Rational Ruijsenaars–Schneider model with cosmological constant

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Plan

1. Ruijsenaars–Schneider models
2. Rational Ruijsenaars–Schneider model with a cosmological constant
3. Qualitative dynamics and reductions
4. The issue of integrability
5. Open problems

Poincaré group in $1+1$ in dimensions

Lorentz transformation in $1+1$ dimensions

$$\begin{pmatrix} ct' \\ x' \end{pmatrix} = \begin{pmatrix} \cosh\phi & \sinh\phi \\ \sinh\phi & \cosh\phi \end{pmatrix} \begin{pmatrix} ct \\ x \end{pmatrix}$$

Introducing the rapidity θ which links to the velocity v of a reference frame

$$\phi = \frac{\theta}{c}, \qquad \tanh\frac{\theta}{c} = -\frac{v}{c}$$

one gets the conventional form of the Lorentz boost

$$t' = \frac{t - \frac{v}{c^2}x}{\sqrt{1-\left(\frac{v}{c}\right)^2}}, \qquad x' = \frac{x - vt}{\sqrt{1-\left(\frac{v}{c}\right)^2}}$$

The generator of the infinitesimal Lorentz transformation jointly with temporal and spatial translations obey the Poincaré algebra in $1+1$ dimensions

$$H = \partial_t, \qquad P = \partial_x, \qquad K = \frac{1}{c^2}x\partial_t + t\partial_x,$$

$$[H,P] = 0, \qquad [H,K] = P, \qquad [P,K] = \frac{1}{c^2}H$$

Ruijsenaars–Schneider models

The energy and momentum of a free massive relativistic particle

$$E = \frac{mc^2}{\sqrt{1-\left(\frac{v}{c}\right)^2}} = mc^2\cosh\frac{\theta}{c}, \qquad P = \frac{mv}{\sqrt{1-\left(\frac{v}{c}\right)^2}} = mc\sinh\frac{\theta}{c}$$

Hamiltonian mechanics of N free particles $(\theta \to \frac{p}{m}, \{x_i, p_j\} = \delta_{ij})$

$$H = mc^2\sum_{i=1}^{N}\cosh\left(\frac{p_i}{mc}\right), \qquad P = mc\sum_{i=1}^{N}\sinh\left(\frac{p_i}{mc}\right), \qquad K = -m\sum_{i=1}^{N}x_i,$$

$$\{H, P\} = 0, \qquad \{H, K\} = P, \qquad \{P, K\} = \frac{1}{c^2}H$$

Interaction (S. Ruijsenaars, H. Schneider, Annals Phys. 170 (1986) 370)

$$H = mc^2\sum_{i=1}^{N}\cosh\left(\frac{p_i}{mc}\right)\prod_{k\neq i}f(x_i - x_k),$$

$$P = mc\sum_{i=1}^{N}\sinh\left(\frac{p_i}{mc}\right)\prod_{k\neq i}f(x_i - x_k), \qquad K = -m\sum_{i=1}^{N}x_i$$

The structure relations of the Poincaré algebra are satisfied provided $f(x)$ is an even function of its argument and

$$\sum_{i=1}^{N} \partial_i \prod_{k \neq i} f^2(x_i - x_k) = 0$$

A few particular solutions

$$f_r(x) = \left(1 + \frac{g^2}{m^2 c^2 x^2}\right)^{\frac{1}{2}} \qquad \text{rational}$$

$$f_h(x) = \left(1 + \frac{\sin^2\left(\frac{\nu g}{mc}\right)}{\sinh^2(\nu x)}\right)^{\frac{1}{2}} \qquad \text{hyperbolic}$$

$$f_{tr}(x) = \left(1 + \frac{\sinh^2\left(\frac{\nu g}{mc}\right)}{\sin^2(\nu x)}\right)^{\frac{1}{2}} \qquad \text{trigonometric}$$

where g and ν are constants.

Ruijsenaars–Schneider models

In the nonrelativistic limit

$$\lim_{c\to\infty}(H_{rel}-mc^2N)=H_{nr}$$

the Ruijsenaars–Schneider models reduce to the Calogero and Sutherland systems

$$V(x)\sim\sum_{i<j}\frac{g^2}{(x_i-x_j)^2},\ V(x)\sim\sum_{i<j}\frac{g^2}{\sinh^2(x_i-x_j)},\ V(x)\sim\sum_{i<j}\frac{g^2}{\sin^2(x_i-x_j)}$$

By this reason, the former are conventionally regarded as relativistic generalizations of the latter.

The Poincaré algebra can be viewed as a contraction of the (anti) de Sitter algebra (H. Bacry, J. Levy–Leblond, J. Math. Phys. 9 (1968) 1605)

$$[H,P]=\pm\frac{1}{R^2}K,\qquad [H,K]=P,\qquad [P,K]=\frac{1}{c^2}H$$

in which $R\to\infty$. In physics literature, $\pm\frac{1}{c^2R^2}$ is identified with the cosmological constant.

A natural question arises whether the Ruijsenaars–Schneider models can be extended so as to include a cosmological constant.

Rational RS model with a cosmological constant

A free particle realization of the anti de Sitter algebra

$$H = mc^2 F(x) \cosh \frac{p}{mc}, \qquad P = mcF(x) \sinh \frac{p}{mc}, \qquad K = -mx,$$

$$c^2 R^2 F(x) F'(x) - x = 0, \qquad \rightarrow \qquad F(x) = \sqrt{1 + \frac{x^2}{c^2 R^2}}$$

The equation of motion

$$\ddot{x} + \frac{x}{R^2} = 0$$

can alternatively be obtained from the geodesic equations associated with

$$\left(1 + \frac{x^2}{c^2 R^2}\right) \left(c^2 dt^2 - dx^2\right)$$

The latter is prompted by the mass–shell condition

$$\left(\frac{H}{cF(x)}\right)^2 - \left(\frac{P}{F(x)}\right)^2 = m^2 c^2$$

Rational RS model with a cosmological constant

Many–body generalization

$$H = mc^2 \sum_{i=1}^{N} \sqrt{1 + \frac{x_i^2}{c^2 R^2}} \cosh\left(\frac{p_i}{mc}\right) \prod_{k \neq i} f(x_i - x_k),$$

$$P = mc \sum_{i=1}^{N} \sqrt{1 + \frac{x_i^2}{c^2 R^2}} \sinh\left(\frac{p_i}{mc}\right) \prod_{k \neq i} f(x_i - x_k), \quad K = -m \sum_{i=1}^{N} x_i$$

The structure relations of the anti de Sitter algebra yield

$$\sum_{i=1}^{N} \partial_i \prod_{j \neq i} f^2(x_i - x_j) = 0, \qquad \sum_{i=1}^{N} \left(2x_i - \partial_i \left(x_i^2 \prod_{j \neq i} f^2(x_i - x_j)\right)\right) = 0$$

The potentials above can be checked against the additional condition. It is straightforward to verify that only the rational model passes the hurdle.
In the nonrelativistic limit one reproduces the Calogero model in the harmonic trap

$$H_{nr} = \frac{1}{2m} \sum_{i=1}^{N} p_i^2 + \sum_{i<j}^{N} \frac{g^2}{m(x_i - x_j)^2} + \frac{m}{2R^2} \sum_{i=1}^{N} x_i^2$$

Newton–Hooke group and nonrelativistic cosmology

Consider the harmonic oscillator in three spatial dimensions

$$S = \int dt \left(\frac{1}{2} \dot{x}^i \dot{x}^i - \frac{1}{2R^2} x^i x^i \right)$$

A temporal translation $t' = t + a$ along with spatial translations and the Newton–Hooke boosts (one usually assumes that $t << R$)

$$x'^i = x^i + b^i \cos\left(\frac{t}{R}\right) = x^i + b^i \left(1 - \frac{1}{2!}\left(\frac{t}{R}\right)^2 + \dots\right)$$

$$x'^i = x^i + v^i R \sin\left(\frac{t}{R}\right) = x^i + v^i R \left(\frac{t}{R} - \frac{1}{3!}\left(\frac{t}{R}\right)^3 + \dots\right)$$

give rise to the Newton–Hooke algebra

$$[H, P^i] = -\frac{1}{R^2} K^i, \qquad [H, K^i] = P^i, \qquad [P^i, K^j] = 0$$

$$H = \partial_t, \qquad P^i = \cos\left(\frac{t}{R}\right)\partial^i, \qquad K^i = R \sin\left(\frac{t}{R}\right)\partial^i$$

A particle moves along an ellipse, which can be regarded as a free fall in nonrelativistic spacetime with universal cosmological attraction.

Equations of motion

$$\ddot{x}_i = mc \sum_{j \neq i} W(x_i - x_j) \left(\dot{x}_i \dot{x}_j + y_i y_j \right)$$
$$- \left(\frac{x_i}{R^2} - mc^3 \left(1 + \frac{x_i^2}{c^2 R^2} \right) \sum_{j \neq i} W(x_i - x_j) \right) \prod_{k \neq i} f_r^2(x_i - x_k)$$

where

$$y_i = \sqrt{\dot{x}_i^2 + c^2 \left(1 + \frac{x_i^2}{c^2 R^2} \right) \prod_{k \neq i} f_r^2(x_i - x_k)}$$

$$W(x_i - x_j) = \begin{cases} 0 , & i = j \\ \dfrac{1}{mc\,(x_i - x_j) \left(1 + \frac{(mc)^2 (x_i - x_j)^2}{g^2} \right)}, & i \neq j \end{cases}$$

Qualitative dynamics

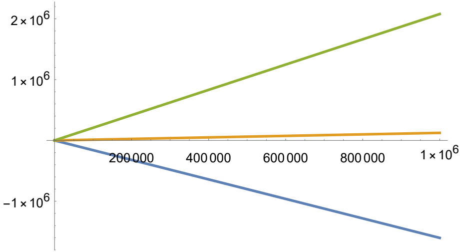

Figure: The graph of x_i versus t for the three–body rational Ruijsenaars–Schneider system with $m = c = g = 1$, $x_1(0) = 1$, $\dot{x}_1(0) = 0.1$ (blue), $x_2(0) = 2$, $\dot{x}_2(0) = 0.2$ (orange), $x_3(0) = 3$, $\dot{x}_3(0) = 0.3$ (green) and $t \in [0, 10^6]$.

Qualitative dynamics

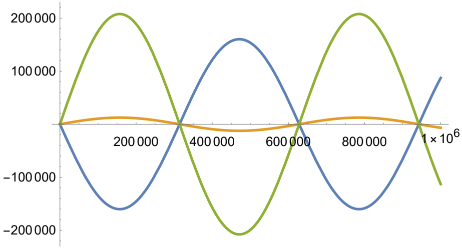

Figure: The graph of x_i versus t for the three–body rational Ruijsenaars–Schneider system with the cosmological constant derived from $R = 10^5$, and $m = c = g = 1$, $x_1(0) = 1$, $\dot{x}_1(0) = 0.1$ (blue), $x_2(0) = 2$, $\dot{x}_2(0) = 0.2$ (orange), $x_3(0) = 3$, $\dot{x}_3(0) = 0.3$ (green), $t \in [0, 10^6]$.

Reduced models

Introducing the subsidiary functions

$$\lambda_i^{\pm} = \left(\frac{mc^2}{2} e^{\pm \frac{p_i}{mc}} \sqrt{1 + \frac{x_i^2}{c^2 R^2}} \prod_{k \neq i} f_r(x_i - x_k) \right)^{\frac{1}{2}}$$

one can represent the Hamiltonian in the manifestly positive–definite form

$$H = \sum_{i=1}^{N} \left(\lambda_i^+ \lambda_i^+ + \lambda_i^- \lambda_i^- \right)$$

The algebra of the subsidiary functions

$$\{\lambda_i^+, \lambda_j^+\} = \frac{1}{2} W(x_i - x_j) \lambda_i^+ \lambda_j^+, \qquad \{\lambda_i^-, \lambda_j^-\} = -\frac{1}{2} W(x_i - x_j) \lambda_i^- \lambda_j^-$$

$$\{\lambda_i^+, \lambda_j^-\} = -\frac{1}{2} \lambda_i^+ \lambda_i^- \left(\frac{1}{mc\left(1 + \frac{x_i^2}{c^2 R^2}\right)} \frac{x_i}{c^2 R^2} - \sum_{k \neq i} W(x_i - x_k) \right) \delta_{ij}$$

A salient feature of the original Ruijsenaars–Schneider systems is that they are integrable. The integrability is maintained if one considers reduced models governed by either $H^+ = \sum_{i=1}^{N} \lambda_i^+ \lambda_i^+$ or $H^- = \sum_{i=1}^{N} \lambda_i^- \lambda_i^-$. In each case, the canonical equations of motion

$$\dot{x}_i = \pm \frac{1}{mc} \left(\lambda_i^\pm\right)^2, \qquad \dot{\lambda}_i^\pm = \pm \sum_{j \neq i} W(x_i - x_j) \lambda_i^\pm \left(\lambda_j^\pm\right)^2$$

can be rewritten in the geodesic form

$$\ddot{x}_i = 2mc \sum_{j \neq i} W(x_i - x_j) \dot{x}_i \dot{x}_j$$

Note that $\dot{x}_i > 0$ for the dynamical system governed by H^+ (right moving modes) and $\dot{x}_i < 0$ for the model defined by H^- (left moving modes). Although such models degenerate to $\ddot{x}_i = 0$ in the nonrelativistic limit, they have been extensively studied in the past. Most notably, they can be put into the Lax form. The explicit dependence on the cosmological constant is absent for the reduced systems, which correlates with the fact that in each respective case one has modes of one and the same type only, their relative motion being insensible to the universal cosmological attraction.

Reduced models originating from the rational, hyperbolic, and trigonometric Ruijsenaars–Schneider systems read

$$\ddot{x}_i = \sum_{j \neq i} W(x_i - x_j)\dot{x}_i \dot{x}_j$$

where $W(x)$ is one of the potentials

$$W(x) = \left[\frac{2g^2}{x(g^2 + x^2)}, \frac{2g^2 \nu \coth \nu x}{g^2 + \sinh^2 \nu x}, \frac{2g^2 \nu \cot \nu x}{g^2 + \sin^2 \nu x} \right]$$

The limits $g \to \infty$ and $g \to 1$ (with $\nu \to \frac{\nu}{2}$) result in the simpler systems

$$W(x) = \left[\frac{2}{x}, \frac{2}{\sin x}, 2 \cot x, \frac{2}{\sinh x}, 2 \coth x \right]$$

which have been extensively studied in the past (F. Calogero, Classical many-body problems amenable to exact treatments, Lecture Notes in Physics: Monographs 66, Springer, 2001).

Integrability of RS models is usually demonstrated by considering the Poisson–commuting set of functions

$$S_1^+ = \sum_{i=1}^{N} e^{\frac{p_i}{mc}} \prod_{j\neq i} f(x_i - x_j),$$

$$S_2^+ = \sum_{i<j}^{N} e^{\frac{p_i}{mc}+\frac{p_j}{mc}} \prod_{k\neq i,j} f(x_i - x_k) f(x_j - x_k),$$

$$S_3^+ = \sum_{i<j<k}^{N} e^{\frac{p_i}{mc}+\frac{p_j}{mc}+\frac{p_k}{mc}} \prod_{l\neq i,j,k} f(x_i - x_l) f(x_j - x_l) f(x_k - x_l), \quad \ldots$$

verifying that S_i^-, $i = 1, \ldots, N$, which follow from S_i^+ by reversing the sign of each p_i, can be algebraically built from S_i^+: $S_i^- = S_{N-i}^+ / S_N^+$, with $S_0^+ = 1$, and finally observing that the RS Hamiltonian is a linear combination of S_1^+ and S_1^-. Because S_i^+ commute under the Poisson bracket, any member of the set can be chosen to define a Hamiltonian of an integrable system of the RS type. In particular, $S_1^{\pm}$ govern the dynamics of the right/left movers discussed above.

The issue of integrability

If a cosmological constant is present, a natural modification of S_i^+, $i = 1, \ldots, N$, reads

$$\mathcal{S}_1^+ = \sum_{i=1}^{N} e^{\frac{p_i}{mc}} \sqrt{1 + \frac{x_i^2}{c^2 R^2}} \prod_{j \neq i} f_r(x_i - x_j),$$

$$\mathcal{S}_2^+ = \sum_{i<j}^{N} e^{\frac{p_i}{mc} + \frac{p_j}{mc}} \sqrt{\left(1 + \frac{x_i^2}{c^2 R^2}\right)\left(1 + \frac{x_j^2}{c^2 R^2}\right)} \prod_{k \neq i,j} f_r(x_i - x_k) f_r(x_j - x_k),$$

$$\mathcal{S}_3^+ = \sum_{i<j<k}^{N} e^{\frac{p_i}{mc} + \frac{p_j}{mc} + \frac{p_k}{mc}} \sqrt{\left(1 + \frac{x_i^2}{c^2 R^2}\right)\left(1 + \frac{x_j^2}{c^2 R^2}\right)\left(1 + \frac{x_k^2}{c^2 R^2}\right)}$$
$$\times \prod_{l \neq i,j,k} f_r(x_i - x_l) f_r(x_j - x_l) f_r(x_k - x_l), \quad \ldots$$

while $\mathcal{S}_i^-$ are obtained by reversing the sign of each p_i.

The issue of integrability

In contrast to the flat case, $\mathcal{S}_i^-$ are no longer expressible in terms of $\mathcal{S}_i^+$

$$\{\mathcal{S}_1^+, \mathcal{S}_1^-\} = -\frac{2}{mc^3 R^2} \sum_{i=1}^{N} x_i$$

Rewriting the Hamiltonian in terms of $\mathcal{S}_1^\pm$

$$H = \frac{mc^2}{2} \left(\mathcal{S}_1^+ + \mathcal{S}_1^-\right)$$

and taking into account the Poisson brackets

$$\{\mathcal{S}_1^\pm, \sum_{i=1}^{n} x_i\} = \mp \frac{1}{mc} \mathcal{S}_1^\pm$$

one can readily construct a conserved quantity

$$I_1 = \mathcal{S}_1^+ \mathcal{S}_1^- - \frac{1}{c^2 R^2} \left(\sum_{i=1}^{n} x_i\right)^2$$

which links to the Casimir invariant of the anti de Sitter algebra.

The issue of integrability

Because the Hamiltonian is symmetric under the interchange of $\mathcal{S}_1^+$ and $\mathcal{S}_1^-$, it seems natural to search for other integrals of motion in the form

$$\mathcal{S}_2^+ + \mathcal{S}_2^- + \ldots, \qquad \mathcal{S}_3^+ + \mathcal{S}_3^- + \ldots, \qquad \text{etc.}$$

where $\ldots$ designate extra contributions needed to ensure the commutativity with H.

It appears that the missing contributions can be built with the use of the elementary monomials in x_i

$$M_1 = \sum_{i=1}^{N} x_i, \qquad M_2 = \sum_{i<j}^{n} x_i x_j, \qquad M_3 = \sum_{i<j<k}^{n} x_i x_j x_k, \qquad \ldots$$

In particular, a direct inspection of the three–body case reveals the following constants of motion

$$I_2 = \mathcal{S}_2^+ + \mathcal{S}_2^- + \frac{2}{c^2 R^2} M_2, \qquad I_3 = \mathcal{S}_3^+ + \mathcal{S}_3^- - \frac{m}{cR^2}\{\mathcal{S}_1^+, M_3\} + \frac{m}{cR^2}\{\mathcal{S}_1^-, M_3\}$$

where the last two terms are compactly written in terms of the Poisson brackets.

1. A transparent algebraic scheme enabling one to build Poisson commuting and functionally independent integrals of motion for the system under consideration remains a challenge.

2. It appears that one has to reconsider the issue of representing the equations of motion of the complete rational Ruijsenaars–Schneider model in flat space (not just the right or left movers) in the Lax form.

3. A matrix model progenitor of the rational Ruijsenaars–Schneider model with a cosmological constant is worth studying.

4. The construction of action–angle variables for the case at hand is an interesting open problem.

Thanks for your attention!