Hadron momentum spectra from analytical solutions of relativistic hydrodynamics

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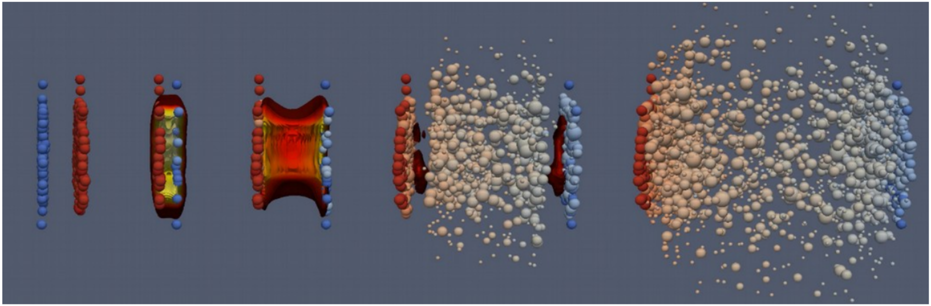
Introduction

- Hydrodynamics: successfully applied to heavy-ion collisions.
- At high collision energies, particle spectra can be described by assuming emission from Bjorken expanding cylindrical firebal.
- At lower collision energies, Bjorken symmetry is broken.
- Hydrodynamics inspired models used to study spectra and yield.
- Analytical solutions of relativistic hydrodynamics in simple symmetric cases useful.
- Serve as benchmarks for testing more realistic hydrodynamic simulation codes.
- **Analytical expressions of spectra for two useful geometries.**
- Direct application to analysis of heavy-ion data.

Boost invariance in heavy-ion collisions

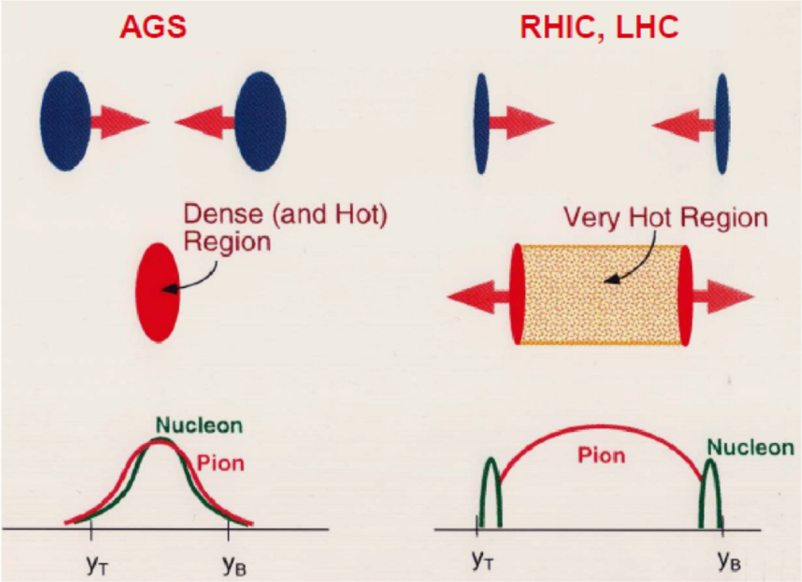
- At very high energy collisions, boost invariance is good symmetry.



[Ref: MADAI collaboration, Hannah Petersen and Jonah Bernhard]

- Signature of boost invariance symmetry is reflected in flat rapidity spectrum of hadrons in collisions above $\sqrt{s_{NN}} = 200$ GeV.
- Boost invariant blast wave model is applicable in collisions at high energy. Cylindrical symmetry.

Boost invariance breaking in lower collision energy



S. Nagamiya, Entropy 24 (2022) 482.

Relativistic hydrodynamics

- Assuming a system with no conserved charges, the hydrodynamical equations are

$$\partial_\mu T^{\mu\nu}(x) = 0.$$

- For non-dissipative evolution of the system where the energy-momentum tensor takes the form

$$T^{\mu\nu} = (\varepsilon + p)u^\mu u^\nu - p g^{\mu\nu}.$$

- The local fluid four-velocity is

$$u^\mu(x) = \gamma(x)(1, \vec{v}(x)), \qquad \gamma(x) = 1/\sqrt{1 - \vec{v}^2(x)}.$$

- Using thermodynamic expressions $Ts = \varepsilon + p$ and $dp = s\, dT$,

$$u^\mu \partial_\mu (T u^\nu) - \partial^\nu T = 0.$$

- For non-dissipative fluids, the entropy conservation implies

$$\frac{\partial}{\partial t}(s\gamma) + \vec{\nabla} \cdot (s\gamma\vec{v}) = 0.$$

Spherical solution

- It is convenient to work in the (t, r, θ, ϕ) co-ordinates.
- The fluid velocity vector is given by $\vec{v} = (v_r, 0, 0)$.
- The hydrodynamic equations become

$$\frac{\partial}{\partial t}(T\gamma v_r) + \frac{\partial}{\partial r}(T\gamma) = 0,$$
$$\frac{\partial}{\partial t}(s\gamma) + \frac{1}{r^2}\frac{\partial}{\partial r}(r^2 s\gamma v_r) = 0.$$

- We assume an equation of state of the form $p = c_s^2\,\varepsilon$, with the speed of sound c_s kept constant. Therefore $s \propto T^{1/c_s^2}$.
- The solution is obtained as

$$T(r,t) = T_0\left[\frac{t_0^2 - r_0^2}{t^2 - r^2}\right]^{3c_s^2/2},$$

$$s(r,t) = s_0\left[\frac{t_0^2 - r_0^2}{t^2 - r^2}\right]^{3/2}, \qquad v_r = \frac{r}{t}.$$

Cylindrical solution (boost invariant)

- It is convenient to work in the (t, ρ, ϕ, z) coordinates.
- The fluid velocity vector is given by $\vec{v} = (v_\rho, 0, v_z)$.
- The hydrodynamic equations become

$$
\begin{aligned}
\frac{\partial}{\partial t}(T\gamma v_\rho) + \frac{\partial}{\partial \rho}(T\gamma) - v_z \left[\frac{\partial}{\partial \rho}(T\gamma v_z) - \frac{\partial}{\partial z}(T\gamma v_\rho)\right] &= 0, \\
\frac{\partial}{\partial t}(T\gamma v_z) + \frac{\partial}{\partial z}(T\gamma) - v_\rho \left[\frac{\partial}{\partial z}(T\gamma v_\rho) - \frac{\partial}{\partial \rho}(T\gamma v_z)\right] &= 0, \\
\frac{\partial}{\partial t}(s\gamma) + \frac{1}{\rho}\frac{\partial}{\partial \rho}(\rho s\gamma v_\rho) + \frac{\partial}{\partial z}(s\gamma v_z) &= 0.
\end{aligned}
$$

- The solution is obtained as

$$
T(\rho, z, t) = T_0 \left[\frac{t_0^2 - \rho_0^2 - z_0^2}{t^2 - \rho^2 - z^2}\right]^{3c_s^2/2}, \qquad v_\rho(\rho, z, t) = \frac{\rho}{t},
$$

$$
s(\rho, z, t) = s_0 \left[\frac{t_0^2 - \rho_0^2 - z_0^2}{t^2 - \rho^2 - z^2}\right]^{3/2}, \qquad v_z(\rho, z, t) = \frac{z}{t}.
$$

Freeze-out and momentum spectra

- The hadron momentum spectra can be derived by applying the Cooper-Frye prescription

$$E\frac{dN}{d^3p} = \frac{g}{(2\pi)^3}\int p_\mu d\Sigma^\mu f(x,p).$$

- Semi-classical generalization of Maxwell-Jüttner distribution

$$f_{\rm eq}(x,p) = \frac{1}{\exp[\beta(u\cdot p)]+\epsilon} = \sum_{n=1}^{\infty}\epsilon_n \exp[-n\beta(u\cdot p)],$$

  where $\epsilon_n \equiv (-\epsilon)^{n-1}$ with $\epsilon_1 = 1$ for classical MB statistics.

- $(3+1)$-dimensional proper time $\tau_3 = \sqrt{t^2 - r^2} = \sqrt{t^2-\rho^2-z^2}$.
- Analytical solution for temperature in both cases

$$T(\tau_3) = T_0\left(\frac{\tau_{30}}{\tau_3}\right)^{3c_s^2}.$$

- $\tau_3 = \text{const.} \implies$ constant temperature freeze-out hypersurface.

Spectra: spherical geometry

- For the spherical case,

$$u \cdot p = \gamma(\zeta)\, E - \gamma(\zeta)\, v_r(\zeta)\, p \cos\theta,$$

$$p \cdot d\Sigma = \left(E\, \frac{dr}{d\zeta} - p\cos\theta\, \frac{dt}{d\zeta} \right) r^2(\zeta) \sin\theta\, d\zeta\, d\theta\, d\phi.$$

- Here $0 < \zeta < 1$ such that $r(0) = 0$ and $r(1) = R$.
- The particle spectra is given by

$$E\frac{dN}{d^3p} = \frac{gR^3}{2\pi^2} \sum_{n=1}^{\infty} \epsilon_n \int_0^1 e^{-n\beta E\sqrt{\nu^2+\chi^2}/\nu} \left[E\, \frac{\sinh(na\chi)}{na\chi} + T\nu \left(\frac{\sinh(na\chi) - na\chi\cosh(na\chi)}{n^2 a\chi\sqrt{\nu^2+\chi^2}} \right) \right] \chi^2\, d\chi.$$

- Here $\chi \equiv r/R$, $\nu \equiv \tau_{3\mathrm{f}}/R$, and $a \equiv p/(T\nu)$. Average radial velocity:

$$\langle v_r \rangle = \frac{\int v_r \sqrt{d\Sigma_\mu d\Sigma^\mu}}{\int \sqrt{d\Sigma_\mu d\Sigma^\mu}} = \frac{1 + \nu^2 \log\left(\frac{\nu^2}{1+\nu^2}\right)}{\sqrt{1+\nu^2} - \nu^2 \log\left[\frac{\nu}{\sqrt{1+\nu^2}-1}\right]}.$$

Spectra: cylindrical geometry

- For the cylindrical case,

$$u \cdot p = m_T \cosh(\eta_T) \cosh(y - \eta_s) - p_T \sinh(\eta_T) \cos(\phi - \phi_p),$$

$$p \cdot d\Sigma = \left[m_T \cosh(y - \eta_s) \frac{d\rho}{d\zeta} - p_T \cos(\phi - \phi_p) \frac{d\tau}{d\zeta} \right] \times \rho(\zeta)\, \tau(\zeta) d\zeta\, d\phi\, d\eta_s$$

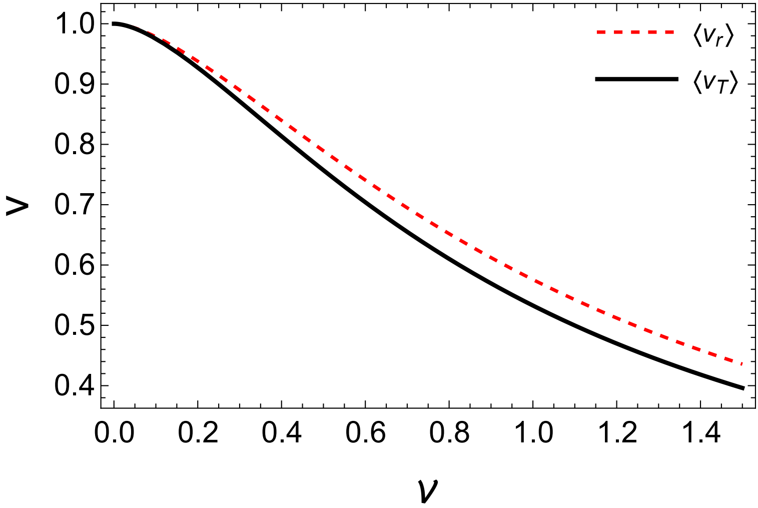
- Here $0 < \zeta < 1$ such that $\rho(0) = 0$ and $\rho(1) = R$.
- The particle spectra is given by

$$E \frac{dN}{d^3p} = \frac{gR^3}{2\pi^2} \sum_{n=1}^{\infty} \epsilon_n \int_0^1 \Bigg[m_T \sqrt{\nu^2 + \chi^2} I_0\left(\frac{n\beta p_T \chi}{\nu}\right) K_1\left(n\beta m_T \sqrt{1 + \frac{\chi^2}{\nu^2}}\right) - \chi\, p_T I_1\left(\frac{n\beta p_T \chi}{\nu}\right) K_0\left(n\beta m_T \sqrt{1 + \frac{\chi^2}{\nu^2}}\right) \Bigg] \chi\, d\chi,$$

- Here $\chi \equiv \rho/R$ and $\nu \equiv \tau_{3f}/R$. Average transverse velocity:

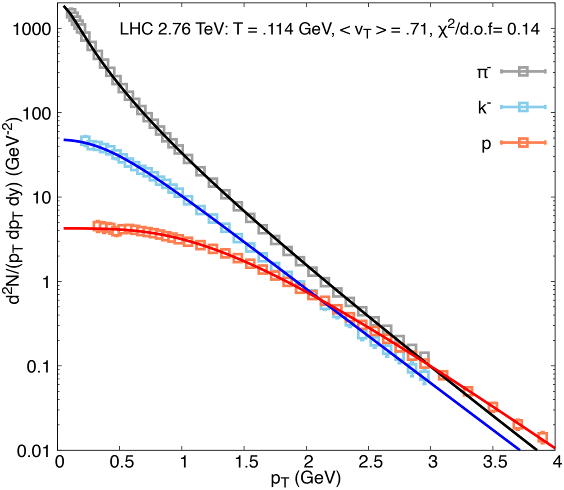
$$\langle v_r \rangle = \frac{\int v_r \sqrt{d\Sigma_\mu d\Sigma^\mu}}{\int \sqrt{d\Sigma_\mu d\Sigma^\mu}} = \sqrt{1 + \nu^2} - \nu^2 \log\left[\frac{\nu}{\sqrt{1 + \nu^2} - 1} \right].$$

Average radial and transverse velocity



[M. S. Ali et. al., arXiv:2403.00624]

p_T spectra at LHC



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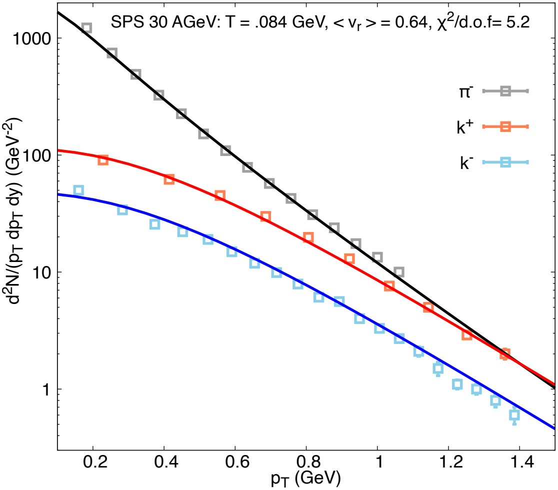
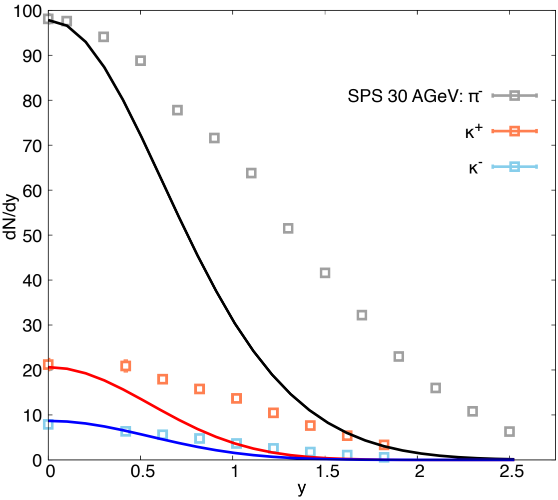
p_T spectra at SPS



[M. S. Ali et. al., arXiv:2403.00624]

Rapidity spectra at SPS



[M. S. Ali et. al., arXiv:2403.00624]

Summary

- Hydrodynamics inspired blast-wave models are commonly employed in analyzing heavy-ion data.
- Analytical hydrodynamic results for spectra obtained.
- Simple form of the expressions: easy to use in fitting algorithms.
- Symmetry in radial/transverse direction: calculation of flow coefficients not possible (yet).
- Another assumption is of constant c_s^2. Maybe interpreted as an averaged value of c_s^2 for that collision.
- Demonstrated the application of these analytical results to study spectra.
- Analytical hydrodynamic results useful to establish a benchmark for more realistic simulations.

Outlook

- Break radial/transverse symmetry perturbatively to allow for calculation of flow coefficients. Work in progress.
- Improvement required to calculate rapidity spectra reliably.
- Makes the results applicable for all collision energy range.
- Solution: combination of spherical + cylindrical geometry. Work in progress.
- Extract an average speed-of-sound by fitting experimental data.
- Information about the equation of state.
- Dissipation needs to be included in the solutions.
- A detailed phenomenology of heavy-ion collision experiment with these results needed.
- **The results find potential applications at NICA.**

Thank you!