Exact relations in $\mathcal{N} = 1$ supersymmetric theories at the three-loop level

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O.H., K.Stepanyantz arXiv:2501.06500 (2025)

NSVZ relation

An attractive feature of supersymmetric theories is the smaller number of ultraviolet divergences. This is ensured by the existence of non-renormalization theorems, which are closely related to some exact all-loop relations between renormalization group functions (RGFs). One example of such relations is the NSVZ relation,

$$\beta(\alpha,\lambda) = -\frac{\alpha^2\Big(3C_2 - T(R) + C(R)_i{}^j(\gamma_\phi)_j{}^i(\alpha,\lambda)/r\Big)}{2\pi(1 - C_2\alpha/2\pi)},$$

where α and λ are the gauge and Yukawa coupling constants, respectively, and we use the notations

$$\mathrm{tr}\,(T^A T^B) \equiv T(R)\,\delta^{AB}; \qquad (T^A)_i{}^k (T^A)_k{}^j \equiv C(R)_i{}^j;$$
$$f^{ACD} f^{BCD} \equiv C_2 \delta^{AB}; \qquad r \equiv \delta_{AA} = \dim G.$$

V.Novikov, M.A.Shifman, A.Vainshtein, V.I.Zakharov, Nucl.Phys. **B 229** (1983) 381; Phys.Lett. **B 166** (1985) 329; M.A.Shifman, A.I.Vainshtein, Nucl.Phys. **B 277** (1986) 456; D.R.T.Jones, Phys.Lett. **B 123** (1983) 45.

Exact relation between β-functions from NSVZ equations

In theories with multiple gauge couplings, from the NSVZ relations it may follow that the couplings do not run independently. By eliminating anomalous dimensions, in some cases it is possible to obtain equations relating their β-functions to each other.

For example, two all-loop renormalization group invariants (RGIs) were constructed for the Minimal Supersymmetric Standard Model (MSSM) in

D. Rystsov and K. Stepanyantz, Phys. Rev. D **111** (2025) no.1, 016012.

For the case $\mathcal{N} = 1$ SQCD+SQED, the exact relations between running of the gauge couplings were obtained in

A. L. Kataev and K. V. Stepanyantz, arXiv:2410.12070 (2024).

Since the underlying NSVZ relations are scheme dependent, any exact relations between β-functions derived from them are scheme dependent as well.

Scheme dependence of the NSVZ relations

The NSVZ relations are scheme dependent starting from the three-loop approximation for β-functions and from the two-loop approximation for anomalous dimensions. For example, in the $\overline{\text{DR}}$-scheme the NSVZ relations in this and subsequent orders of perturbation theory are no longer satisfied.

I. Jack, D. R. T. Jones and C. G. North, Phys. Lett. B **386** (1996) 138; I. Jack, D. R. T. Jones and C. G. North, Nucl. Phys. B **486** (1997) 479;

It is convenient to use a Higher covariant Derivatives (HD) regularization,

A. A. Slavnov, Nucl. Phys. B **31** (1971) 301; A. A. Slavnov, Theor.Math.Phys. **13** (1972) 1064 [Teor. Mat. Fiz. 13 (1972) 174].

since in this case the NSVZ relations are valid in all loops if RGFs are defined in terms of the bare couplings. For RGFs defined in terms of the renormalized couplings, the NSVZ relations will only hold in a certain class of schemes, such as the HD+MSL scheme, which implies adding only powers of Λ/μ when renormalizing couplings and matter superfields.

A. L. Kataev and K. V. Stepanyantz, Nucl. Phys. B **875** (2013) 459; K. Stepanyantz, Eur. Phys. J. C **80** (2020) no.10, 911.

NSVZ relations for theories with multiple gauge couplings

The same statements about the scheme dependence of the NSVZ relations are also valid for $\mathcal{N}=1$ supersymmetric theories with multiple gauge couplings. In the case where the gauge group is a direct product of n subgroups G_K, each of which is either simple or $U(1)$,

$$G = \prod_K G_K = G_1 \times G_2 \times \ldots \times G_n,$$

It is convenient to numerate with index a the matter superfields ϕ_a that lie in irreducible representations R_{aK} of simple subgroups G_K and have charges q_{aK} with respect to subgroups $U(1)$. They will be renormalized as follows:

$$\phi_a = (\sqrt{Z})_a{}^b \phi_{b,R},$$

where $\phi_{b,R}$ are the renormalized superfields.

NSVZ relations for theories with multiple gauge couplings

β-functions and anomalous dimensions in terms of the bare couplings are defined respectively as

$$\beta_K(\alpha_0,\lambda_0) \equiv \frac{d\alpha_{0K}}{d\ln\Lambda}\Big|_{\alpha,\lambda=\text{const}}; \qquad \gamma_a{}^b(\alpha_0,\lambda_0) \equiv -\frac{d\ln Z_a{}^b}{d\ln\Lambda}\Big|_{\alpha,\lambda=\text{const}},$$

where capital Latin indices numerate subgroups G_K, α_0 and λ_0 denote the bare gauge and Yukawa couplings, respectively, and Λ is an ultraviolet cut-off.

β-functions and anomalous dimensions in terms of the renormalized couplings are defined respectively as

$$\widetilde{\beta}_K(\alpha,\lambda) \equiv \frac{d\alpha_K}{d\ln\mu}\Big|_{\alpha_0,\lambda_0=\text{const}}; \qquad \widetilde{\gamma}_a{}^b(\alpha,\lambda) \equiv \frac{d\ln Z_a{}^b}{d\ln\mu}\Big|_{\alpha_0,\lambda_0=\text{const}},$$

where α and λ denote the renormalized gauge and Yukawa couplings, respectively, and μ is the renormalization point. It should be noted that the latest RGFs depend not only on regularization, but also on the subtraction scheme.

NSVZ relations for theories with multiple gauge couplings

For theories with multiple gauge couplings, the generalization of the NSVZ relations is as follows:

$$\frac{\beta_K(\alpha_0,\lambda_0)}{{\alpha_0}_K{}^2} = -\frac{1}{2\pi(1 - C_2(G_K)\alpha_{0K}/2\pi)}\Big[3C_2(G_K) - \sum_a T_{aK}(1 - {\gamma_a}^a(\alpha_0,\lambda_0))\Big],$$

where RGFs are defined in terms of the bare couplings,

$$C_2(G_K)\delta^{A_K B_K} = f^{A_K C_K D_K} f^{B_K C_K D_K}; \quad T_K(R_{aK})\delta^{A_K B_K} = (T_a^{A_K} T_a^{B_K})_{i_K}{}^{i_K};$$

$$T_{aK} = \begin{cases} \delta_{i_1}{}^{i_1}\cdot\ldots\cdot\delta_{i_{K-1}}{}^{i_{K-1}} T_K(R_{ak})\delta_{i_{K+1}}{}^{i_{K+1}}\cdot\ldots\cdot\delta_{i_n}{}^{i_n}, & G_K \text{ is simple;}\\ \delta_{i_1}{}^{i_1}\cdot\ldots\cdot\delta_{i_{K-1}}{}^{i_{K-1}} q_{aK}^2\delta_{i_{K+1}}{}^{i_{K+1}}\cdot\ldots\cdot\delta_{i_n}{}^{i_n}, & G_K = U(1).\end{cases}$$

M. A. Shifman, Int. J. Mod. Phys. A **11** (1996), 5761; D. Korneev, D. Plotnikov, K. Stepanyantz and N. Tereshina, JHEP **10** (2021), 046.

If general expressions for the anomalous dimensions of matter superfields are known up to the n-th order inclusively, then using these relations one can obtain a general expression for the gauge β-functions in the $n{+}1$ order.

General expression for the two-loop anomalous dimensions

For a theory with a single gauge coupling constant, the general expression for the two-loop anomalous dimension can be written in the form

$$\gamma_i{}^j(\alpha_0,\lambda_0) = -\frac{d\ln Z_i{}^j}{d\ln\Lambda}\Big|_{\alpha,\lambda=\text{const}} = -\frac{\alpha_0}{\pi}C(R)_i{}^j + \frac{1}{4\pi^2}\lambda^*_{0imn}\lambda_0^{jmn}$$
$$+\frac{\alpha_0^2}{2\pi^2}\Big[[C(R)^2]_i{}^j - 3C_2 C(R)_i{}^j\Big(\ln a_\varphi + 1 + \frac{A}{2}\Big) + T(R)C(R)_i{}^j\Big(\ln a + 1 + \frac{A}{2}\Big)\Big]$$
$$-\frac{\alpha_0}{8\pi^3}\lambda^*_{0lmn}\lambda_0^{jmn}C(R)_i{}^l(1-B+A) + \frac{\alpha_0}{4\pi^3}\lambda^*_{0imn}\lambda_0^{jml}C(R)_l{}^n(1+B-A)$$
$$-\frac{1}{16\pi^4}\lambda^*_{0iac}\lambda_0^{jab}\lambda^*_{0bde}\lambda_0^{cde} + O(\alpha_0^3,\alpha_0^2\lambda_0^2,\alpha_0\lambda_0^4,\lambda_0^6),$$

where the regularization parameters a and a_φ are present, which are the proportionality coefficients between the parameter Λ and the masses of the Pauli–Villars superfields,

$$M_\varphi = a_\varphi\Lambda;\quad M = a\Lambda,$$

as well as regularization parameters

$$A \equiv \int\limits_0^\infty dx\,\ln x\,\frac{d}{dx}\Big(\frac{1}{R(x)}\Big);\quad B \equiv \int\limits_0^\infty dx\,\ln x\,\frac{d}{dx}\Big(\frac{1}{F^2(x)}\Big).$$

A. Kazantsev and K. Stepanyantz, JHEP **06** (2020) 108.

General expression for the two-loop anomalous dimensions

The generalization of this expression for the case of a theory with several coupling constants can be presented in the form

$$\gamma_a{}^b(\alpha_0,\lambda_0) = -\frac{d\ln Z_a{}^b}{d\ln\Lambda}\Big|_{\alpha,\lambda=\text{const}} = -\sum_K \frac{\alpha_{0K}}{\pi} C(R_{aK})\delta_a{}^b + \frac{1}{4\pi^2}(\lambda_0^*\lambda_0)_a{}^b$$

$$+\sum_{KL}\frac{\alpha_{0K}\alpha_{0L}}{2\pi^2}C(R_{aK})C(R_{aL})\delta_a{}^b - \sum_K \frac{3\alpha_{0K}^2}{2\pi^2}C_2(G_K)C(R_{aK})(\ln a_{\varphi,K}+1+\frac{A}{2})\delta_a{}^b$$

$$+\sum_K \frac{\alpha_{0K}^2}{2\pi^2}C(R_{aK})\sum_c \mathbf{T}_{cK}(\ln a_K+1+\frac{A}{2})\delta_a{}^b - \sum_K \frac{\alpha_{0K}}{8\pi^3}(\lambda_0^*\lambda_0)_a{}^b C(R_{aK})$$

$$\times(1-B+A) + \sum_K \frac{\alpha_{0K}}{4\pi^3}(\lambda_0^* C_K\lambda_0)_a{}^b(1+B-A) - \frac{1}{16\pi^4}(\lambda_0^*[\lambda_0^*\lambda_0]\lambda_0)_a{}^b$$

$$+O(\alpha_0^3,\alpha_0^2\lambda_0^2,\alpha_0\lambda_0^4,\lambda_0^6),$$

where we use the notations

$$(\lambda_0^*\lambda_0)_a{}^b\delta_{i_a}{}^{j_b} = \sum_{cd}\lambda^*_{0i_a m_c n_d}\lambda_0^{j_b m_c n_d};$$

$$(\lambda_0^* C_K\lambda_0)_a{}^b\delta_{i_a}{}^{j_b} = \sum_{cd}\lambda^*_{0i_a m_c n_d}C(R_{dK})\lambda_0^{j_b m_c n_d};$$

$$(\lambda_0^*[\lambda_0^*\lambda_0]\lambda_0)_a{}^b\delta_{i_a}{}^{j_b} = \sum_{cdefg}\lambda^*_{0i_a k_e l_f}\lambda_0^{j_b k_e p_g}\lambda^*_{0p_g m_c n_d}\lambda_0^{l_f m_c n_d}.$$

General expression for the tree-loop β-functions

Substituting the general expression for the two-loop anomalous dimensions into the NSVZ relations, one can obtain a general expression for three-loop β-functions for $\mathcal{N}=1$ supersymmetric theories with multiple gauge couplings,

$$\frac{\beta_K(\alpha_0,\lambda_0)}{\alpha_{0K}^2} = -\frac{1}{2\pi}\Big\{ -Q_K - \frac{\alpha_{0K}}{2\pi}C_2(G_K)Q_K - \sum_a\sum_L \frac{\alpha_{0L}}{\pi}T_{aK}C(R_{aL})$$
$$+\frac{1}{4\pi^2}\sum_{abc}T_{aK}\lambda^*_{0i_am_bn_c}\lambda_0^{i_am_bn_c} - \sum_a\sum_L\frac{\alpha_{0K}\alpha_{0L}}{2\pi^2}T_{aK}C_2(G_K)C(R_{aL})$$
$$-\frac{\alpha_{0K}^2}{4\pi^2}C_2^2(G_K)Q_K - \sum_a\sum_L\frac{\alpha_{0L}^2}{2\pi^2}T_{aK}C(R_{aL})\Big(3C_2(G_L)\ln a_{\varphi,L} - \sum_b T_{bL}\ln a_L$$
$$-Q_L\Big(1+\frac{A}{2}\Big)\Big) + \sum_a\sum_{MN}\frac{\alpha_{0M}\alpha_{0N}}{2\pi^2}T_{aK}C(R_{aM})C(R_{aN}) - \sum_{abc}\sum_L\frac{\alpha_{0L}}{8\pi^3}T_{aK}\times$$
$$\times C(R_{aL})\lambda^*_{0i_am_bn_c}\lambda_0^{i_am_bn_c}\Big(1+A-B\Big) + \sum_{abc}\sum_L\frac{\alpha_{0L}}{4\pi^3}T_{aK}\lambda^*_{0i_am_bn_c}C(R_{cL})\times$$
$$\times\lambda_0^{i_am_bn_c}\Big(1+B-A\Big) + \sum_{abc}\frac{\alpha_{0K}}{8\pi^3}T_{aK}C_2(G_K)\lambda^*_{0i_am_bn_c}\lambda_0^{i_am_bn_c}$$
$$-\frac{1}{16\pi^4}\sum_{abcdef}T_{aK}\lambda^*_{0i_am_bn_c}\lambda_0^{i_am_bp_d}\lambda^*_{0p_dk_el_f}\lambda_0^{n_ck_el_f}\Big\} + O\Big(\alpha_0^3,\alpha_0^2\lambda_0^2,\alpha_0\lambda_0^4,\lambda_0^6\Big).$$

General expression for the tree-loop β-functions

Using the renormalization group equations and the expression for the β-functions in terms of the bare couplings, it is possible to obtain an expression for the same β-functions in terms of the renormalized couplings. In the special case where the Yukawa terms are absent, it has the form

$$\frac{\widetilde{\beta}_K(\alpha)}{\alpha_K^2} = -\frac{1}{2\pi}\Big\{ -Q_K - \frac{\alpha_K}{2\pi}C_2(G_K)Q_K - \sum_{a,L}\frac{\alpha_L}{\pi}T_{aK}C(R_{aL})$$
$$-\sum_{a,L}\frac{\alpha_K\alpha_L}{2\pi^2}T_{aK}C_2(G_K)C(R_{aL}) - \frac{\alpha_K^2}{4\pi^2}C_2(G_K)Q_K \times$$
$$\times\Big(C_2(G_K) + b_{2,K} - b_{1,K}\Big) + \sum_{a,M,N}\frac{\alpha_M\alpha_N}{2\pi^2}T_{aK}C(R_{aM})C(R_{aN})$$
$$-\sum_{a,L}\frac{\alpha_L^2}{2\pi^2}T_{aK}C(R_{aL})\Big[3C_2(G_L)\ln a_{\varphi,L} - \sum_b T_{bL}\ln a_L$$
$$-b_{1,L} - Q_L\Big(b_{2,KL} + 1 + \frac{A}{2}\Big)\Big]\Big\} + O(\alpha^3),$$

where where $Q_K = \sum_a T_{aK} - 3C_2(G_K)$ and $b_{1,K}$, $b_{2,K}$ and $b_{2,KL}$ are the integration constants that determine a renormalization prescription.

The aim of our study is to perform a three-loop verification of the exact relations between running of the gauge couplings obtained for $\mathcal{N}=1$ SQCD+SQED.

This model has the gauge group $G\times U(1)$ and is described by the action

$$S=\frac{1}{2g^2}\,\mathrm{Re}\,\mathrm{tr}\int d^4x\,d^2\theta\,W^aW_a+\frac{1}{4e^2}\,\mathrm{Re}\int d^4x\,d^2\theta\,\boldsymbol{W}^a\boldsymbol{W}_a$$
$$+\sum_{\mathsf{a}=1}^{N_f}\frac{1}{4}\int d^4x\,d^4\theta\,\Big(\phi_{\mathsf{a}}^{+}e^{2V+2q_{\mathsf{a}}\boldsymbol{V}}\phi_{\mathsf{a}}+\widetilde{\phi}_{\mathsf{a}}^{+}e^{-2V^T-2q_{\mathsf{a}}\boldsymbol{V}}\widetilde{\phi}_{\mathsf{a}}\Big),$$

where the gauge superfield V and the coupling constant $\alpha_s\equiv g^2/4\pi$ correspond to the simple subgroup G, and the chiral matter superfields ϕ_{a} and $\widetilde{\phi}_{\mathsf{a}}$ lie in its representations R and $\bar{R}$, respectively. The gauge superfield $\boldsymbol{V}$ and the coupling constant $\alpha\equiv e^2/4\pi$ correspond to the subgroup $U(1)$, and the matter superfields ϕ_{a} and $\widetilde{\phi}_{\mathsf{a}}$ have with respect to it the opposite charges $q_{\mathsf{a}}e$ and $-q_{\mathsf{a}}e$, respectively.

Exact relations between β-functions in $\mathcal{N}=1$ SQCD+SQED

The evolution of the couplings α and α_s is determined respectively by the β-functions

$$\beta(\alpha_s,\alpha) = \frac{d\alpha}{d\ln\mu}\Big|_{\alpha_{s0},\alpha_0=\text{const}}; \qquad \beta_s(\alpha_s,\alpha) = \frac{d\alpha_s}{d\ln\mu}\Big|_{\alpha_{s0},\alpha_0=\text{const}}.$$

In the simplest case where the charges $q_{\text{a}}=1$ for all matter superfields and the representation R is irreducible, the anomalous dimensions are the same for all $\text{a}=1,\ldots,N_f$ and can be eliminated from the NSVZ relations. In this way an exact relation between $\beta(\alpha_s,\alpha)$ and $\beta_s(\alpha_s,\alpha)$ can be obtained,

$$\Big(1-\frac{\alpha_s C_2}{2\pi}\Big)\frac{\beta_s(\alpha_s,\alpha)}{\alpha_s^2} = -\frac{3C_2}{2\pi} + \frac{T(R)}{\dim R}\cdot\frac{\beta(\alpha_s,\alpha)}{\alpha^2}.$$

A. L. Kataev and K. V. Stepanyantz, arXiv:2410.12070 (2024).

If the charges q_{a} are different for different flavors, the relation between the β-functions can no longer be obtained, but in the limit $\alpha\to 0$ there is an equation relating the β-function $\beta_s(\alpha_s)$ to the Adler D-function.

The Adler D-function is defined as

$$D(\alpha_s) \equiv -\frac{3\pi}{2}\frac{d}{d\ln\mu}\left(\frac{1}{\alpha}\right)\Big|_{\alpha_{s0},\alpha_0=\text{const},\ \alpha\to 0} = \frac{3\pi}{2}\lim_{\alpha\to 0}\frac{\beta(\alpha_s,\alpha)}{\alpha^2}.$$

This function encodes quantum corrections to the $U(1)$ coupling constant provided by the interaction corresponding to the subgroup G. The exact NSVZ-like relation for this function was derived in

M. Shifman and K. Stepanyantz, Phys. Rev. Lett. **114** (2015) no.5, 051601;
M. Shifman and K. V. Stepanyantz, Phys. Rev. D **91** (2015), 105008.

By comparing them with the NSVZ relation for a theory with gauge group G, one can eliminate anomalous dimensions and obtain an exact relation

$$\beta_s(\alpha_s) = -\frac{\alpha_s^2}{2\pi(1-C_2\alpha_s/2\pi)}\left[3C_2 - \Big(\sum_{\mathsf{a}=1}^{N_f} q_{\mathsf{a}}^2\Big)^{-1}\cdot\frac{4\,T(R)N_f D(\alpha_s)}{3\dim R}\right].$$

The exact relation between $\beta(\alpha_s,\alpha)$ and $\beta_s(\alpha_s,\alpha)$ holds only when $q_{\mathsf{a}}=1$ for all $\mathsf{a}=1,\ldots,N_f$. Therefore, to verify it, it is necessary to find the three-loop beta functions for $\mathcal{N}=1$ SQCD+SQED for this case.

It is convenient to use the previously obtained general expression for the three-loop β-functions of multi-charged $\mathcal{N}=1$ supersymmetric theories under the HD regularization supplemented by an arbitrary renormalization prescription.

For the β-function encoding running of the U(1) coupling constant this gives

$$\frac{\beta(\alpha_s,\alpha)}{\alpha^2}=\frac{N_f\,\mbox{dim}\,R}{\pi}\Big\{1+\frac{\alpha}{\pi}+\frac{\alpha_s}{\pi}C(R)-\frac{1}{2\pi^2}\Big(\alpha+\alpha_s C(R)\Big)^2 -\frac{\alpha^2}{\pi^2}N_f\,\mbox{dim}\,R\Big(\ln a_1+1+\frac{A}{2}+\widetilde{d}_2-d_1\Big)+\frac{3\alpha_s^2}{2\pi^2}C_2C(R)\Big(\ln a_\varphi+1+\frac{A}{2}+d_2 -b_{11}\Big)-\frac{\alpha_s^2}{\pi^2}N_fC(R)T(R)\Big(\ln a_G+1+\frac{A}{2}+d_2-b_{12}\Big)+O(\alpha_s^3,\alpha_s^2\alpha,\alpha_s\alpha^2,\alpha^3)\Big\},$$

where the set of parameters d_i, b_i, b_{ij} determines a subtraction scheme.

For the β-function corresponding to the coupling constant α_s, the result is as follows:

$$\frac{\beta_s(\alpha_s,\alpha)}{\alpha_s^2} = -\frac{1}{2\pi}\Big(3C_2 - 2N_f T(R)\Big) + \frac{\alpha}{\pi^2}N_f T(R) + \frac{\alpha_s}{4\pi^2}\Big(-3(C_2)^2 + 2N_f C_2 T(R) + 4N_f C(R)T(R)\Big) - \frac{\alpha^2}{\pi^3}(N_f)^2 T(R)\,\text{dim}\,R\Big(\ln a_1 + 1 + \frac{A}{2} + \widetilde{b}_{21} - d_1\Big) - \frac{1}{2\pi^3}N_f T(R)\Big(\alpha + \alpha_s C(R)\Big)^2 + \frac{\alpha\alpha_s}{2\pi^3}N_f C_2 T(R) - \frac{3\alpha_s^2}{8\pi^3}(C_2)^3\Big(1 + 3b_{21} - 3b_{11}\Big) + \frac{\alpha_s^2}{4\pi^3}N_f (C_2)^2 T(R)\Big(1 + 3b_{21} - 3b_{11} + 3b_{22} - 3b_{12}\Big) + \frac{3\alpha_s^2}{2\pi^3}N_f C_2 C(R) \times T(R)\Big(\ln a_\varphi + \frac{4}{3} + \frac{A}{2} + b_{23} - b_{11}\Big) - \frac{\alpha_s^2}{2\pi^3}(N_f)^2 C_2 T(R)^2 (b_{22} - b_{12}) - \frac{\alpha_s^2}{\pi^3}(N_f)^2 C(R) T(R)^2\Big(\ln a_G + 1 + \frac{A}{2} + b_{23} - b_{12}\Big) + O(\alpha_s^3, \alpha_s^2\alpha, \alpha_s\alpha^2, \alpha^3).$$

In the $\overline{\text{DR}}$ scheme, the corresponding result for the β-functions takes the form:

$$\frac{\beta(\alpha_s,\alpha)}{\alpha^2}\Big|_{\overline{\text{DR}}} = \frac{N_f\,\text{dim}\,R}{\pi}\Big\{1+\frac{\alpha}{\pi}+\frac{\alpha_s}{\pi}C(R)-\frac{1}{2\pi^2}\Big(\alpha+\alpha_s C(R)\Big)^2$$
$$-\frac{3\alpha^2}{4\pi^2}N_f\,\text{dim}\,R+\frac{9\alpha_s^2}{8\pi^2}C_2C(R)-\frac{3\alpha_s^2}{4\pi^2}N_f C(R)T(R)+O(\alpha_s^3,\alpha_s^2\alpha,\alpha_s\alpha^2,\alpha^3)\Big\};$$

$$\frac{\beta_s(\alpha_s,\alpha)}{\alpha_s^2}\Big|_{\overline{\text{DR}}} = -\frac{1}{2\pi}\Big(3C_2-2N_fT(R)\Big)+\frac{\alpha}{\pi^2}N_fT(R)+\frac{\alpha_s}{4\pi^2}\Big(-3(C_2)^2$$
$$+2N_fC_2T(R)+4N_fC(R)T(R)\Big)-\frac{3\alpha^2}{4\pi^3}(N_f)^2T(R)\,\text{dim}\,R-\frac{1}{2\pi^3}N_fT(R)\times$$
$$\times\Big(\alpha+\alpha_sC(R)\Big)^2+\frac{\alpha\alpha_s}{2\pi^3}N_fC_2T(R)-\frac{21\alpha_s^2}{32\pi^3}(C_2)^3+\frac{5\alpha_s^2}{8\pi^3}N_f(C_2)^2T(R)$$
$$+\frac{13\alpha_s^2}{8\pi^3}N_fC_2C(R)T(R)-\frac{\alpha_s^2}{8\pi^3}(N_f)^2C_2T(R)^2-\frac{3\alpha_s^2}{4\pi^3}(N_f)^2C(R)T(R)^2$$
$$+O(\alpha_s^3,\alpha_s^2\alpha,\alpha_s\alpha^2,\alpha^3).$$

By substituting the found β-functions into the relation being tested, one can find the constraints that the finite constants d_i, b_i, b_{ij} must satisfy in order for it to remain valid:

$$\widetilde{b}_{21} = \widetilde{d}_2; \qquad b_{21} = b_{11}; \qquad b_{22} = b_{12}; \qquad b_{23} = d_2.$$

From this it is obvious that the exact relation between the β-functions will be satisfied in the HD+MSL scheme for any values of the HD regularization parameters, since in this scheme all finite constants d_i, b_i, b_{ij} are equal to zero.

In the $\overline{\text{DR}}$ scheme, on the contrary, in the considered approximation this relation will not be satisfied, which was to be expected, since starting from the three-loop approximation the NSVZ relations are not satisfied in it.

The "minimal" scheme in which the relation between the β-functions holds

The integration constants d_i, b_i, b_{ij} can be chosen in such a way that the relation between the β-functions will be satisfied and RGFs will take the simplest form. Such a scheme can be obtained by imposing the additional constrains

$$b_{12} = b_{11} + \ln\frac{a_G}{a_\varphi}; \qquad d_2 = b_{11} - \ln a_\varphi - 1 - \frac{A}{2}; \qquad \widetilde{d}_2 = d_1 - \ln a_1 - 1 - \frac{A}{2}.$$

In this "minimal" scheme the β-functions take the form

$$\frac{\beta(\alpha_s,\alpha)}{\alpha^2} = \frac{N_f\,\text{dim}\,R}{\pi}\Big\{1 + \frac{\alpha}{\pi} + \frac{\alpha_s}{\pi}C(R) - \frac{1}{2\pi^2}\Big(\alpha + \alpha_s C(R)\Big)^2$$
$$+O(\alpha_s^3, \alpha_s^2\alpha, \alpha_s\alpha^2, \alpha^3)\Big\};$$

$$\frac{\beta_s(\alpha_s,\alpha)}{\alpha_s^2} = -\frac{1}{2\pi}\Big(3C_2 - 2N_f T(R)\Big) + \Big(1 + \frac{\alpha_s C_2}{2\pi}\Big)\Big[\frac{\alpha}{\pi^2}N_f T(R)$$
$$+\frac{\alpha_s}{4\pi^2}\Big(-3(C_2)^2 + 2N_f C_2 T(R) + 4N_f C(R)T(R)\Big)\Big] - \frac{1}{2\pi^3}N_f T(R)\times$$
$$\times\Big(\alpha + \alpha_s C(R)\Big)^2 + O(\alpha_s^3, \alpha_s^2\alpha, \alpha_s\alpha^2, \alpha^3).$$

Tree-loop verification of the exact relation between $\beta_s(\alpha_s)$ and the Adler D-function

In the case where the charges q_a are different for different flavors, only the exact relation between $\beta_s(\alpha_s)$ and $D(\alpha_s)$ can be written instead of the exact relation between $\beta(\alpha_s, \alpha)$ and $\beta_s(\alpha_s, \alpha)$.

The tree-loop $\beta_s(\alpha_s)$ is the β-function of $\mathcal{N} = 1$ SQCD and can be found from the obtained $\beta_s(\alpha_s, \alpha)$ by taking the limit $\alpha \to 0$,

$$\beta_s(\alpha_s) = -\frac{\alpha_s^2}{2\pi}\Big(3C_2 - 2N_f T(R)\Big) + \frac{\alpha_s^3}{4\pi^2}\Big(- 3(C_2)^2 + 2N_f C_2 T(R) + 4N_f C(R)\, T(R)\Big) + \frac{\alpha_s^4}{8\pi^3}\Big[- 3(C_2)^3\Big(1 + 3b_{21} - 3b_{11}\Big) + 2N_f (C_2)^2\, T(R) \times \times\Big(1 + 3b_{21} - 3b_{12} + 3b_{22} - 3b_{11}\Big) - 4N_f C(R)^2\, T(R) - 4(N_f)^2 C_2 T(R)^2 \times \times\Big(b_{22} - b_{12}\Big) + 4N_f C_2\, C(R)\, T(R)\Big(3\ln a_\varphi + 4 + \frac{3A}{2} + 3b_{23} - 3b_{11}\Big) - 8(N_f)^2 C(R) T(R)^2\Big(\ln a_G + 1 + \frac{A}{2} + b_{23} - b_{12}\Big)\Big] + O(\alpha_s^5).$$

Tree-loop verification of the exact relation between $\beta_s(\alpha_s)$ and the Adler D-function

The tree-loop $D(\alpha_s)$ in the case of using HD regularization supplemented by an arbitrary renormalization prescription was obtained in

A. L. Kataev, A. E. Kazantsev and K. V. Stepanyantz, Nucl. Phys. B **926 (2018), 295.**

In our notations it takes the form

$$D(\alpha_s) = \frac{3}{2}\sum_{\mathsf{a}=1}^{N_f} q_{\mathsf{a}}^2 \,\text{dim}\, R \Big\{ 1 + \frac{\alpha_s}{\pi} C(R) - \frac{\alpha_s^2}{2\pi^2} C(R)^2 + \frac{3\alpha_s^2}{2\pi^2} C_2 C(R) \Big(\ln a_\varphi + 1 + \frac{A}{2} + d_2 - b_{11} \Big) - \frac{\alpha_s^2 N_f}{\pi^2} C(R)\, T(R) \Big(\ln a_G + 1 + \frac{A}{2} + d_2 - b_{12} \Big) + O(\alpha_s^3) \Big\}.$$

The exact relation between $\beta_s(\alpha_s)$ and $D(\alpha_s)$ remains valid under the conditions

$$b_{21} = b_{11}; \qquad b_{22} = b_{12}; \qquad b_{23} = d_2.$$

Again, it can be shown that this exact relation holds in the HD+MSL scheme and is not satisfied in the $\overline{\text{DR}}$ scheme.

The "minimal" scheme in which the relation between $\beta_s(\alpha_s)$ and the Adler D-function holds

Again, it is also possible to impose additional constrains

$$b_{12} = b_{11} + \ln\frac{a_G}{a_\varphi}; \qquad d_2 = b_{11} - \ln a_\varphi - 1 - \frac{A}{2}$$

which finite constants must satisfy together with the conditions under which the exact relation between $\beta_s(\alpha_s)$ and the Adler D-function is valid. This allows to construct a "minimal" scheme in which $\beta_s(\alpha_s)$ and $D(\alpha_s)$ take the simplest form

$$\begin{aligned}
\beta_s(\alpha_s) &= -\frac{\alpha_s^2}{2\pi}\Big(3C_2 - 2N_f T(R)\Big) + \frac{\alpha_s^3}{4\pi^2}\Big(1 + \frac{\alpha_s C_2}{2\pi}\Big)\Big(-3(C_2)^2 \\
&+ 2N_f C_2 T(R) + 4N_f C(R)\, T(R)\Big) - \frac{\alpha_s^4}{2\pi^3} N_f C(R)^2\, T(R) + O(\alpha_s^5); \\
D(\alpha_s) &= \frac{3}{2}\sum_{\mathsf{a}=1}^{N_f} q_{\mathsf{a}}^2 \,\mathrm{dim}\, R\left\{1 + \frac{\alpha_s}{\pi}C(R) - \frac{\alpha_s^2}{2\pi^2}C(R)^2 + O(\alpha_s^3)\right\}.
\end{aligned}$$

Conclusion

Two exact all-loop relations inherent in $\mathcal{N}=1$ SQCD+SQED were verified in the three-loop approximation, where the scheme dependence becomes essential.

For this purpose, a general expression for the three-loop gauge β-functions was obtained for $\mathcal{N}=1$ supersymmetric theories with multiple gauge couplings in the case of using HD regularization supplemented by an arbitrary renormalization prescription. This general expression was obtained for β-functions in terms of the bare couplings and also, in the special case where the Yukawa couplings are absent, in terms of the renormalized couplings.

Using the obtained general expression, the three-loop β-functions for $\mathcal{N}=1$ SQCD+SQED were calculated. Substituting the result into the considered exact relation, a class of schemes in which this relation holds was found.

As it was expected, this class of renormalization prescriptions includes the HD+MSL scheme and does not include the $\overline{\text{DR}}$ scheme.

Thank you for the attention!