce Longitudinal Cooling Gain (most general case)



$$\frac{\delta p_\omega}{p_0} = \frac{\Delta p_{ext\omega}}{p_0} + \int e^{-i\omega T_{20}}\left[1 - A(\omega)e^{-i\omega T_0}\right] G(x,\omega)\tilde{\psi}_{2\omega}(x)dx$$

A(ω)=0 – Palmer cooling, A(ω)=1 – filter cooling

- On other hand, pickup signal at frequency ω depends on hor. particle coordinate ($X = D\Delta p/p \equiv Dx$)

$$U_{pickup\,\omega} = I_0 \int Z_p(x,\omega)\psi_{2\omega}(x,\omega)dx \xrightarrow[Z_{ampl}=50\Omega]{\delta E_{kicker\,\omega}=eU_{kicker\,\omega}\left(Z_k(\omega)/Z_{ampl}\right)} \boxed{G(x,\omega) = \frac{eI_0 Z_p(x,\omega)Z_k(\omega)}{\gamma\beta^2 mc^2 Z_{ampl}} K(\omega)}$$

- Combining we obtain Eq. for $\tilde{f}_{2\omega}(x)$

$$\tilde{\psi}_{2\omega}(x)\left[e^{i\omega T_1(x)} - e^{-i\omega T_2(x)}\right] + \frac{d\psi_0(x)}{dx}\left[\frac{\Delta p_{ext\omega}}{p_0} + e^{-i\omega T_{20}}\left[1 - A(\omega)e^{-i\omega T_0}\right]\int dx'\tilde{\psi}_{2\omega}(x')G(x',\omega)\right] = 0$$

- Solving we obtain pickup signal excited by external perturbation

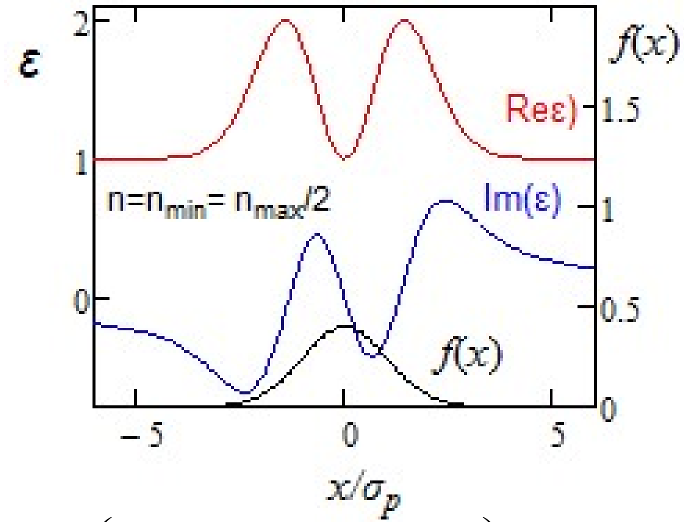
$$\boxed{S_\omega \equiv \int dx'\tilde{\psi}_{2\omega}(x')G(x',\omega) = -\frac{1}{\varepsilon(\omega)}\frac{\Delta p_{ext\omega}}{p_0}\int_{\delta\to 0_+} \frac{d\psi_0(x)}{dx}\frac{G(x',\omega)e^{i\omega T_2(x)}}{e^{i\omega T(x)} - (1-\delta)}dx}$$

$$\varepsilon(\omega) = 1 + \left(1 - A(\omega)e^{-i\omega T_0}\right)\int_{\delta\to 0_+} \frac{d\psi_0(x)}{dx}\frac{G(x,\omega)e^{i\omega(T_2(x)-T_2(0))}}{e^{i\omega T(x)} - (1-\delta)}dx$$

$$\boxed{\xrightarrow[\text{overlap}]{\substack{\text{Far away}\\ \text{from band}}} \varepsilon(\omega) = 1 + \left(1 - A(n\omega_0)e^{2\pi i n\eta y}\right)\frac{1}{2\pi i n\eta}\int_{\delta\to 0_+}\frac{d\psi_0(x)}{dx}\frac{G(x,n\omega_0)}{x - y - i\delta\,\mathrm{sign}(n\eta)}dx, \quad \omega = n\omega_0(1-\eta y)}$$

Discussion: Signal Suppression in Long. Cooling

- For cooling of fixed number of particles, signal suppression is negligible (*i.e.* $\varepsilon \approx 1$) at the beginning and becomes important with cooling
  - Simplified formula can be used
- For particle accumulation the signal suppression is negligible at the process beginning and becomes important at full intensity if system operates near or at band overlap
  - Exact formula has to be used



- Palmer cooling: $G(x,\omega) = -G_n' x$ - near n^{th} harmonic

$$\varepsilon(y) = 1 - \frac{G_n'}{2\pi i n\eta} \int_{\delta\to 0_+} \frac{d\psi_0(x)}{dx} \frac{x\,dx}{x - y - i\delta\,\mathrm{sign}(n\eta)} = 1 - \frac{G_n'}{2\pi i n\eta} \int_{\delta\to 0_+} \frac{d\psi_0(x)}{dx}\left(1 + \frac{y}{x - y - i\delta\,\mathrm{sign}(n\eta)}\right) dx$$

$$\boxed{\varepsilon(y) = 1 + i\frac{G_n'}{2\pi n\eta} y \int_{\delta\to 0_+} \frac{d\psi_0(x)}{dx} \frac{dx}{x - y - i\delta\,\mathrm{sign}(n\eta)}}$$ where $y = \Delta\omega_n / (\eta n \omega_0)$

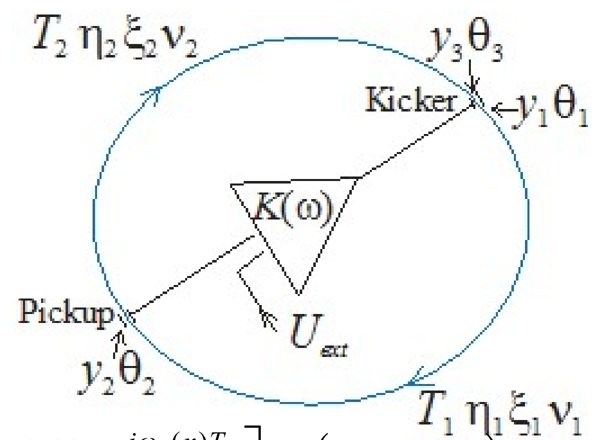
- Filter cooling: $G(x,\omega) = -iG(n\omega_0) = -iG_n$ - near n-*th* harmonic

$$\left(1 - A(n\omega_0)e^{2\pi i n\eta y}\right) \xrightarrow[2\pi i n\eta y \ll 1]{A(n\omega_0)=1} -2\pi i n\eta\, y \;\; \Rightarrow \;\; \boxed{\varepsilon(y) = 1 + iG_n y \int_{\delta\to 0_+} \frac{d\psi_0(x)}{dx} \frac{dx}{x - y - i\delta\,\mathrm{sign}(n\eta)}}$$

- For constant gain: suppression is decreased with harmonic number as ~1/n for Palmer cooling, and stays the same in Filter cooling

Theory of Longitudinal Stochastic Cooling

Cooling force



- The gain was introduced as

$$\frac{\delta p_\omega}{p_0} = \frac{\Delta p_{ext\omega}}{p_0} + \int e^{-i\omega T_2(0)}\left[1 - A(\omega)e^{-i\omega T_0}\right] G(x,\omega)\tilde{\psi}_{2\omega}(x)dx$$

For single particle: $\psi(x,t) = \delta(x - x_0)\dfrac{e}{T(x)}\displaystyle\sum_{n=-\infty}^{\infty}\delta(t - t_0 - nT(x))$

summing for all harmonics yields: $\dfrac{\delta p}{p_0} = \displaystyle\sum_n e^{i\omega_n(x)\left(T_2(x)-T_{20}\right)}\left[1 - A(\omega_n(x))e^{-i\omega_n(x)T_0}\right] G\left(x,\omega_n(x)\right)$

⇨ Cooling force $F(x) \equiv \dfrac{dx}{dt} = \dfrac{1}{T_0}\displaystyle\sum_{n=-\infty}^{\infty}\frac{G_1\left(x,\omega_n(x)\right)}{\varepsilon\left(\omega_n(x)\right)}\left(1 - A\left(\omega_n(x)\right)e^{-i\omega_n(x)T_0}\right)e^{i\omega_n(x)\left(T_2(x)-T_{20}\right)}$

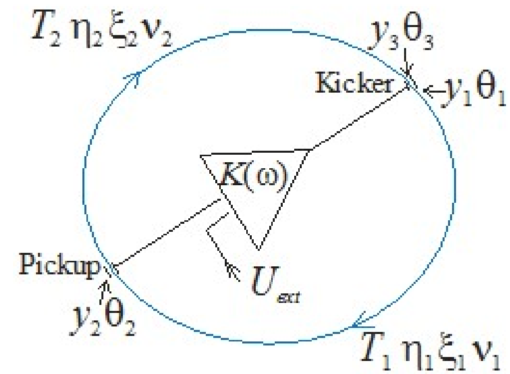
where we additionally accounted for signal suppression

Diffusion

- To obtain diffusion one needs
  - find noise spectral density at the pickup
  - Multiply by the transfer function (responses of pickup and kicker, & amplifier gain)
  - account for signal suppression
  - find effect of kicker noise on particle motion

Equations Describing Longitudinal Stochastic Cooling

- Finally we can write



$$\frac{\partial \psi}{\partial t}+\frac{\partial}{\partial x}\left(F(x)\psi\right)=\frac{1}{2}\frac{\partial}{\partial x}\left(D(x)\frac{\partial \psi}{\partial x}\right)$$

$\psi(x)$ is the distribution function, $x \equiv \Delta p / p$

$$F(x)\equiv\frac{dx}{dt}=\frac{1}{T_0}\sum_{n=-\infty}^{\infty}\frac{G\left(x,\omega_n\right)}{\varepsilon\left(\omega_n\right)}\left(1-A\left(\omega_n\right)e^{-i\omega_n T_0}\right)e^{i\omega_n T_0 \eta_2 x}, \qquad \omega_n = n\omega_0\left(1-\eta x\right)$$

$$D(x)=\frac{N}{T_0}\sum_{n=-\infty}^{\infty}\left|G(x,\omega_n)\right|^2\left|\left(1-A\left(\omega_n\right)e^{-i\omega_n T_0}\right)\right|^2\begin{cases}\dfrac{\psi\left(x\right)}{\left|n\eta\right|\left|\varepsilon\left(\omega_n\right)\right|^2}, & \text{no band overlap}\\ 1, & \text{complete band overlap}\end{cases}$$

$$\varepsilon(\omega)=1+\left(1-A(\omega)e^{-i\omega T_0}\right)N\int_{\delta\to 0_+}\frac{d\psi(x)}{dx}\frac{G\left(x,\omega\right)e^{i\omega T_0\eta_2 x}}{e^{i\omega T_0(1+\eta x)}-(1-\delta)}dx$$

- Amplifier noise is not accounted (insignificant in most of real systems)
- Note: the theory is built on the same principle as plasma theory – which is a perturbation theory (large number of particles in the Debye sphere versus large number of particles in the sample)

Equations Describing Transverse Stochastic Cooling

- Fokker-Planck equation in the action-phase variables describes transverse cooling in the case of linear transverse motion

$$\frac{\partial \psi}{\partial t}+\lambda_\perp(x)\frac{\partial}{\partial I}\left(I\psi\right)=D_\perp(x)\frac{\partial}{\partial I}\left(I\frac{\partial \psi}{\partial I}\right)\xrightarrow{\times I\ \&\ \text{Integrating}}\frac{\partial \overline{I(x,t)}}{\partial t}-\lambda_\perp(x)\overline{I(x,t)}=D_\perp(x)$$

$\lambda(x)$ and $D(x)$ do not depend on I

$\psi(x,I)$ is the distribution function, $\psi_\parallel(x)=\int \psi(x,I)dI$

$$\lambda_\perp(x)=\frac{1}{T_0}\sum_{n=-\infty}^{\infty}\mathrm{Re}\left(\frac{G_\perp\left(\omega_{n\perp}(x)\right)}{i\varepsilon_\perp\left(\omega_{n\perp}(x)\right)}e^{i\omega_n\left(T_2(x)-T_{20}\right)-2\pi i\nu_2(x)}\right),$$

$$\omega_{n\perp}(x)=\frac{2\pi n}{T(x)}-\nu(x)\approx\omega_0\left(n(1-\eta x)-(\nu+\xi x)\right) \quad .$$

$$D_\perp(x)=\sum_{n=-\infty}^{\infty}\frac{1}{\left|\varepsilon_\perp(\omega_{n\perp}(x))\right|^2}\left(\frac{\pi\beta_k}{2T_0^{\,2}}\left(\frac{e\left|Z_{k\perp}\left(\omega_{n\perp}(x)\right)\right|}{mc^2\beta^2\gamma Z_{ampl}}\right)^2 P_{\perp U}\left(\omega_{n\perp}(x)\right)+\left|G_\perp(\omega_{n\perp}(x))\right|^2\frac{\overline{I(x)}N\psi_\parallel(x)}{2T_0\left|\nu'(x)+\eta(x)n\right|}\right)$$

$$\varepsilon_\perp(\omega)=1-\frac{G_\perp(\omega)N}{2}\int\limits_{\delta\to 0_+}\frac{\left[e^{-i\omega T(x)}\sin\left(2\pi\nu_2(x)\right)+\sin\left(2\pi\nu_1(x)\right)\right]e^{i\omega\left(T_2(x)-T_{20}\right)}}{\cos\left(\omega T(x)\right)-\cos\left(2\pi\nu(x)\right)+i\delta\sin\left(\omega T(x)\right)}\psi_\parallel(x)dx$$

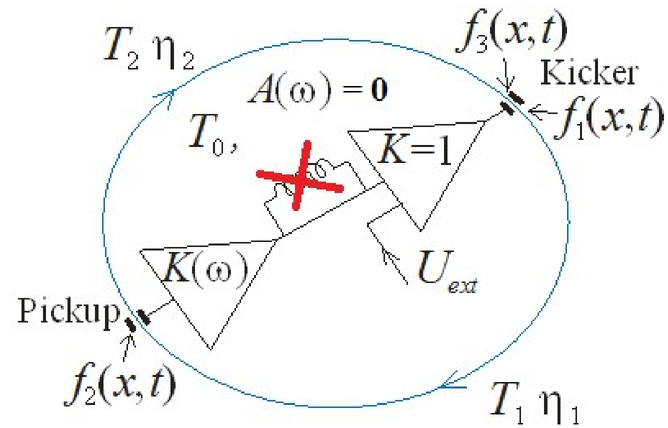
- Amplifier noise is referenced to the pickup output
  - Negligible in most of real systems

Cooling Force and Cooling Range for Palmer Cooling

- Palmer cooling: $G(x,\omega) = -xG'(\omega); \quad A(\omega) = 0$

$$F(x) = -\frac{1}{T_0}\sum_{n=-\infty}^{\infty} G'(n\omega_0)\, e^{i\omega_n T_0 \eta_2 x}$$

- P-K partial slip factor: $\Delta t = T_0\eta_2 x \equiv T_0\eta_2 \dfrac{\Delta p}{p}$



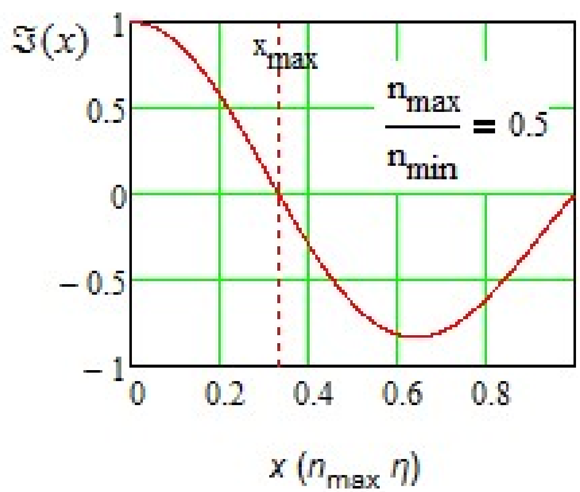
- For a rectangular band with perfect phasing ($G'(\omega) = G',\ \omega \in [\omega_{\min}, \omega_{\max}]; \quad \mathrm{Im}(G') = 0$)

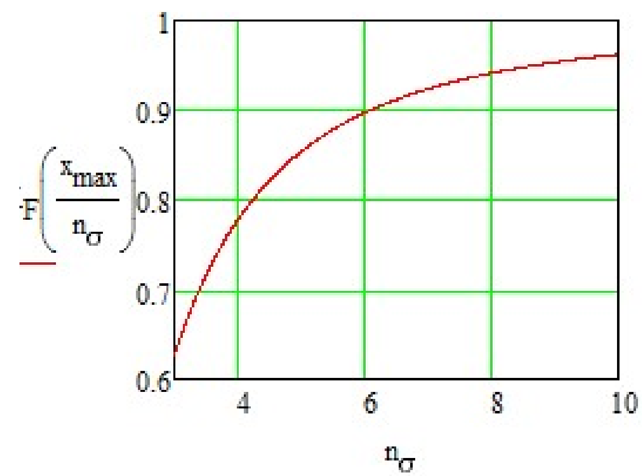
$$F(x) = -\frac{2G'x}{T_0}(n_{\max} - n_{\min})\,\Im(x)$$

  - The cooling range: $x_{\max} = \dfrac{1}{2(n_{\max}+n_{\min})\eta_2}$

$$\Im(x) = \frac{\sin(2\pi n_{\max}\eta_2 x) - \sin(2\pi n_{\min}\eta_2 x)}{2\pi(n_{\max}-n_{\min})\eta_2 x}$$

- Cooling range ≡ “Bad mixing”?
  - Good lifetime requires the cooling range >4σ
- η_2 can be controlled by machine optics and cooling range can be ∞





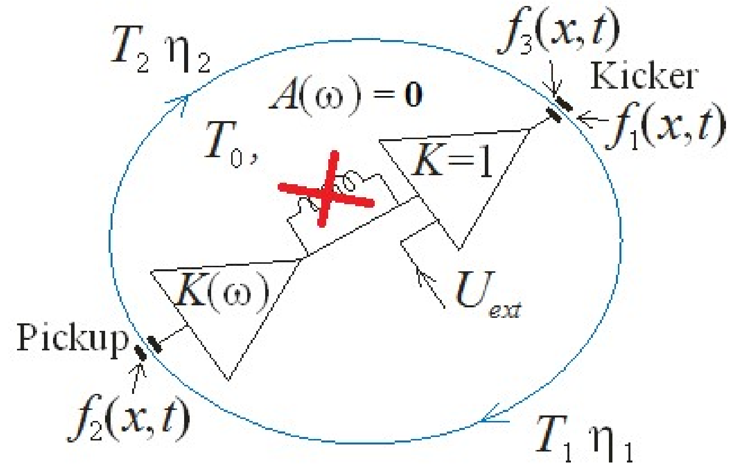
Form-factor of cooling rate reduction ($K_F(\sigma)$) for Gaussian distribution as function of $n_\sigma = x_{max}/\sigma$

Cooling Force and Range for Transient Time Cooling

- Transient time cooling is the only method which can work at optical frequencies
  - FC requires notch filter
- Transient time cooling:

$$G(x,\omega_n) = -iG_F(\omega); \quad A(\omega) = 0$$

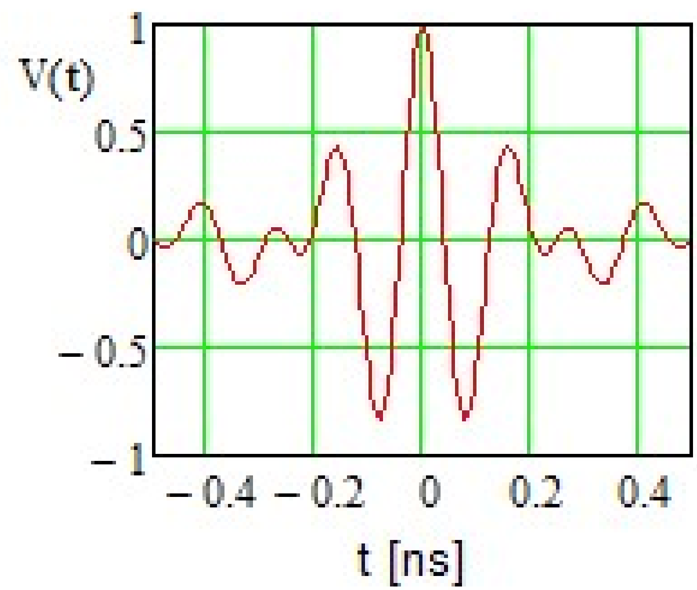
$$F(x) = \frac{1}{T_0}\sum_{n=-\infty}^{\infty} iG_F(n\omega_0)e^{2\pi i n\eta_2 x}$$



- For a rectangular band

$(G_F(\omega) = G_0, \omega \in [\omega_{\min}, \omega_{\max}])$

$$F(x) = \frac{2G_0}{T_0}\frac{\sin(\pi\eta_2(n_{\max}-n_{\min})x)}{\pi\eta_2 x}\sin(\pi\eta_2(n_{\max}+n_{\min})x)$$



  - The cooling range: $x_{\max} = \dfrac{1}{\eta_2(n_{\max}+n_{\min})}$
    - Does not depend on η
    - And is determined by η_2

Optimal Gain and Maximum Damping Rate

- Optimum depends on particle distribution, technical and other limitations

$$\int x^2 dx \left(\frac{\partial \psi}{\partial t} + \frac{\partial}{\partial x}\left(F(x)\psi\right) = \frac{1}{2}\frac{\partial}{\partial x}\left(D(x)\frac{\partial \psi}{\partial x}\right)\right) \quad \rightarrow \quad \frac{d\overline{x^2}}{dt} = -2\frac{G}{G_{ref}}\overline{F} + \left(\frac{G}{G_{ref}}\right)^2 \overline{D}$$

where $\overline{F} = -\int xF(x)\psi(x)dx \,, \quad \overline{D} = \int \psi(x)\frac{d}{dx}\left(xD(x)\right)dx$

Differentiating over G yields optimal gain => optimal damping

$$\left.\frac{d\overline{x^2}}{dt}\right|_{\max} = \frac{\overline{F}^2}{\overline{D}} \quad \Rightarrow \quad \left.\frac{d\overline{x^2}}{dt}\right|_{\max} = \frac{\left(\int dx\, x\psi(x)\frac{1}{T_0}\sum_{n=-\infty}^{\infty}\frac{G(x,\omega_n)}{\varepsilon(\omega_n)}\left(1-A(\omega_n)e^{-i\omega_n T_0}\right)e^{i\omega_n T_2 \eta_2 x}\right)^2}{\int dx\,\psi(x)\frac{d}{dx}\left(\frac{N}{T_0}\sum_{n=-\infty}^{\infty}\left|G(x,\omega_n)\right|^2\left|\left(1-A(\omega_n)e^{-i\omega_n T_0}\right)\right|^2\frac{\psi(x)}{|n\eta|\left|\varepsilon(\omega_n)\right|^2}\right)}$$

- ♦ Signal suppression (1/ε) affects both diffusion and cooling force and can be neglected

- For major fraction of particles

$$\left.\frac{d\overline{x^2}}{dt}\right|_{\max} \propto \frac{\left(\sum_{n=0}^{\infty}\mathrm{Re}\left(G(\omega_n)\right)\right)^2}{\sum_{n=0}^{\infty}\left|G(\omega_n)/n\right|}$$

Replacing summation by integration we introduce the effective bandwidth

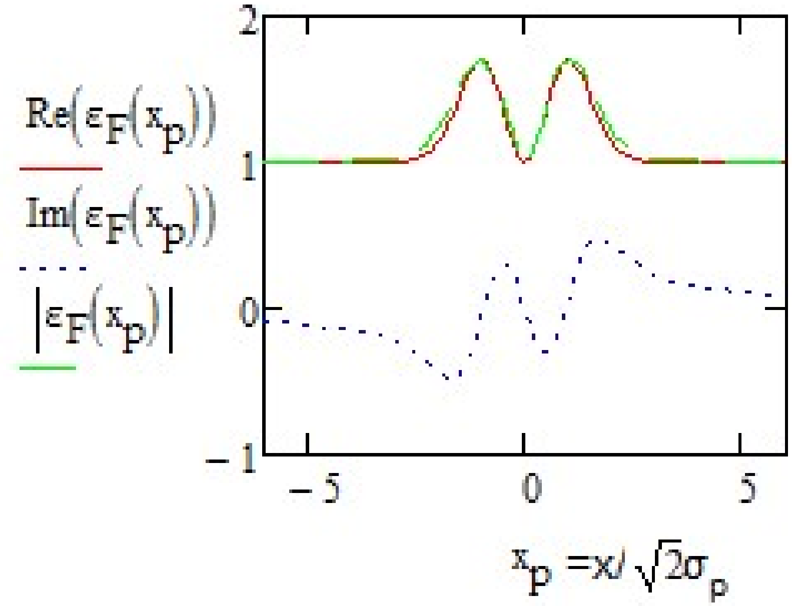
$$\boxed{W_{eff} = \sqrt{\frac{\left(\int_0^{\infty}\mathrm{Re}\left(G(f)\right)df\right)^2}{\int_0^{\infty}\left|G(f)\right|^2 df/f}}}$$

Optimal Damping Rate for Transient Cooling

- Small amp. oscillations, Gaussian distribution, continuous beam & Rectangular band
  - Cooling range: $x_{\max} \approx \dfrac{1}{\eta_2 \left(n_{\max} + n_{\min} \right)}$
  - Diffusion is much larger than for filter cooling
    - Noncompetitive to the filter cooling in the case of non-overlapped bands
      - at optimum: $$\lambda_{TTC} \approx \lambda_{FC} \left(\frac{\eta_2}{\eta} \right)^2$$
    - For completely overlapped bands
      - Diffusion does not depend on momentum deviation and momentum spread: $D = 2NG_0{}^2 W$
      - signal suppression is negligible

$$\boxed{\lambda_{opt} \approx \frac{2\pi^2 W}{N n_\sigma{}^2}} \qquad W = \frac{n_{\max} - n_{\min}}{T_0}, \quad n_\sigma = \frac{x_{\max}}{\sigma_p}.$$

Back to Signal Suppression



Filter Cooling

- For $G_n \approx const\ , n \in [n_{\min}, n_{\max}]$ the effective gain is proportional to n
  $\Rightarrow$ signal suppression does not depend on n
- Cooling force: $F_n \approx \mathrm{Re}\left(F_{n0}(x) / \varepsilon_n(\mathrm{x})\right)$
- Diffusion: $D_n \approx D_{n0}(x) / \left|\varepsilon_n(\mathrm{x})\right|^2$
- Damping rate:

$$\left.\frac{d\overline{x^2}}{dt}\right|_{\max} = \frac{\left\langle \sum_{n=-\infty}^{\infty} \frac{G(n\omega_0)}{\varepsilon_n(x)} \left(1-e^{-2\pi i n \eta_2 x}\right) e^{2\pi i n \eta_2 x} \right\rangle_x^2}{\left\langle x\psi(x) \sum_{n=-\infty}^{\infty} \left|\frac{G(n\omega_0)}{\varepsilon_n(x)}\right|^2 \frac{\left|1-e^{-2\pi i n \eta_2 x}\right|^2}{|n|} \right\rangle_x}$$

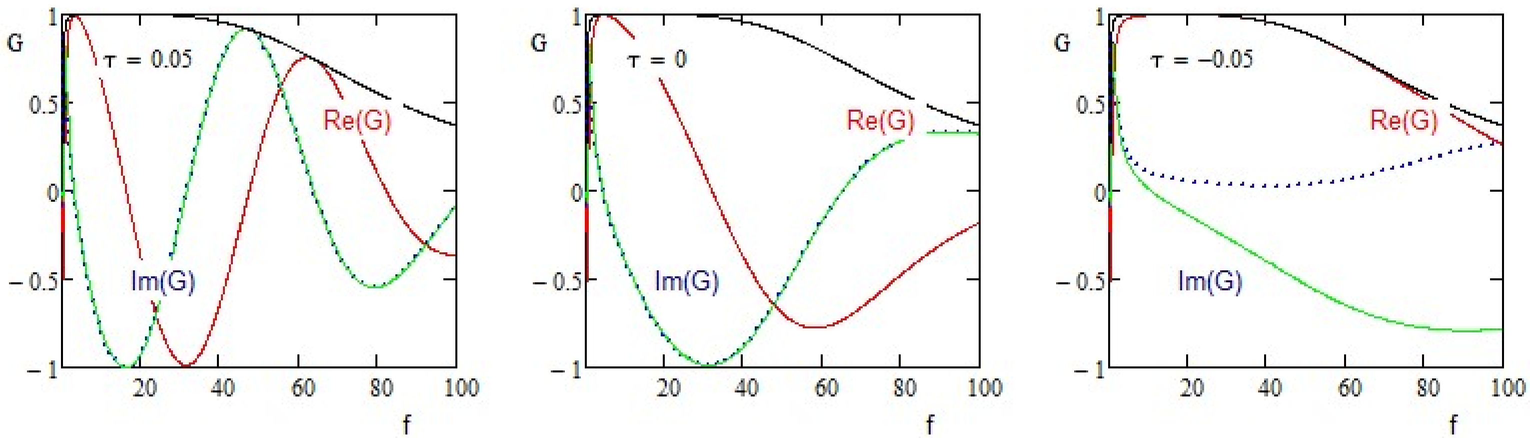
  - Numerical computation with signal suppression at optimal gain and for rectangular band yields only a few % correction

Palmer cooling and Transient time cooling

- Qualitatively - similar picture
- Signal suppression is reduced when bands start to overlap
  - negligible for completely overlapped bands

Causality in Stochastic Cooling

- Causality binds the real and imaginary parts of system response
- The same as for the medium permeability, the Kramers-Kronig relations bind the real and imaginary parts of the gain for an amplifier
- It is true for any system where causality works
  - But there are no causality limitations in a stochastic cooling system
    - Changing delay in the cable we can deliver signal earlier than particle will arrive



Real and imaginary parts of system response for LPF*HPF*Delay (4^{th} order Bessel filters)

Negative delay makes a flat phase response but breaks Kramers-Kronig relationship

$$G''(\omega) = \frac{1}{\pi} \int_{-\infty}^{\infty} \frac{G'(x)}{x-\omega} dx$$

Typical Stochastic Cooling Block Diagram