SIMPLE NONLOCAL DE SITTER GRAVITY

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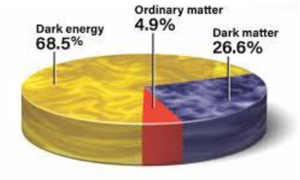
Based mainly on joint work with I. Dimitrijevic, Z. Rakic and J. Stankovic:
PLB 797 (2019) 134848; arXiv:1906.07560 [gr-qc].
JHEP 12 (2022) 054; arXiv:2206.13515 [gr-qc].
Symmetry **2024**, *16, 544; arXiv:2404.05848 [physics.gen-ph].*

Introduction

Standard Model of Cosmology (ΛCDM model)

- General Relativity (GR) is classical theory of gravitation at all scales from the Solar system to the universe as a whole: $R_{\mu\nu} - \frac{1}{2} R g_{\mu\nu} = 8\pi G T_{\mu\nu}(DE + DM + OM)$.
- At the current cosmic time the universe consists of 68 % of dark energy (DE), 27 % of dark matter (DM) and only 5 % of ordinary matter (OM).
- DE = Λ, DM = CDM, ordinary matter.
- DE causes accelerated expansion of the universe (1998), DM is responsible for galaxy dynamics (1930th).



Standard Model of Cosmology

Problems:

- DE and DM are not yet discovered in any experiment.
- GR is not confirmed on galaxy and larger cosmic scales without assumption of DE and DM. GR – singularities, problems with quantization.

Possible solution:

- There is a sense to look for a modified (extended GR) gravity.

$$R_{\mu\nu} - \frac{1}{2} R\, g_{\mu\nu} + ... = 8\pi G\, T_{\mu\nu}(OM)$$

- There is no theoretical principle that could tell us in what direction to make modification of GR. Hence, many attempts!
- There are many directions to modify GR.

Generalizations of general relativity

- Einstein equation and Einstein-Hilbert action

$$R_{\mu\nu} - \frac{1}{2} R\, g_{\mu\nu} = 8\pi G\, T_{\mu\nu}$$

$$S = \int d^4x \frac{\sqrt{-g}}{16\pi G} R + \int d^4x \sqrt{-g}\, \mathcal{L}(matter)$$

- What does mean modification of GR?

$$R \to f(R, \Lambda, R_{\mu\nu}, R^{\alpha}_{\mu\beta\nu}, \Box, \ldots), \quad \Box = \nabla^{\mu}\nabla_{\mu} = \frac{1}{\sqrt{-g}} \partial_{\mu} \sqrt{-g} g^{\mu\nu} \partial_{\nu}$$

Generalizations of general relativity

- $f(R)$ modified gravity

$$S = \int d^4x \frac{\sqrt{-g}}{16\pi G} f(R) + \int d^4x \sqrt{-g}\, \mathcal{L}(matter)$$

- nonlocal modified gravity

$$S = \int d^4x \frac{\sqrt{-g}}{16\pi G} f(R, \Lambda, \Box, \Box^{-1}, ...) + \int d^4x \sqrt{-g}\, \mathcal{L}(matter)$$

- Here we consider nonlocal approach to modification of GR.

Nonlocal de Sitter gravity $\sqrt{dS}$

- Our nonlocal de Sitter gravity model

$$
\begin{aligned}
S &= \frac{1}{16\pi G}\int\Big(R-2\Lambda+\sqrt{R-2\Lambda}\,\mathcal{F}(\Box)\,\sqrt{R-2\Lambda}\Big)\sqrt{-g}\,d^4x \\
&= \frac{1}{16\pi G}\int\sqrt{R-2\Lambda}\,F(\Box)\,\sqrt{R-2\Lambda}\Big)\sqrt{-g}\,d^4x
\end{aligned}
$$

where $\mathcal{F}(\Box)=\sum_{n=1}^{+\infty}\big(f_n\Box^n+f_{-n}\Box^{-n}\big)$, $F(\Box)=1+\mathcal{F}(\Box)$ and Λ is cosmological constant. Motivation: string theory (ordinary and p-adic), mimics od DE and DM.

- Simple and natural construction of nonlocal term:

$$R-2\Lambda=\sqrt{R-2\Lambda}\,\sqrt{R-2\Lambda}\to\sqrt{R-2\Lambda}\,\mathcal{F}(\Box)\,\sqrt{R-2\Lambda}$$

- Invariance: $\sqrt{R-2\Lambda}\to-\sqrt{R-2\Lambda}$
- $F(\Box)$ is dimensionless nonlocal operator. Only one parameter, Λ.
- We consider nonlocal modification without matter sector, but we obtain effect of dark matter and dark energy at the cosmological scale. Also rotation curves of spiral galaxies.

Action for a class of models:

$$S = \frac{1}{16\pi G}\int_M \Big(R - 2\Lambda + P(R)\mathcal{F}(\Box)Q(R)\Big)\sqrt{-g}\, d^4x$$

where $P(R)$ and $Q(R)$ are some differentiable functions of scalar curvature R.

Equations of motion (EoM):

$$\begin{aligned} & G_{\mu\nu} + \Lambda g_{\mu\nu} - \frac{1}{2} g_{\mu\nu} P(R)\mathcal{F}(\Box)Q(R) + (R_{\mu\nu} - K_{\mu\nu})\,\Phi \\ & + \frac{1}{2}\sum_{n=1}^{\infty} f_n \sum_{\ell=0}^{n-1} \big(g_{\mu\nu} g^{\alpha\beta}\partial_\alpha \Box^\ell P(R)\partial_\beta \Box^{n-1-\ell} Q(R) \\ & - 2\partial_\mu \Box^\ell P(R)\partial_\nu \Box^{n-1-\ell} Q(R) + g_{\mu\nu}\Box^\ell P(R)\Box^{n-\ell} Q(R)\big) = 0, \end{aligned}$$

where $K_{\mu\nu} = \nabla_\mu\nabla_\nu - g_{\mu\nu}\Box$, $\Phi = P'(R)\mathcal{F}(\Box)Q(R) + Q'(R)\mathcal{F}(\Box)P(R)$, and $'$ denotes derivative on R.

A way to solve EoM

- $P(R) = Q(R) = \sqrt{R - 2\Lambda}$
- $\Box\sqrt{R - 2\Lambda} = q\sqrt{R - 2\Lambda}, \quad \Box^{-1}\sqrt{R - 2\Lambda} = q^{-1}\sqrt{R - 2\Lambda}, \quad q \neq 0$
  $\mathcal{F}(\Box)\,\sqrt{R - 2\Lambda} = \mathcal{F}(q)\,\sqrt{R - 2\Lambda}$
- Very simple form of EoM

$$(G_{\mu\nu} + \Lambda g_{\mu\nu})\,(1 + \mathcal{F}(q)) + \frac{1}{2}\mathcal{F}'(q) S_{\mu\nu}(\sqrt{R - 2\Lambda}, \sqrt{R - 2\Lambda}) = 0$$

where

$$S_{\mu\nu}(P, P) = g_{\mu\nu}\left(\nabla^{\alpha} P\, \nabla_{\alpha} P + P\Box P\right) - 2\nabla_{\mu} P\, \nabla_{\nu} P, \quad P = \sqrt{R - 2\Lambda}$$

- Equations of motion are satisfied with conditions:
  $\mathcal{F}(q) = -1$ and $\mathcal{F}'(q) = 0.$

Exact cosmological solutions

- The universe is homogeneous and isotropic space at cosmic scale with FLRW metric

$$ds^2 = -dt^2 + a^2(t)\Big(\frac{dr^2}{1-kr^2} + r^2 d\theta^2 + r^2 \sin^2\theta \, d\phi^2\Big)$$

k=0 (flat space), k= +1 (closed space), k=-1 (open space)

- We have to find $a(t)$ which solves nonlinear equation:

$$\Box\sqrt{R-2\Lambda} = q\sqrt{R-2\Lambda} \qquad (q = \eta\Lambda)$$

$$\Box = -\frac{\partial^2}{\partial t^2} - 3H(t)\frac{\partial}{\partial t}, \quad H(t) = \frac{\dot a}{a},$$

$$R(t) = 6\left(\frac{\ddot a}{a} + \Big(\frac{\dot a}{a}\Big)^2 + \frac{k}{a^2}\right), \quad k \in \{0, +1, -1\}.$$

- Then $\mathcal{F}(\Box)\sqrt{R-2\Lambda} = \mathcal{F}(q)\sqrt{R-2\Lambda}$.

Exact cosmological solutions

- Solution of $\Box\sqrt{R-2\Lambda} = q\sqrt{R-2\Lambda}$ leads to the equations of motion

$$(G_{\mu\nu} + \Lambda g_{\mu\nu})\,(1+\mathcal{F}(q)) + \frac{1}{2}\mathcal{F}'(q)S_{\mu\nu}(\sqrt{R-2\Lambda}, \sqrt{R-2\Lambda}) = 0$$

have solutions if $\mathcal{F}(q) = -1, \quad \mathcal{F}'(q) = 0.$

- An example of nonlocal operator

$$\mathcal{F}(\Box) = e\Big(a\,\frac{\Box}{q}\,e^{(-\frac{\Box}{q})} + b\,\frac{q}{\Box}\,e^{(-\frac{q}{\Box})}\Big), \quad q = \eta\Lambda \neq 0, \;\; a+b = -1$$

where η is dimensionless parameter depending of a concrete cosmological solution.

Exact cosmological solutions

- $a_1(t) = A\, t^{\frac{2}{3}}\, e^{\frac{\Lambda}{14}t^2}, \quad (k = 0,\ \Lambda \neq 0)$
- $a_2(t) = A\, e^{\frac{\Lambda}{6}t^2}, \quad (k = 0,\ \Lambda \neq 0)$
- $a_3(t) = A\, \cosh^{\frac{2}{3}}\left(\sqrt{\frac{3\Lambda}{8}}\, t\right), \quad (k = 0,\ \Lambda > 0)$
- $a_4(t) = A\, \sinh^{\frac{2}{3}}\left(\sqrt{\frac{3\Lambda}{8}}\, t\right), \quad (k = 0,\ \Lambda > 0)$
- $a_5(t) = A\left(1 + \sin\left(\sqrt{-\frac{3\Lambda}{2}}\, t\right)\right)^{\frac{1}{3}}, \quad (k = 0,\ \Lambda < 0)$
- $a_6(t) = A\left(1 - \sin\left(\sqrt{-\frac{3\Lambda}{2}}\, t\right)\right)^{\frac{1}{3}}, \quad (k = 0,\ \Lambda < 0)$
- $a_7(t) = A\, \sin^{\frac{2}{3}}\left(\sqrt{-\frac{3\Lambda}{8}}\, t\right), \quad (k = 0,\ \Lambda < 0)$
- $a_8(t) = A\, \cos^{\frac{2}{3}}\left(\sqrt{-\frac{3\Lambda}{8}}\, t\right), \quad (k = 0,\ \Lambda < 0)$
- $a_9(t) = A\, e^{\pm\sqrt{\frac{\Lambda}{6}}t}, \quad (k = \pm 1,\ \Lambda > 0)$
- $a_{10}(t) = A\, \cosh^{\frac{1}{2}}\left(\sqrt{\frac{3\Lambda}{2}}\, t\right), \quad (k = \pm 1,\ \Lambda > 0)$
- $a_{11}(t) = A\, \sinh^{\frac{1}{2}}\left(\sqrt{\frac{3\Lambda}{2}}\, t\right), \quad (k = \pm 1,\ \Lambda > 0)$
- + some anisotropic cosmological solutions. arXiv:2307.00621 [gr-qc].

Cosmological parameters and $a(t) = A\, t^{\frac{2}{3}}\, e^{\frac{\Lambda}{14}t^2}$

Planck 2018 data for the ΛCDM universe:

- $H_0 = (67.40 \pm 0.50)$ km/s/Mpc – Hubble parameter;
- $\Omega_m = 0.315 \pm 0.007$ – matter density parameter;
- $\Omega_\Lambda = 0.685$ – Λ density parameter;
- $t_0 = (13.801 \pm 0.024) \cdot 10^9$ yr – age of the universe;
- $w_0 = -1.03 \pm 0.03$ – ratio of pressure to energy density.
- $\Lambda = 3H_0^2\Omega_\Lambda = 0.98 \cdot 10^{-35} s^{-2}$.

$$H(t) = \frac{2}{3}\frac{1}{t} + \frac{1}{7}\Lambda t \qquad H_0 = \frac{2}{3}\frac{1}{t_0} + \frac{1}{7}\Lambda t_0$$

Cosmological parameters and $a(t) = A\, t^{\frac{2}{3}}\, e^{\frac{\Lambda}{14} t^2}$

Solution $a(t) = A\, t^{\frac{2}{3}}\, e^{\frac{\Lambda}{14} t^2}, \quad (k = 0,\ \Lambda \neq 0)$

$$H(t) = \frac{2}{3}\frac{1}{t} + \frac{1}{7}\Lambda t \qquad H_0 = \frac{2}{3}\frac{1}{t_0} + \frac{1}{7}\Lambda t_0$$

- mimics dark matter $t^{\frac{2}{3}}$ and dark energy $e^{\frac{\Lambda}{14} t^2}$
- $H_0, t_0 : \quad \Lambda_1 = 1.05 \cdot 10^{-35}\, s^{-2}$
- $\bar{\rho}_1(t_0) = \frac{3}{8\pi G}\Big(H_0^2 - \frac{\Lambda_1}{3}\Big) = 2.26 \times 10^{-30}\ \frac{g}{cm^3}$
- $\rho(t_0) = \frac{3}{8\pi G}\Big(H_0^2 - \frac{\Lambda}{3}\Big) = 2.68 \times 10^{-30}\ \frac{g}{cm^3}$
- $\rho_c = \frac{3}{8\pi G} H^2(t_0) = 8.51 \times 10^{-30}\ \frac{g}{cm^3}$
-

$$\Omega_{\Lambda_1} = \frac{\rho_{\Lambda_1}}{\rho_c} = 0.734, \quad \Omega_\Lambda = \frac{\rho_\Lambda}{\rho_c} = 0.685, \quad \Delta\Omega_\Lambda = \Omega_{\Lambda_1} - \Omega_\Lambda = 0.049$$

$$\Omega_m = \frac{\rho(t_0)}{\rho_c} = 0.315, \quad \Omega_{m_1} = \frac{\bar{\rho}_1(t_0)}{\rho_c} = 0.266, \quad \Delta\Omega_m = \Omega_m - \Omega_{m_1} = 0.049.$$

Cosmological parameters and $a(t) = A\, t^{\frac{2}{3}}\, e^{\frac{\Lambda}{14} t^2}$

H_0 tension

t_0 $[10^9$ years$]$	Λ $[\times 10^{-35} s^{-2}]$	H_0 $[km/s/Mpc]$	ρ_c $[\times 10^{-27} kg/m^3]$	ρ_Λ	ρ	Ω_Λ	Ω
13.8	0.98	66.069	8.19917	5.84225	2.35691	0.712543	0.287457
13.8	1.04938	67.4	8.53286	6.25588	2.27698	0.733152	0.266848
13.2001	0.98	67.4	8.53286	5.84225	2.6906	0.684678	0.315322
36.2737	0.98	67.4	8.53286	5.84225	2.6906	0.684678	0.315322
13.8	1.34129	73	10.010	7.9961	2.01359	0.798836	0.201164
11.3324	0.98	73	10.010	5.84225	4.16743	0.58366	0.41634
42.2519	0.98	73	10.010	5.84225	4.16743	0.58366	0.41634

- $H_0 = 67.40 \pm 0.50$ km/s Mpc   Planck 2018
- $H_0 = 73 \pm 1 km/sMpc$ Sn Ia

Cosmological parameters and $a(t) = A\, t^{\frac{2}{3}}\, e^{\frac{\Lambda}{14}t^2}$

Effective energy density and pressure:

- $\bar{\rho} = \frac{2t^{-2} + \frac{9}{98}\Lambda^2 t^2 - \frac{9}{14}\Lambda}{12\pi G}, \quad \bar{p} = \frac{\Lambda}{56\pi G}\left(1 - \frac{3}{7}\Lambda t^2\right)$
- $\bar{w}_0 = \frac{\bar{p}_o}{\bar{\rho}_0} = 0.058$
- $\bar{w} = \frac{\bar{p}}{\bar{\rho}} \to -1 \quad \text{when} \quad t \to \infty$
- $t \to 0 : \ \bar{\rho} \to \infty, \quad \bar{p} \to \frac{\Lambda}{56\pi G}.$
- One can also compute time (t_m) when the Hubble parameter has minimum value H_m, i.e. $t_m = 21.1 \cdot 10^9$ yr and $H_m = 61.72$ km/s/Mpc.
- Beginning of the universe expansion acceleration was at $t_a = 7.84 \cdot 10^9$ yr, or in other words at 5.96 billion years ago.

Localization of nonlocality: $a(t) = A\ t^{\frac{2}{3}}\ e^{\frac{\Lambda}{14}t^2}$

$$S = \frac{1}{16\pi G}\int R\sqrt{-g}\,\mathrm{d}^4x + \frac{1}{8\pi G}\int(-\frac{1}{2}\,\nabla_\mu\varphi\nabla^\mu\varphi - V(\varphi))\sqrt{-g}\,d^4x.$$

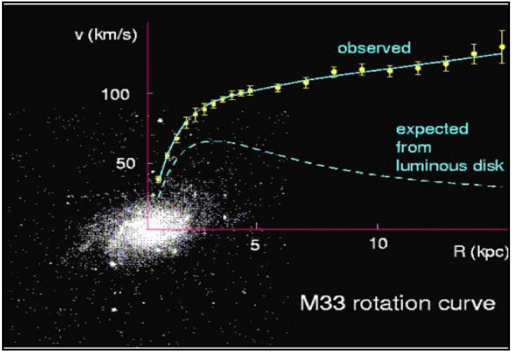
$$\frac{1}{16\pi G}G_{\mu\nu} + \frac{1}{8\pi G}\Big(\frac{1}{4}g_{\mu\nu}\nabla^\rho\varphi\nabla_\rho\varphi + \frac{1}{2}g_{\mu\nu}V(\varphi) - \frac{1}{2}\nabla_\mu\varphi\nabla_\nu\varphi\Big) = 0.$$

$$G_{\mu\nu} = 8\pi G T_{\mu\nu}, \qquad \Box\varphi = V'(\phi)$$

$$8\pi G\,\rho = \frac{1}{2}\dot\varphi^2 + V(\varphi), \qquad 8\pi G\,p = \frac{1}{2}\dot\varphi^2 - V(\varphi),$$

$$8\pi G(\rho + p) = \dot\varphi^2 = \frac{4}{3}t^{-2} - \frac{2}{7}\Lambda,$$
$$4\pi G(\rho - p) = V(t) = -\frac{2\Lambda}{7} + \frac{3\Lambda^2t^2}{49} + \frac{2}{3t^2}.$$

- Dark matter ?


v (km/s)
100
50
observed
expected
from
luminous disk
5
10
R (kpc)
M33 rotation curve

Rotation curves for spiral galaxies

$$ds^2 = -A(r)dt^2 + B(r)dr^2 + r^2 d\theta^2 + r^2 \sin^2\theta d\varphi^2, \qquad (c = 1).$$

Equation that should be solved

$$\Box\sqrt{R - 2\Lambda} = q\sqrt{R - 2\Lambda}$$

$$\Box u(r) = \frac{1}{B(r)}\left(\triangle u(r) + \frac{1}{2}\left(\frac{A'(r)}{A(r)} - \frac{B'(r)}{B(r)}\right)u'(r)\right) = qu(r), \quad u(r) = \sqrt{R - 2\Lambda}$$

$$R = \frac{2}{r^2} - \frac{1}{B(r)}\Big(\frac{2}{r^2} + \frac{2A'(r)}{rA(r)} - \frac{A'(r)^2}{2A(r)^2} - \frac{2B'(r)}{rB(r)} - \frac{A'(r)B'(r)}{2A(r)B(r)} + \frac{A''(r)}{A(r)}\Big)$$

$$\triangle = \frac{1}{r^2}\frac{\partial}{\partial r}\left[r^2\frac{\partial}{\partial r}\right] = \frac{\partial^2}{\partial r^2} + \frac{2}{r}\frac{\partial}{\partial r}$$

Rotation curves for spiral galaxies

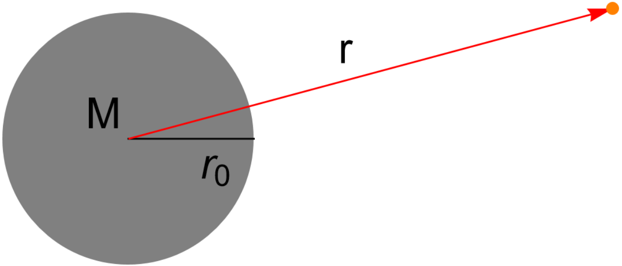


Figure: We consider the Schwarzschild-de Sitter metric of nonlocal $\sqrt{dS}$ gravity at the distances far from a spherically symmetric massive body.

Rotation curves for spiral galaxies

$$B(r) = \frac{1}{A(r)}$$

After linearization (weak field approximation):

$$A(r) = 1 - \frac{\mu}{r} - \frac{\Lambda r^2}{3} + \frac{\delta}{\sqrt{q}r}\Big(1 + e^{-\sqrt{q}\,r}\Big) - \frac{2\delta}{qr^2}\Big(1 - e^{-\sqrt{q}\,r}\Big), \quad q = \zeta\Lambda$$

Finally:

$$v(r) = c\sqrt{\frac{GM}{c^2 r} - \frac{\Lambda r^2}{3} + \frac{\delta}{\sqrt{q}r}\Big(\frac{2}{\sqrt{q}r} - \frac{1}{2}\Big) - \delta\Big(\frac{1}{2} + \frac{3}{2\sqrt{q}r} + \frac{2}{qr^2}\Big)e^{-\sqrt{q}\,r}}.$$

Rotation curves for spiral galaxies

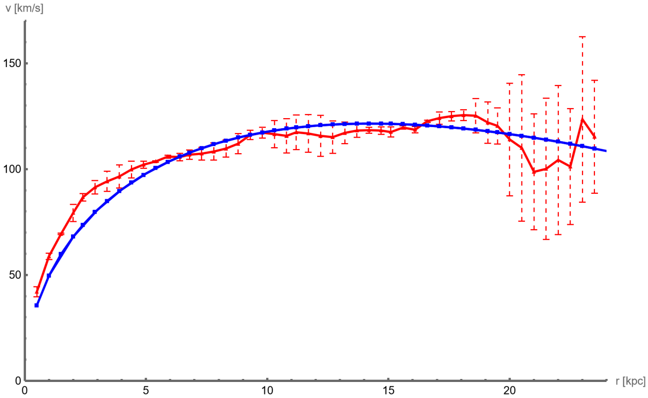


Figure: Rotation curve for the galaxy M33. Red points are measured observational values and blue line is computed $v(r)$ by our Formula, where $\delta = 5.7 \times 10^{-6}$, $\zeta = 3.62 \times 10^{10}$, $\Lambda = 10^{-52}\ \mathrm{m}^{-2}$, and $M = 1.5 \times 10^3 M_\odot$.

Rotation curves for spiral galaxies

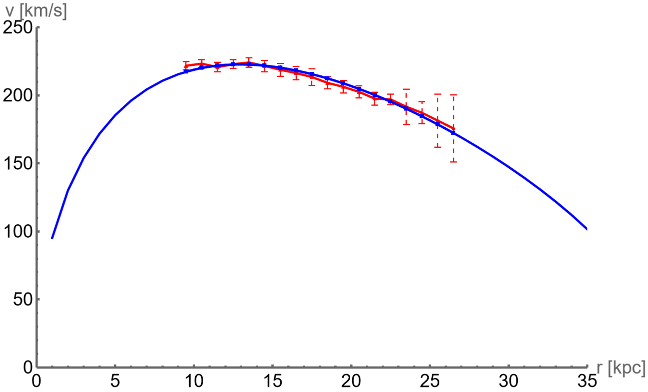


Figure: Rotation curve for the Milky Way galaxy. Red points are measured observational values and blue line is computed $v(r)$ by our Formula, where $\delta = 1.9 \times 10^{-5}$, $\zeta = 4.4 \times 10^{10}$, $\Lambda = 10^{-52}\mathrm{m}^{-2}$, and $M = 4.28 \times 10^{6} M_{\odot}$.

Conclusions and perspective

- We introduced and analyzed nonlocal de Sitter gravity model $\sqrt{dS}$

$$S = \frac{1}{16\pi G} \int_M \Big(R - 2\Lambda + \sqrt{R - 2\Lambda}\, \mathcal{F}(\Box)\, \sqrt{R - 2\Lambda}\Big) \sqrt{-g}\, d^4x$$

  as very simple and interesting model in several aspects.
- Model set up and EoM are relatively very simple.
- We found 11 exact cosmological (flat, closed and open) solutions. Some of them are nonsingular bounce, and also cyclic.
- All solutions are new and do not exist in the local de Sitter case.
- The most interesting is exact vacuum cosmological solution

$$a(t) = A\, t^{\frac{2}{3}} e^{\frac{\Lambda}{14} t^2}, \quad \Lambda \neq 0, \;\; k = 0$$

  which mimics dark matter and dark energy. Computed cosmological parameters are in good agreement with observations.
- We also get description of galaxy rotation curves without dark matter.
- The next step is testing this model at other space-time scales and phenomena.

Some more references

- I. Dimitrijevic, B. Dragovich, Z. Rakic, J. Stankovic, *Nonlocal de Sitter gravity and its exact cosmological solutions,* JHEP 12 (2022) 054; arXiv:2206.13515 [gr-qc].
- I. Dimitrijevic, B. Dragovich, A. S. Koshelev, Z. Rakic, J. Stankovic, *Cosmological solutions of a nonlocal square root gravity*, Phys. Lett. B 797 (2019) 134848; arXiv:1906.07560 [gr-qc].
- I. Dimitrijevic, B. Dragovich, A. S. Koshelev, Z. Rakic, J. Stankovic, *Some cosmological solutions of a new nonlocal gravity model*, Symmetry 12, 917 (2020), arXiv:2006.16041 [gr-qc].
- I. Dimitrijevic, B. Dragovich, Z. Rakic and J. Stankovic, *New cosmological solutions of a nonlocal gravity model*, Symmetry **2022** 14 (2022) 3, arXiv:2112.06312 [gr-qc].
- I. Dimitrijevic, B. Dragovich, Z. Rakic, J. Stankovic, *Variations of infinite derivative modified gravity,* Springer Proc. in Mathematics & Statistics **263** (2018) 91–111.
- I. Dimitrijevic, B. Dragovich, J. Grujic, A.S. Koshelev, Z. Rakic, *Cosmology of modified gravity with a nonlocal* $f(R)$, Filomat 33 (2019) 1163–1178, arXiv:1509.04254[hep=th].
- I. Dimitrijevic, B. Dragovich, A. S. Koshelev, J. Stankovic, Z. Rakic, *On nonlocal modified gravity and its cosmological solutions,* Springer Proc. in Mathematics & Statistics **191** (2016) 35–51.
- I. Dimitrijevic, B. Dragovich, Z. Rakic and J. Stankovic, *The Schwarzschild–de Sitter metric of nonlocal* $\sqrt{dS}$ *gravity*, Symmetry **2024** 16, 544 (2024).
- S. Capozziello and F. Bajardi, *Nonlocal gravity cosmology: An overview*, Int. J. Mod. Phys. D **31** (2022) 2230009, arXiv:2201.04512 [gr-qc].
- Jiao, Y; Hammer, F.; Wang, H.; Wang, J.; Amram, P.; Chemin, L.; Yang, Y. Detection of the Keplerian decline in the Milky Way rotation curve. *Astron. Astrophys.* **2023**, *678*, A208.
- Kam, S.Z.; Carignan, C.; Chemin, L.; Foster, T.; Elson, E.; Jarrett, T.H. HI kinematics and mass distribution of Messier 33. *Astron. J.* **2017**, *154*, 41.

THANK YOU FOR YOUR ATTENTION!