Harmonic superspace in action: unravelling $\mathcal{N} = 2$ higher spins (continuation of the PMMP'24 talk)

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Outline

- $\mathcal{N} = 2$ harmonic superspace
- $\mathcal{N} = 2$ supergravity in HSS
- $\mathcal{N} = 2$ higher spins
- Superconformal couplings
- Fully consistent higher-spin hypermultiplet coupling
- On AdS background
- Further prospects

$\mathcal{N} = 2$ harmonic superspace

$\mathcal{N} = 2$ Harmonic Superspace (HSS) has been discovered in Galperin, Ivanov, Kalitzin, Ogievetsky, Sokatchev, (1984). It is the product

$$(x^m, \theta_{\alpha\, i}, \bar\theta^k_{\dot\beta}) \otimes S^2$$

Here, the internal two-sphere $S^2 \sim SU(2)_R/U(1)_R$ is represented, in a parametrization-independent way, by the lowest (isospinor) $SU(2)_R$ harmonics which are treated as new cordinates:

$$S^2 \in (u^+_i, u^-_k), \quad u^{+i}u^-_i = 1, \quad u^\pm_i \to e^{\pm i\lambda} u^\pm_i$$

The superfields given on HSS (harmonic $\mathcal{N} = 2$ superfields) admit the harmonic expansions on S^2, with the set of all symmetrized products of u^+_i, u^-_i as the basis. Such an expansion is fully specified by the harmonic $U(1)$ charge of the given superfield.

The main merit of HSS is that it contains an invariant subspace, the $\mathcal{N}=2$ analytic HSS, with only half the original Grassmann coordinates.

One can pass to the **analytic basis** in HSS

$$\{(x^m_A, \theta^+_\alpha, \bar\theta^+_{\dot\alpha}, u^\pm_i)\,\theta^-_\alpha, \bar\theta^-_{\dot\alpha}\} \equiv \{\left(\zeta^M, u^\pm_i\right), \theta^-_\alpha, \bar\theta^-_{\dot\alpha}\}\,,$$
$$x^m_A = x^m - 2i\theta^{(i}\sigma^m\bar\theta^{k)}u^+_i u^-_k\,,\quad \theta^\pm_\alpha = \theta^i_\alpha u^\pm_i\,,\ \bar\theta^\pm_{\dot\alpha} = \bar\theta^i_{\dot\alpha}u^\pm_i$$

Then the set of coordinates

$$(x^m_A, \theta^+_\alpha, \bar\theta^+_{\dot\alpha}, u^\pm_i) \equiv \left(\zeta^M, u^\pm_i\right),$$

is closed under both $\mathcal{N}=2$ supersymmetry transformations and $SU(2)_R$ and is real with respect to the special involution defined as the product of the ordinary complex conjugation and the antipodal map (Weyl reflection) of S^2.

The superfields given on the analytic subspace can be defined in the basis-independent way by the new (Grassmann-analyticity) conditions

$$D^+_\alpha\Phi(Z,u) = \bar D^+_{\dot\alpha}\Phi(Z,u) = 0 \ \leftrightarrow \left(\frac{\partial}{\partial\theta^-_\alpha}, \frac{\partial}{\partial\bar\theta^-_{\dot\alpha}}\right)\Phi(\zeta^M, u^\pm_i, \theta^-_\alpha, \bar\theta^-_{\dot\alpha}) = 0$$
$$D^+_\alpha = u^+_i D^i_\alpha\,,\ \bar D^+_{\dot\alpha} = u^+_i \bar D^i_{\dot\alpha}\,,\quad \Phi = \varphi(\zeta^M, u^\pm_i)$$

All $\mathcal{N}=2$ theories of interest **(SYM, matter, supergravities, superextension of higher spins,...)** are described by analytic superfields $\varphi^{+q}(\zeta^M, u^\pm_i)$ as the fundamental geometric objects.

$\mathcal{N} = 2$ **SYM constraints** are solved as follows. One projects

$$(\mathcal{D}^i_\alpha, \bar{\mathcal{D}}^i_{\dot\alpha}) \Leftrightarrow (\mathcal{D}^\pm_\alpha, \bar{\mathcal{D}}^\pm_{\dot\alpha}), \qquad \mathcal{D}^\pm_\alpha = u^\pm_i \mathcal{D}^i_\alpha\,, \; \bar{\mathcal{D}}^\pm_{\dot\alpha} = u^\pm_i \bar{\mathcal{D}}^i_{\dot\alpha} \tag{1}$$

Then the constraints are rewritten as

$$\{\mathcal{D}^+_\alpha, \mathcal{D}^+_\beta\} = \{\bar{\mathcal{D}}^+_{\dot\alpha}, \bar{\mathcal{D}}^+_{\dot\beta}\} = \{\mathcal{D}^+_\alpha, \bar{\mathcal{D}}^+_{\dot\beta}\} = 0 \tag{2}$$

After taking off u^{+i}, the standard constraints $\{\mathcal{D}^{(i}_\alpha, \bar{\mathcal{D}}^{j)}_{\dot\beta}\} = \{\mathcal{D}^{(i}_\alpha, \mathcal{D}^{j)}_\beta\} = 0$ are recovered in view of arbitrariness of u^+_i. How to reduce the procedure of converting with harmonics to a new kind of differential constraint?
This can be done with the new differential operators, the harmonic derivatives

$$\begin{aligned} &\partial^{\pm\pm} = u^{\pm i}\frac{\partial}{\partial u^{\mp i}}, \; \partial^0 = u^{+i}\frac{\partial}{\partial u^{+i}} - u^{-i}\frac{\partial}{\partial u^{-i}}\,, \\ &[\partial^{++}, \partial^{--}] = \partial^0\,, \quad [\partial^0, \partial^{\pm\pm}] = \pm 2\partial^{\pm\pm}\,, \end{aligned} \tag{3}$$

One adds to (2) new constraints

$$[\partial^{++}, \mathcal{D}^+_\alpha] = [\partial^{++}, \bar{\mathcal{D}}^+_{\dot\alpha}] = 0 \Rightarrow \mathcal{D}^+_{\alpha,\dot\alpha} = u^+_i \mathcal{D}^i_{\alpha,\dot\alpha}\,, \tag{4}$$

using $\partial^{++}u^+_i = 0, \partial^{++}u^-_i = u^+_i$. Now one can treat $\mathcal{D}^+_{\alpha,\dot\alpha}$ to have an unconstrained dependence on $u^\pm$ while the linear dependence appears as a result of imposing extra constraints (4).

The extended set of constraints (2) and (4), besides the standard SYM constraints, admits a new solution. By making some similarity gauge-like transformation (with a general harmonic suiperfield as a parameter, so called "bridge") and simultaneously passing to the analytic basis in HSS, one can solve the integrability conditions (2) and (4) as

$$(\mathcal{D}^+_\alpha, \bar{\mathcal{D}}^+_{\dot\alpha}) \Rightarrow (\partial^+_\alpha, \bar\partial^+_{\dot\alpha}), \quad \partial^{++} \Rightarrow D^{++} + iV^{++}(\zeta, u), \tag{5}$$
$$D^{++} = \partial^{++} - 2i\theta^+\sigma^a\bar\theta^+\partial_a + \theta^{+\alpha}\partial^+_\alpha + \bar\theta^{+\dot\alpha}\bar\partial^+_{\dot\alpha}$$

In other words, (2) implies the spinorial connections in $\mathcal{D}^+_{\alpha,\dot\alpha}$ to be a "pure gauge" in the full HSS, while D^{++} in the new frame acquires some harmonic connection V^{++} which is unconstrained **analytic** by the constraint (4), $V^{++} \Rightarrow V^{++}(\zeta, u)$. It is just the fundamental gauge prepotential of $\mathcal{N} = 2$ SYM theory. It carries the analytic gauge freedom,

$$\delta V^{++} = D^{++}\Lambda + i[V^{++}, \Lambda],$$

which can be used to reduce V^{++} to the Wess-Zumino form

$$V^{++} = (\theta^+)^2\phi + (\bar\theta^+)^2\bar\phi + i\theta^+\sigma^m\bar\theta A_m + [(\theta^+)^2\bar\theta^+_{\dot\alpha}\bar\psi^{\dot\alpha i}u^-_i + \text{c.c.}] + (\theta^+)^4 D^{(ik)}u^-_i u^-_k$$

These fields form just the off-shell $\mathcal{N} = 2$ vector supermultiplet.

The off-shell action can be constructed using the second (non-analytic) harmonic connection V^{--} related to V^{++} by the harmonic flatness condition

$$D^{++}V^{--} - D^{--}V^{++} + i[V^{++}, V^{--}] = 0$$

The action for the abelian case is

$$S_v \sim \int d^{12}Zdu \; V^{++}V^{--}$$

For non-abelian case the action looks more complicated since it contains harmonic non-localities (B. Zupnik, 1987). Its variation is much simpler

$$\delta S_v \sim \int d^{12}Zdu \, \mathrm{Tr}(\delta V^{++}V^{--}),$$

implying, e.g., rather simple form of the equation of motion,

$$(D^+)^2(\bar{D}^+)^2V^{--} = 0 \, . \qquad (6)$$

Hypermultiplet. The on-shell constraints on the hypermultiplet in the standard $\mathcal{N}=2$ superspace are rewritten in HSS as

$$D^{(i}_{\alpha,\dot{\alpha}} q^{k)} = 0\,, \Leftrightarrow \; (a)\, D^{+}_{\alpha,\dot{\alpha}} q^{+} = 0,\; (b)\, D^{++} q^{+} = 0 \tag{7}$$

Indeed, eq. (7)b implies $q^{+} = u^{+}_{k} q^{k}$, then eq (7)a yields the standard constraints for $\mathcal{N}=2$ superfield q^{k} (that amount to eqs. of motion). Once the standard hypermultiplet constraints have been rewritten as differential conditions in HSS, one can pass to the analytic basis where $D^{+}_{\alpha,\dot{\alpha}}$ become "short" and (7)a simply implies that q^{+} is analytic in this basis:

$$(7)a \Rightarrow q^{+} = q^{+}(\zeta, u) \tag{8}$$

As any analytic $\mathcal{N}=2$ superfield, $q^{+}(\zeta, u)$ is off-shell and involves the ∞ number of fields. The whole dynamics proves to be concentrated in (7)b:

$$(7)b \Rightarrow D^{++} q^{+} = (\partial^{++} - 2i\theta^{+}\sigma^{a}\bar{\theta}^{+}\partial_{a}) q^{+} = 0 \tag{9}$$

This equation nullifies all fields in $q^{+}(\zeta, u)$ except for those entering at zero and first degrees of $\theta^{+}_{\alpha}, \bar{\theta}^{+}_{\dot{\alpha}}$ and at first and zero powers in harmonics:

$$q^{+} \Rightarrow q^{i}(x) u^{+}_{i} + \theta^{+}_{\alpha}\psi^{\alpha}(x) + \bar{\theta}^{+}_{\dot{\alpha}}\chi^{\dot{\alpha}}(x) \tag{10}$$

These are just physical bosonic and fermionic fields and for them (9) implies the standard free massless equations of motion ($\Box q^{i}(x) = 0$, etc).

The splitting of the hypermultiplet constraints into the kinematical and dynamical parts in the analytic basis of HSS entails a remarkable consequence. Now the dynamical constraint (7)(b) can ne derived as an equation of motion from the off-shell action:

$$D^{++}q^{+} = 0 \quad \text{from} \quad S_q^{free} \sim \int d\zeta^{(-4)} du \, (q^{+}D^{++}\bar{q}^{+} - \bar{q}^{+}D^{++}q^{+}) \qquad (11)$$

Thus the Grassmann harmonic analyticity allowed to construct off-shell action for the hypermultiplet which was not possible in the framework of standard superspaces. It became possible just at cost of admitting an infinite number of auxiliary fields, the cardinally new feature brought about by the harmonic superspace formalism.

Now it is straightforward to write the most general action for interacting hypermultiplets:

$$S_q^{gen} \sim \int d\zeta^{(-4)} du \left(q^{+A}D^{++}\bar{q}^{+}_A - \bar{q}^{+}_A D^{++}q^{+A} + \mathcal{L}^{+4}(q^{+}, \bar{q}^{+}, u^{\pm}) \right) \qquad (12)$$

where $\mathcal{L}^{+4}(q^{+}, \bar{q}^{+}, u^{\pm})$ is an arbitrary function of its arguments, the “hyper Kähler potential”, the true analog of Kähler potential of $\mathcal{N} = 1$ supersymmetric matter. It was proven in Alvarez-Gaume, Freedman, 1980, 1981 that any $\mathcal{N} = 2, 4D$ supersymmetric matter Lagrangian contains as its bosonic “core” just sigma model with hyper-Kähler target space. So Lagrangians (12) provide an efficient way of the **explicit** construction of HK metrics (Galperin, Ivanov, Ogievetsky, Sokatchev, 1986).

$\mathcal{N} = 2$ supergravity in HSS

The fundamental gauge group of Einstein $\mathcal{N} = 2$ supergravity in HSS is general diffeomorphisms of the analytic superspace, such that the harmonics themselves remain untouched

$$\delta\zeta^M = \Lambda^M(\zeta, u)\,,\ \delta u_i^{\pm} = 0\,,\quad M := (\alpha\dot{\beta}, 5\,, \hat{\alpha}+),\ \hat{\alpha} := (\alpha, \dot{\alpha})$$

One more coordinate x^5 was added in order to describe *massive* q^{+a} hypermultiplets. Nothing depends on x^5 while the x^5 dependence of q^{+a} is assumed to be trivial, $q^{+a} \to (e^{im\,x^5\sigma^3})^a_b q^{+b}$.
To ensure the gauge invariance of the q^+ Lagrangian, we need to gauge-covariantize the harmonic derivative D^{++}:

$$\frac{1}{2}q^{+a}D^{++}q_a^+ \ \Rightarrow\ \frac{1}{2}q^{+a}\mathcal{D}^{++}q_a^+\,,\ \mathcal{D}^{++} = \partial^{++} + H^{++M}\partial_M\,, \tag{13}$$

The gauge transformation of H^{++M} reads

$$\delta H^{++M} = \mathcal{D}^{++}\lambda^M - \lambda^N\partial_N H^{++M} \tag{14}$$

It can be checked that the q^+ Lagrangian $\frac{1}{2}q^{+a}\mathcal{D}^{++}q_a^+$ is transformed by a total derivative under (14) and the transformation $\delta q^{+a} = -\frac{1}{2}(\partial_{\alpha\dot{\alpha}}\lambda^{\alpha\dot{\alpha}} - \partial_{\hat{\alpha}}\lambda^{\hat{\alpha}})q^{+a} - \lambda^N\partial_N q^{+a}$, so the action is invariant.

Using the gauge transformations of H^{++M}, one can pass to WZ gauge with $(40+40)$ off-shell degrees of freedom, which is just the off-shell content of the so called “minimal” $\mathcal{N}=2$ supergravity Fradkin, Vasiliev, 1979; de Wit, van Holten, 1979. The invariant superfield action for H^{++M} can be also constructed. In the linearized approximation:

$$S_Y \sim \int dud^4xd^8\theta[G^{++\alpha\dot{\alpha}}G^{--}_{\alpha\dot{\alpha}} + G^{++5}G^{--5}]\,,$$
$$G^{\pm\pm\alpha\dot{\alpha}} = h^{\pm\pm\alpha\dot{\alpha}} + 2i(h^{\pm\pm\alpha+}\bar{\theta}^{-\dot{\alpha}} + \theta^{-\alpha}h^{\pm\pm\dot{\alpha}+}),\; G^{\pm\pm5} = h^{\pm\pm5} - 2ih^{\pm\pm\hat{\alpha}+}\theta^{-}_{\hat{\alpha}}\,,$$
$$H^{++\alpha\dot{\alpha}} = h^{++\alpha\dot{\alpha}} - 4i\theta^{+\alpha}\bar{\theta}^{+\dot{\alpha}}\,,\; H^{++5} = h^{++5} + i(\theta^{\hat{+}})^2\,,$$
$$D^{++}G^{--\alpha\dot{\alpha},5} = D^{--}G^{++\alpha\dot{\alpha},5}\,.$$

Conformal $\mathcal{N}=2$ supergravity was also formulated in HSS and various versions of $\mathcal{N}=2$ Einstein supergravities through the compensating procedure by the appropriate compensating superfields were reproduced.

It was rather surprising that the unconstrained superfield formulations of the higher-spin $\mathcal{N}=2$ supergravities could be formulated in HSS (for the first time!) as a direct generalization of the HSS formulation of Einstein $\mathcal{N}=2$ supergravity (Buchbinder, Ivanov, Zaigraev, 2021 - 2024)

$\mathcal{N} = 2$ higher spins

Let me limit the presentation by $\mathcal{N} = 2$ spin **3**. Like in $\mathcal{N} = 2$ supergravity, the basic analytic superfields form a triad, with additional spinorial indices:

$$h^{++(\alpha\beta)(\dot\alpha\dot\beta)}(\zeta)\,,\; h^{++\alpha\dot\alpha}(\zeta),\; h^{++(\alpha\beta)\dot\alpha+}(\zeta),\; h^{++(\dot\alpha\dot\beta)\alpha+}(\zeta)\,,$$

$$\delta h^{++(\alpha\beta)(\dot\alpha\dot\beta)} = D^{++}\lambda^{(\alpha\beta)(\dot\alpha\dot\beta)} + 2i\big[\lambda^{+(\alpha\beta)(\dot\alpha}\bar\theta^{+\dot\beta)} + \theta^{+(\alpha}\bar\lambda^{+\beta)(\dot\alpha\dot\beta)}\big],$$

$$\delta h^{++\alpha\dot\alpha} = D^{++}\lambda^{\alpha\dot\alpha} - 2i\big[\lambda^{+(\alpha\beta)\dot\alpha}\theta^{+}_{\beta} + \bar\lambda^{+(\dot\alpha\dot\beta)\alpha}\bar\theta^{+}_{\dot\beta}\big],$$

$$\delta h^{++(\alpha\beta)\dot\alpha+} = D^{++}\lambda^{+(\alpha\beta)\dot\alpha}\,,\; \delta h^{++(\dot\alpha\dot\beta)\alpha+} = D^{++}\lambda^{+(\dot\alpha\dot\beta)\alpha}$$

The bosonic physical fields in the WZ gauge are collected in

$$h^{++(\alpha\beta)(\dot\alpha\dot\beta)} = -2i\theta^{+\rho}\bar\theta^{+\dot\rho}\Phi^{(\alpha\beta)(\dot\alpha\dot\beta)}_{\rho\dot\rho} + \ldots$$

$$h^{++\alpha\dot\alpha} = -2i\theta^{+\rho}\bar\theta^{+\dot\rho}C^{\alpha\dot\alpha}_{\rho\dot\rho} + \ldots$$

The physical gauge fields are $\Phi^{(\alpha\beta)(\dot\alpha\dot\beta)}_{\rho\dot\rho}$ (spin 3), $C^{\alpha\dot\alpha}_{\rho\dot\rho}$ (spin 2) and $\psi^{(\alpha\beta)(\dot\alpha\dot\beta)i}_{\gamma}$ (spin 5/2). Other fields are auxiliary. On shell, $\mathbf{(3, 5/2, 5/2, 2)}$.

The linearized gauge action has the form quite similar to the spin **2** action

$$S_{s=3} = \int d^4x d^8\theta du \left\{ G^{++(\alpha\beta)(\dot\alpha\dot\beta)} G^{--}_{(\alpha\beta)(\dot\alpha\dot\beta)} + G^{++\alpha\dot\beta} G^{--}_{\alpha\dot\beta} \right\},$$

$$G^{++(\alpha\beta)(\dot\alpha\dot\beta)} = h^{++(\alpha\beta)(\dot\alpha\dot\beta)} + 2i\left[h^{++(\alpha\beta)(\dot\alpha+}\bar\theta^{-\dot\beta)} - h^{++(\dot\alpha\dot\beta)(\alpha+}\theta^{-\beta)}\right],$$

$$G^{++\alpha\dot\beta} = h^{++\alpha\dot\beta} - 2i\left[h^{++(\alpha\beta)\dot\beta+}\theta^{-}_{\beta} - \bar\theta^{-}_{\dot\alpha} h^{++(\dot\alpha\dot\beta)\alpha+}\right],$$

$$D^{++}G^{--(\alpha\beta)(\dot\alpha\dot\beta)} - D^{--}G^{++(\alpha\beta)(\dot\alpha\dot\beta)} = 0\,, \; D^{++}G^{--\alpha\dot\beta} - D^{--}G^{++\alpha\dot\beta} = 0$$

The actions for higher spins are constructed quite analogously. The on-shell spin contents of $\mathcal{N} = 2$ higher-spin multiplets can be summarized as

$\underline{spin\ 1}: \ 1, (1/2)^2, (0)^2$

$\underline{spin\ 2}: \ 2, (3/2)^2, 1$

$\underline{spin\ 3}: \ 3, (5/2)^2, 2$

....

$\underline{spin\ s}: \ s, (s-1/2)^2, s-1$

Each spin enters the direct sum of these multiplets twice, in accord with the general Vasiliev theory of $4D$ higher spins. The off-shell contents of the spin **s** multiplet is described by the formula $8[\mathbf{s}^2 + (\mathbf{s}-\mathbf{1})^2]_B + 8[\mathbf{s}^2 + (\mathbf{s}-\mathbf{1})^2]_F$.

Superconformal couplings

- Free conformal higher-spin actions in $4D$ Minkowski space were pioneered by Fradkin & Tseytlin, 1985; Fradkin & Linetsky, 1989, 1991. Since then, a lot of works on (super)conformal higher spins appeared (e.g., Segal, 2003, Kuzenko *et al*, 2017, 2023 ......).
- (Super)conformal higher-spin theories are considered as a basis for all other types of higher-spin models. Non-conformal ones follow from the superconformal ones through couplings to the superfield compensators.
- In (Buchbinder, Ivanov, Zaigraev, arXiv:2404.19016 [hep-th]), we extended the off-shell $\mathcal{N}=2, 4D$ higher spins and their hypermultiplet couplings to the superconformal case. Rigid $\mathcal{N}=2, 4D$ superconformal symmetry plays a crucial role in fixing the structure of the theory.
- $\mathcal{N}=2, 4D$ SCA preserves harmonic analyticity and is a closure of the rigid $\mathcal{N}=2$ supersymmetry and special conformal symmetry

$$\delta_\epsilon \theta^{+\hat{\alpha}} = \epsilon^{\hat{\alpha}i} u^+_i\,,\ \delta_\epsilon x^{\alpha\dot{\alpha}} = -4i\left(\epsilon^{\alpha i}\bar{\theta}^{+\dot{\alpha}} + \theta^{+\alpha}\bar{\epsilon}^{\dot{\alpha}i}\right)u^-_i\,, \hat{\alpha} = (\alpha, \dot{\alpha})\,,$$

$$\delta_k \theta^{+\alpha} = x^{\alpha\dot{\beta}} k_{\beta\dot{\beta}} \theta^{\beta}\,,\ \delta_k x^{\alpha\dot{\alpha}} = x^{\rho\dot{\alpha}} k_{\rho\dot{\rho}} x^{\dot{\rho}\alpha}\,,\ \delta_k u^{+i} = (4i\theta^{+\alpha}\bar{\theta}^{+\dot{\alpha}} k_{\alpha\dot{\alpha}}) u^{-i}$$

- What about the conformal properties of various analytic higher-spin potentials? No problems with the spin **1** potential V^{++}:

$$\delta_{sc} V^{++} = -\hat{\Lambda}_{sc} V^{++}\,, \quad \hat{\Lambda}_{sc} := \lambda^{\alpha\dot{\alpha}}_{sc}\partial_{\alpha\dot{\alpha}} + \lambda^{\hat{\alpha}+}_{sc}\partial_{\hat{\alpha}+} + \lambda^{++}_{sc}\partial^{--}$$

- The cubic vertex $\sim q^{+a}V^{++}Jq^{+}_{a}$ is invariant up to total derivative if

$$\delta_{sk} q^{+a} = -\hat{\Lambda}_{sc} q^{+a} - \frac{1}{2}\Omega q^{+a}\,, \quad \Omega := (-1)^{P(M)}\partial_M \lambda^M$$

Moreover, this vertex is invariant under arbitrary analytic superdiffeomorphisms, $\hat{\Lambda}_{sk} \to \hat{\Lambda}(\zeta)$.

- Situation gets more complicated for **s** ≥ 2. Requiring $\mathcal{N} = 2$ gauge potentials for **s** $= 2$ to be closed under $\mathcal{N} = 2$ SCA necessarily leads to

$$\mathcal{D}^{++} \to \mathcal{D}^{++} + \kappa_2 \hat{\mathcal{H}}^{++}_{(s=2)}\,,$$
$$\hat{\mathcal{H}}^{++}_{(s=2)} := h^{++M}\partial_M = h^{++\alpha\dot{\alpha}}\partial_{\alpha\dot{\alpha}} + h^{++\alpha+}\partial^{-}_{\alpha} + h^{++\dot{\alpha}+}\partial^{-}_{\dot{\alpha}} + h^{(+4)}\partial^{--}$$
$$\delta_{k_{\alpha\dot{\alpha}}} h^{(+4)} = -\hat{\Lambda} h^{(+4)} + 4i h^{++\alpha+}\bar{\theta}^{+\dot{\alpha}} k_{\alpha\dot{\alpha}} + 4i\theta^{+\alpha} h^{++\dot{\alpha}+} k_{\alpha\dot{\alpha}}$$

It is impossible to avoid introducing the extra potential $h^{(+4)}$ for ensuring conformal covariance. The extended set of potentials embodies $\mathcal{N} = 2$ **Weyl multiplet** ($\mathcal{N} = 2$ conformal SG gauge multiplet).

- For $\mathbf{s} \geq 3$ the gauge-covariantization of the free q^{+a} action requires the superfield differential operators having rank $\mathbf{s-1}$ in the derivatives ∂_M,

$$\mathcal{D}^{++} \to \mathcal{D}^{++} + \kappa_s \hat{\mathcal{H}}^{++}_{(s)} (J)^{P(s)}, \quad P(s) = \frac{1+(-1)^{s-1}}{2}$$

- For $\mathbf{s} = 3$:

$$\hat{\mathcal{H}}_{(s=3)} = h^{++MN}\partial_N\partial_M + h^{++}, \quad h^{++MN} = (-1)^{P(M)P(N)} h^{++NM}$$

- $\mathcal{N} = 2$ SCA mixes different entries of h^{++MN}, so we need to take into account all these entries, as distinct from non-conformal case where it was enough to consider, e.g., $h^{++\alpha\dot{\alpha}M}$.
- The spin **3** gauge transformations of q^{+a} and h^{++MN} leaving invariant the action $\sim q^{+a}(D^{++} + \kappa_3 \hat{\mathcal{H}}_{(s=3)})q^+_a$ are

$$\delta^{(s=3)}_\lambda q^{+a} = -\frac{\kappa_3}{2}\{\hat{\Lambda}^M + \frac{1}{2}\Omega^M, \partial_M\}_{AGB} J q^{+a} \equiv -\kappa_3 \hat{\mathcal{U}}_{(s=3)} J q^{+a},$$

$$\delta^{(s=3)}_\lambda \hat{\mathcal{H}}^{++}_{(s=3)} = \left[\mathcal{D}^{++}, \hat{\mathcal{U}}_{(s=3)}\right],$$

$$\hat{\Lambda}^M := \sum_{N \leq M} \lambda^{MN}\partial_N, \; \Omega^M := \sum_{N \leq M} (-1)^{[P(N)+1]P(M)} \partial_N \lambda^{NM},$$

$$\{F_1, F_2\}_{AGB} = [F_1, F_2], \quad \{B_1, B_2\}_{AGB} = \{B_1, B_2\}$$

- All the potentials except $h^{++\alpha\dot{\alpha}M}$ can be put equal to zero using the original extensive gauge freedom:

$$S^{(s=3)}_{int|fixed} = -\frac{\kappa_3}{2}\int d\zeta^{(-4)}\; q^{+a}h^{++\alpha\dot{\alpha}M}\partial_M\partial_{\alpha\dot{\alpha}}Jq^+_a \tag{15}$$

- In such a gauge the superconformal transformations are accompanied by the proper compensating gauge transformations in order to preserve the gauge, so the final SC transformations are nonlinear in $h^{++M\alpha\dot{\alpha}}$.
- Using the linearized gauge transformations of $h^{++\alpha\dot{\alpha}M}$

$$\begin{aligned}
&\delta_\lambda h^{++(\alpha\beta)(\dot{\alpha}\dot{\beta})} = \mathcal{D}^{++}\lambda^{(\alpha\beta)(\dot{\alpha}\dot{\beta})} + 4i\lambda^{+(\alpha\beta)(\dot{\alpha}}\bar{\theta}^{+\dot{\beta})} + 4i\theta^{+(\alpha}\bar{\lambda}^{+\beta)(\dot{\alpha}\dot{\beta})},\\
&\delta_\lambda h^{++(\alpha\beta)\dot{\alpha}+} = \mathcal{D}^{++}\lambda^{+(\alpha\beta)\dot{\alpha}} - \lambda^{++(\alpha\dot{\alpha}}\theta^{+\beta)},\\
&\delta_\lambda h^{++(\dot{\alpha}\dot{\beta})\alpha+} = \mathcal{D}^{++}\lambda^{+(\dot{\alpha}\dot{\beta})\alpha} - \lambda^{++\alpha(\dot{\alpha}}\bar{\theta}^{+\dot{\beta})},\\
&\delta_\lambda h^{(+4)\alpha\dot{\alpha}} = \mathcal{D}^{++}\lambda^{++\alpha\dot{\alpha}} - 4i\bar{\theta}^{+\dot{\alpha}}\lambda^{+\alpha++} + 4i\theta^{+\alpha}\lambda^{+\dot{\alpha}++},
\end{aligned}$$

we can find WZ gauge for the spin 3 gauge supermultiplet

$$\begin{aligned}
h^{++(\alpha\beta)(\dot{\alpha}\dot{\beta})} &= -4i\theta^{+\rho}\bar{\theta}^{+\dot{\rho}}\Phi^{(\alpha\beta)(\dot{\alpha}\dot{\beta})}_{\rho\dot{\rho}} + (\bar{\theta}^+)^2\theta^+\psi^{(\alpha\beta)(\dot{\alpha}\dot{\beta})i}u^-_i\\
&\quad + (\theta^+)^2\bar{\theta}^+\bar{\psi}^{(\alpha\beta)(\dot{\alpha}\dot{\beta})i}u^-_i + (\theta^+)^2(\bar{\theta}^+)^2V^{(\alpha\beta)(\dot{\alpha}\dot{\beta})ij}u^-_i u^-_j\,,\\
h^{++(\alpha\beta)\dot{\alpha}+} &= (\theta^+)^2\bar{\theta}^+_{\dot{\nu}}P^{(\alpha\beta)(\dot{\alpha}\dot{\nu})} + (\bar{\theta}^+)^2\theta^+_{\nu}T^{(\alpha\beta\nu)\dot{\alpha}} + (\theta^+)^4\chi^{(\alpha\beta)\dot{\alpha}i}u^-_i\,,\\
h^{(+4)\alpha\dot{\alpha}} &= (\theta^+)^2(\bar{\theta}^+)^2D^{\alpha\dot{\alpha}}
\end{aligned}$$

- In the **bosonic sector**: the spin $\mathbf{s}=3$ gauge field, $SU(2)$ triplet of conformal gravitons, singlet conformal graviton, spin 1 gauge field and non-standard field which gauges self-dual two-form symmetry:

$$\Phi^{(\alpha\beta\rho)(\dot\alpha\dot\beta\dot\rho)}, \;\; V^{(\alpha\beta)(\dot\alpha\dot\beta)(ij)}, \;\; P^{(\alpha\beta)(\dot\alpha\dot\nu)}, \;\; D^{\alpha\dot\alpha}, \;\; T^{(\alpha\beta\gamma)\dot\alpha}$$

In the **fermionic sector**: conformal spin $5/2$ and spin $3/2$ gauge fields:

$$\psi^{(\alpha\beta\rho)(\dot\alpha\dot\beta)i}, \quad \chi^{(\alpha\beta)\dot\alpha i}$$

- They carry total of $40+40$ off-shell degrees. Starting from $\mathbf{s}=3$, all the component fields are gauge fields, no auxiliary fields are present.

- The sum of conformal spin 2 and spin 3 actions

$$S=-\frac{1}{2}\int d\zeta^{(-4)}\, q^{+a}\left(\mathcal{D}^{++}+\kappa_2\hat{\mathcal{H}}^{++}_{(s=2)}+\kappa_3\hat{\mathcal{H}}^{++}_{(s=3)}J\right)q^+_a$$

is invariant with respect to the (properly modified) spin $\mathbf{3}$ transformations to the leading order in κ_3 and to any order in κ_2. Thus the cubic vertex $(\mathbf{3},\frac{\mathbf{1}}{\mathbf{2}},\frac{\mathbf{1}}{\mathbf{2}})$ is invariant under the gauge transformations of conformal $\mathcal{N}=2$ SG and we obtain the superconformal vertex of the spin $\mathbf{3}$ supermultiplet on *generic* $\mathcal{N}=2$ Weyl SG background.

- The whole consideration can be generalized to the general integer higher-spin s case: $8(2s-1)_B+8(2s-1)_F$ d.o.f. off shell.

Fully consistent higher-spin hypermultiplet coupling

- The superconformal cubic vertices $(\mathbf{s}, \frac{1}{2}, \frac{1}{2})$ are consistent to the leading order in the higher-spin analogs of Einstein constant. In fact, they can be made invariant with respect to gauge transformations also in the next orders.
- Come back to the simplest case of the spin **3** in curved superspace:

$$S_{(s=3)} = -\frac{1}{2}\int d\zeta^{(-4)}\, q^{+a}\left(\mathfrak{D}^{++} + \kappa_3 \hat{\mathcal{H}}^{++}_{(s=3)} J\right) q^+_a$$

It is gauge invariant to the leading order in κ_3. In the next order we have the following spin **3** gauge transformation of the hypermultiplet:

$$\delta^{(s=3)}_\lambda \left(-\frac{\kappa_3}{2} q^{+a}\hat{\mathcal{H}}^{++}_{(s=3)} J q^+_a\right) = -\frac{\kappa_3^2}{4} q^{+a}\left[\hat{\mathcal{H}}^{++}_{(s=3)}, \left\{\hat{\Lambda}^M + \frac{1}{2}\Omega^M, \partial_M\right\}_{AGB}\right] q^+_a$$

We gained the differential operator of the third order in superspace derivatives. It can be compensated (modulo a total derivative) by the proper gauge transformation of the spin **4** superconformal multiplet

$$\kappa_4 \delta^{(s=3)}_\lambda \hat{\mathcal{H}}^{++}_{(s=4)} = -\frac{\kappa_3^2}{4}\left[\hat{\mathcal{H}}^{++}_{(s=3)}, \left\{\hat{\Lambda}^M + \frac{1}{2}\Omega^M, \partial_M\right\}_{AGB}\right]$$

- So the action

$$S_{s=3,4} = -\frac{1}{2}\int d\zeta^{(-4)}\, q^{+a}\left(\mathfrak{D}^{++} + \kappa_3 \hat{\mathcal{H}}^{++}_{(s=3)} J + \kappa_4 \hat{\mathcal{H}}^{++}_{(s=4)}\right) q^+_a$$

respects the spin $\mathbf{s} = 3$ gauge invariance to κ_3^2 order.

- However, the action we started with is not invariant in the $\kappa_3\kappa_4$ order. Then the above procedure should be continued step by step.
- To summarize the procedure, we introduce an analytic differential operator involving **all** integer higher spins:

$$\hat{\mathcal{H}}^{++} := \sum_{s=1}^{\infty} \kappa_s \hat{\mathcal{H}}^{++}_{(s)} (J)^{P(s)}$$

- Then the action of an infinite tower of integer $\mathcal{N} = 2$ superconformal higher spins interacting with the hypermultiplet in an arbitrary $\mathcal{N} = 2$ conformal supergravity background reads:

$$S_{full} = -\frac{1}{2}\int d\zeta^{(-4)}\, q^{+a}\left(\mathcal{D}^{++} + \hat{\mathcal{H}}^{++}\right) q^+_a \qquad (16)$$

- Ascribing the proper gauge transformation to $\hat{\mathcal{H}}^{++}$, one can achieve gauge invariance to any order in the couplings constants. The total hypermultiplet gauge transformation can be written as

$$\delta_\lambda q^{+a} = -\hat{\mathcal{U}}_{hyp} q^{+a} = -\sum_{s=1}^{\infty} \kappa_s \hat{\mathcal{U}}_s \,(J)^{P(s)}\, q^{+a},$$

$$\hat{\mathcal{U}}_s q^{+a} := \sum_{k=s,s-2,\dots} \hat{\mathcal{U}}_s^{(k)} q^{+a}$$

  This transformation acts linearly on the hypermultiplet superfield.

- For the set of gauge superfields we obtain the transformation law:

$$\delta_\lambda \hat{\mathcal{H}}^{++} = \left[\mathcal{D}^{++} + \hat{\mathcal{H}}^{++}, \hat{\mathcal{U}}_{gauge}\right], \qquad \hat{\mathcal{U}}_{gauge} := \sum_{s=1}^{\infty} \kappa_s \hat{\mathcal{U}}_s$$

  It mixes different spins, so it is a non-Abelian deformation of the spin $\mathbf{s}$ transformation laws. In the lowest order, it becomes Abelian and reproduces the sum of transformations of all integer spins $\mathbf{s} \geq 1$.

- The invariance under $\mathcal{N} = 2$ conformal supergravity transformations is automatic. So we have constructed the fully consistent gauge-invariant and conformally invariant interaction of hypermultiplet with an infinite tower of $\mathcal{N} = 2$ higher spins in an arbitrary $\mathcal{N} = 2$ conformal supergravity background.

On AdS background (Ivanov & Zaigraev, in progress)

- It is most interesting to explicitly construct $\mathcal{N}=2$ higher spins in the AdS background, with the superconformal symmetry $SU(2,2|2)$ being broken to the AdS supersymmetry $OSp(2|4;R)$. The latter involves the spinor generators $\Psi^i_\alpha, \Psi^j_{\dot\alpha}$, the Lorentz $SO(1,3)$ generators $L_{(\alpha\beta)}, \bar L_{(\dot\alpha\dot\beta)}$, nonlinear $SO(2,3)/SO(1,3)$ translation generators $R_{\alpha\dot\beta}$ and the internal $SO(2)$ symmetry generator T

$$\{\Psi^i_\alpha, \Psi^j_\beta\} \sim (L, T) \qquad \{\Psi^i_\alpha, \bar\Psi^j_{\dot\beta}\} \sim \varepsilon^{ij} R_{\alpha\dot\beta}$$

- The embedding of the $\mathcal{N}=2$ AdS superalgebra into $SU(2,2|2)$ is realized through the identification (Bandos, Ivanov, Lukierski, Sorokin, 2002)

$$\Psi^i_\alpha = Q^i_\alpha + c^{ik} S_{k\alpha}, \quad \bar\Psi^i_{\dot\alpha} = \overline{\Psi^i_\alpha} = \bar Q_{\dot\alpha i} + c_{ik}\bar S^k_{\dot\alpha},$$
$$c^{ik} = c^{ki} \qquad \overline{c^{ik}} = c_{ik} = \varepsilon_{il}\varepsilon_{kj} c^{lj}$$

- The $SU(2,2|2)$ commutation relations imply for super AdS generators

$$\{\Psi^i_\alpha, \Psi^k_\beta\} = c^{ik} L_{(\alpha\beta)} + 4i\varepsilon_{\alpha\beta}\varepsilon^{ik} T, \quad T := c_{lm} T^{lm}, \quad [J, \Psi^i_\alpha] \sim c^{ik}\Psi_{k\alpha},$$
$$\{\Psi^i_\alpha, \bar\Psi_{\dot\beta k}\} = 2\delta^i_k R_{\alpha\dot\beta}, \quad R_{\alpha\dot\beta} = P_{\alpha\dot\beta} + \frac{1}{2}c^2 K_{\alpha\dot\beta}, \quad c^2 \sim \frac{1}{R^2_{AdS}},$$
$$[R_{\alpha\dot\alpha}, R_{\gamma\dot\gamma}] \sim c^2\left(\varepsilon_{\alpha\gamma} L_{\dot\alpha\dot\gamma} + \varepsilon_{\dot\alpha\dot\gamma} L_{\alpha\gamma}\right), \quad [R_{\alpha\dot\beta}, \Psi^i_\beta] \sim \varepsilon_{\alpha\beta}\bar\Psi^i_{\dot\beta} \ (\text{and c.c.})$$

- The super AdS transformation properties of analytic coordinates:

$$\delta u^{+i} = u^{-i}\big[u^+_k c^{kl}(\epsilon_{\alpha l}\theta^{+\alpha} + \bar\epsilon_{\dot\alpha l}\bar\theta^{+\dot\alpha})\big],$$
$$\delta x^{\alpha\dot\alpha} = -4i\, u^-_i\,\big[\epsilon^{\alpha i}\bar\theta^{+\dot\alpha} + \theta^{+\alpha}\bar\epsilon^{\dot\alpha i} - c^{ik}\big(x^{\alpha\dot\beta}\bar\epsilon_{\dot\beta k}\bar\theta^{+\dot\alpha} + x^{\beta\dot\alpha}\epsilon_{\beta k}\theta^{+\alpha}\big)\big],$$
$$\delta\theta^{+\alpha} = \big(\epsilon^{\alpha i} - x^{\alpha\dot\alpha}c^{ik}\bar\epsilon_{\dot\alpha k}\big)u^+_i - 2i(\theta^+)^2 c^{ki}\epsilon^\alpha_k u^-_i\,,$$
$$\delta\bar\theta^{+\dot\alpha} = \big(\bar\epsilon^{\dot\alpha i} + x^{\alpha\dot\alpha}c^{ik}\epsilon_{\alpha k}\big)u^+_i + 2i(\bar\theta^+)^2 c^{ik}\bar\epsilon^{\dot\alpha}_k u^-_i\,.$$

- The nonlinear AdS translations:

$$\delta x^{\alpha\dot\alpha} = a^{\alpha\dot\alpha} + \frac{1}{2}c^2 a_{\beta\dot\beta}x^{\alpha\dot\beta}x^{\beta\dot\alpha} = a^{\alpha\dot\alpha}\big(1 - \frac{1}{4}c^2x^2\big) + \frac{1}{2}c^2(ax)x^{\alpha\dot\alpha}\,,$$
$$\delta\theta^{+\alpha} = \frac{1}{2}c^2 a_{\beta\dot\alpha}\theta^{+\beta}x^{\alpha\dot\alpha}\,,\quad \delta\bar\theta^{+\dot\alpha} = \frac{1}{2}c^2 a_{\beta\dot\beta}\bar\theta^{+\dot\beta}x^{\beta\dot\alpha}\,,$$
$$\delta u^{+i} = \frac{1}{2}c^2 u^{-i}\, a_{\alpha\dot\alpha}\theta^{+\alpha}\bar\theta^{+\dot\alpha}\,.$$

- The T transformations (with parameter ω) can also be easily found

$$\delta u^{+i} = u^{-i}\, c^{++}\,\omega,\quad \delta x^{\alpha\dot\alpha} = 4ic^{--}\bar\theta^{+\dot\alpha}\theta^{+\alpha}\omega,\quad \delta\theta^{+\alpha} = c^{+-}\theta^{+\alpha}\omega\,,$$
$$c^{\pm\pm} = c^{ik}u^\pm_i u^\pm_k,\quad c^{+-} = c^{ik}u^+_i u^-_k\,.$$

- The first step to constructing off-shell $\mathcal{N}=2$ AdS higher spin theory is to define the super AdS invariant Lagrangian of hypermultiplet, such that it respect no full superconformal invariance, but only the super AdS one. To this end, one needs to define the AdS covariant version of the analyticity-preserving harmonic derivative $\mathcal{D}^{++}$. One way to find it is to require its commutativity with the super AdS generators acting on the analytic harmonic coordinates. Without enering into details, the appropriate $\mathcal{D}^{++}_{AdS}$ acting on $q^{+a}=(q^+,\tilde{q}^+)$ has the structure

$$\mathcal{D}^{++}_{AdS} = \partial^{++} - 4i\hat{\theta}^{+\alpha}\hat{\bar{\theta}}^{+\dot{\alpha}}\nabla_{\alpha\dot{\alpha}} + h^{++}\hat{T} + \mathcal{O}(c)$$

$$\nabla_{\alpha\dot{\alpha}} = (1+c^2x^2)\partial_{\alpha\dot{\alpha}}, \quad h^{++} = i\big[(\hat{\theta}^+)^2 - (\hat{\bar{\theta}}^+)^2\big] + \mathcal{O}(c),$$

$$\hat{T}\,(q^+,\,\tilde{q}^+) = (q^+,\,-\tilde{q}^+)\,,$$

where $\hat{\theta}^+_\alpha, \hat{\bar{\theta}}^+_{\dot{\alpha}})$ are some redefinitions of the original Grassmann coordinates and $\mathcal{O}(c)$ stand for terms vanishing in the limit $c^{ik}\to 0$.
- An extra term $\sim \hat{T}$ in $\mathcal{D}^{++}_{AdS}$ is necessary for breaking superconformal invariance and it produces a mass of q^+ proportional to $1/R^2_{AdS}$. In the properly defined flat limit this term becomes the central charge extension of flat D^{++} and $\hat{T}$ goes just into the derivative ∂_5.
- More details on the AdS invariant q^+ Lagrangians will be given in my work in progress with Nikita Zaigraev (also in the talk by N. Zaigraev today after lunch).

Further prospects

- In the HSS description of $\mathcal{N} = 2$ supersymmetric higher spins there still remains a lot of problems. Some urgent ones are generalizing the theory to AdS type backgrounds (as was stated above) and developing the appropriate quantization methods in $\mathcal{N} = 2$ HSS, like those existing in the case of $\mathcal{N} = 2$ SYM theories.

- How to extend the linearized theory of $\mathcal{N} = 2$ higher spins to its full nonlinear version? The latter is known only for $s \leq 2$ ($\mathcal{N} = 2$ super Yang - Mills and $\mathcal{N} = 2$ supergravities). This problem will seemingly require accounting for ALL higher $\mathcal{N} = 2$ superspins simultaneously (like in the hypermultiplet - higher spins couplings). One can expect the appearance of new interesting supergeometries on this road with many forks.

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- J. Buchbinder, E. Ivanov, N. Zaigraev, *Unconstrained off-shell superfield formulation of $4D, \mathcal{N} = 2$ supersymmetric higher spins*, JHEP 12 (2021) 016; 2109.07639 [hep-th].

THANK YOU FOR ATTENTION!