Elliptical, hyperbolic, and parabolic dislocations

arXiv:2405.09422

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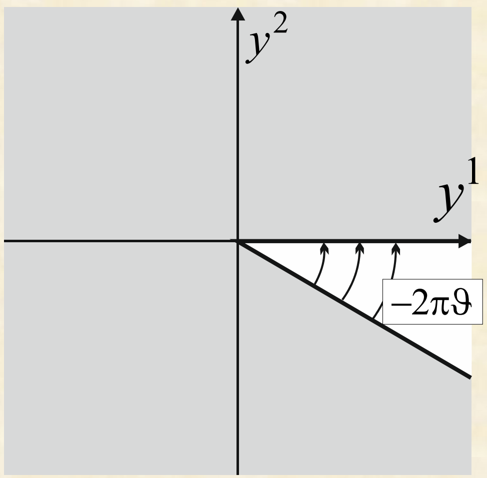
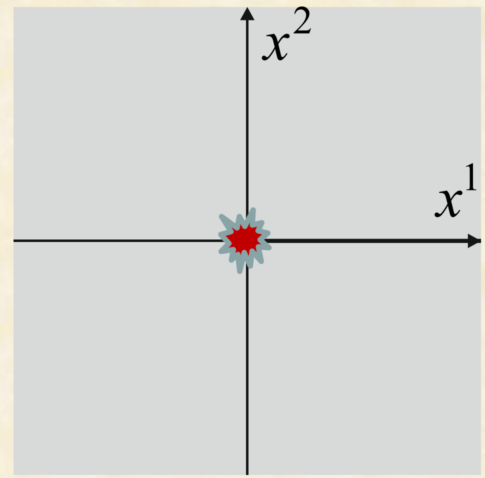
Notation: $(\mathbb{R}^3, \delta_{\alpha\beta}) \ni \ (y^\alpha) \to (x^\alpha) \ \in (\mathbb{M}, g_{\alpha\beta}), \quad \alpha, \beta = 1, 2, 3$

before dislocation is made — dislocation

The wedge dislocation = conical singularity





Metrics: $ds^2 = dr^2 + r^2 d\hat{\varphi}^2$ $\qquad ds^2 = dr^2 + \alpha^2 r^2 d\varphi^2$

$\hat{\varphi} \in (0, 2\alpha\pi) \quad \alpha := 1 + \vartheta, \quad \vartheta$ - deficit angle $\qquad \varphi \in (0, 2\pi), \quad \hat{\varphi} = \alpha\varphi$

Creation of dislocations = the map $y \to x$

The singularity of metric is located at the origin

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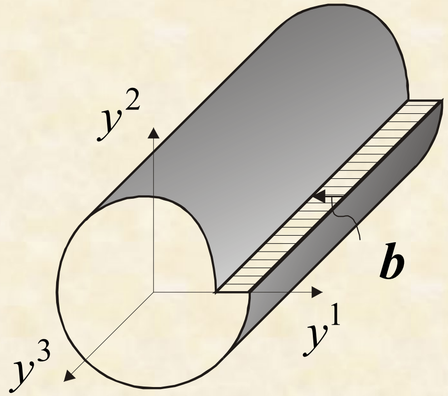
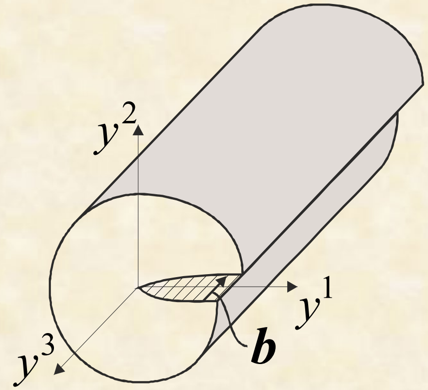
Dislocations

Definition: $\mathbb{M}$ is called a dislocation, if the map $y \to x$ is not continuous

Linear defects:



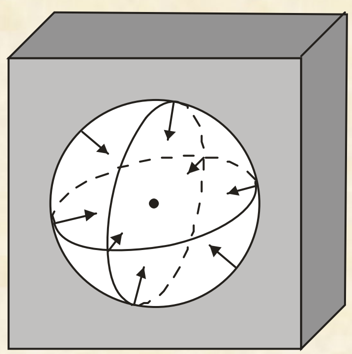
Edge dislocation



Screw dislocation

b - Burgers vector

Point defects:



Vacancy

Geometric theory of defects.

Katanaev, Volovich. Ann.Phys.(1992)
Katanaev. Physics – Uspekhi (2005)

The separable metric

We are looking for point, line, or surface dislocations. Then the manifold $\mathbb{M} \approx \mathbb{R}^3$ must be locally flat $R_{\alpha\beta\gamma\delta}(g) = 0$

Separable metric
$$g_{\alpha\beta} := (\phi_2 + \phi_3) \begin{pmatrix} \dfrac{1}{k_2 + k_3} & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \in [1,2,0] \text{ - class}$$

$$(\phi_2 + \phi_3)(k_2 + k_3) > 0$$

$\phi_2(x^2),\ \phi_3(x^3),\ k_2(x^2),\ k_3(x^3)$ - four arbitrary functions on single arguments

For separable metrics the geodesic Hamilton-Jacobi equation admits complete separation of variables, i.e. the geodesic equations are Liouville integrable. Moreover, variables in the Laplace-Beltrami equation are also separated. Metric $g_{\alpha\beta}$ belongs to class $[1,2,0]$ (1 Killing vector and 2 Killing tensors of second rank) according to the classification proposed in

Katanaev. arXiv (2023); Theor.Math.Phys.(2024)

Main result: all separable locally flat metrics of type $[1,2,0]$ are found and corresponding dislocations are constructed.

Nonzero components of the curvature tensor:

$$R_{1212} = \frac{\phi_2''}{2(k_2+k_3)} - \frac{2(\phi_2+\phi_3)k_2'' + \phi_2' k_2' + \phi_3' k_3'}{4(k_2+k_3)^2} + $$

$$+ \frac{3(\phi_2+\phi_3)k_2'^2}{4(k_2+k_3)^3} - \frac{2\phi_2'^2 - \phi_3'^2}{4(\phi_2+\phi_3)(k_2+k_3)},$$

$$R_{1313} = \frac{\phi_3''}{2(k_2+k_3)} - \frac{2(\phi_2+\phi_3)k_3'' + \phi_2' k_2' + \phi_3' k_3'}{4(k_2+k_3)^2} + $$

$$+ \frac{3(\phi_2+\phi_3)k_2'^2}{4(k_2+k_3)^3} - \frac{2\phi_3'^2 - \phi_2'^2}{4(\phi_2+\phi_3)(k_2+k_3)},$$

$$R_{1213} = \frac{3(\phi_2+\phi_3)k_2'^2}{4(k_2+k_3)^3} - \frac{3\phi_2'\phi_3'}{4(\phi_2+\phi_3)(k_2+k_3)},$$

$$R_{2323} = \frac{1}{2}\left(\phi_2'' + \phi_3'' - \frac{\phi_2'^2 + \phi_3'^2}{\phi_2+\phi_3}\right).$$

The problem is to solve equations $R_{\alpha\beta\gamma\delta} = 0$

A general case

$$R_{2323} = 0 \quad \Leftrightarrow \quad \phi_2''(\phi_2 + \phi_3) + \phi_3''(\phi_2 + \phi_3) - \phi_2'^2 - \phi_3'^2 = 0$$

Differentiate with respect to x^2 and x^3

$$\phi_2'''\phi_3' + \phi_3'''\phi_2' = 0 \quad \Rightarrow \quad \frac{\phi_2'''}{\phi_2'} = -\frac{\phi_3'''}{\phi_3'} = c, \quad c = \text{const}$$

The general solution depends on 4 integration constants r, ω, χ, ψ:

I $\quad \phi_2 = r\cosh(\omega x^2 + \chi), \quad \phi_3 = r\cos(\omega x^3 + \psi) \qquad \phi_2 + \phi_3 \geq 0$

Degenerate cases

II $\quad \phi_2'' = C\,\mathrm{e}^{\lambda x^2}, \quad \phi_3 = 0, \quad C > 0$

III $\quad \phi_2 = a(x^2 - b^2)^2, \quad \phi_3 = a(x^3 - b^3)^2, \quad a > 0, \quad b^2, b^3 \in \mathbb{R}$

IV $\quad \phi_2 = c > 0, \quad \phi_3 = 0$

The general solution

$$R_{1213}=0 \quad \Rightarrow \quad (\phi_2+\phi_3)^2 k_2' k_3' = (k_2+k_3)\phi_2'\phi_3'$$

$$\frac{k_2' k_3'}{\phi_2'\phi_3'}=0$$ - degenerate cases

In general $k_2=\pm\dfrac{E^2}{\phi_2+C}, \quad k_3=\pm\dfrac{E^2}{\phi_3+C}, \quad E\in\mathbb{R}$

$$R_{1212}=R_{1313}=0 \quad \Rightarrow \quad C=\pm r$$

The general solution $k_2=\pm\dfrac{E^2}{\phi_2\pm r}, \quad k_3=\pm\dfrac{E^2}{\phi_3\pm r}$

$$g_{11}^{\pm}=\pm\frac{r^2}{E^2}\left[\cosh(\omega x^2+\chi)\mp 1\right]\left[\cos(\omega x^3+\psi)\pm 1\right]$$

$$g_{22}=g_{33}=r\left[\cosh(\omega x^2+\chi)+\cos(\omega x^3+\psi)\right]$$

Without loss of generality, we set $r=\dfrac{2}{\omega^2}, \quad A:=2r$

Dislocation for g^{+}

Without loss of generality we put $\psi = -\pi$

Introduce new coordinates for g^{+}: $\varphi := \frac{x^1}{E}, \quad u := \frac{\omega x^2 + \chi}{2}, \quad v := \frac{\omega x^3}{2}$

$$ds_{+}^{2} = A^2 \sinh^2 u \, \sin^2 v \, d\varphi^2 + A^2 (\sinh^2 u + \sin^2 v)(du^2 + dv^2)$$

- Euclidean metric in prolate spheroidal coordinates

Denote by $x, y, z \in \mathbb{R}^3$ Cartesian coordinates

Transformation to prolate spheroidal coordinates:

$$x := A \sinh u \sin v \cos\varphi,$$
$$y := A \sinh u \sin v \cos\varphi,$$
$$z := A \cosh u \cos v$$

Dislocation for g^+

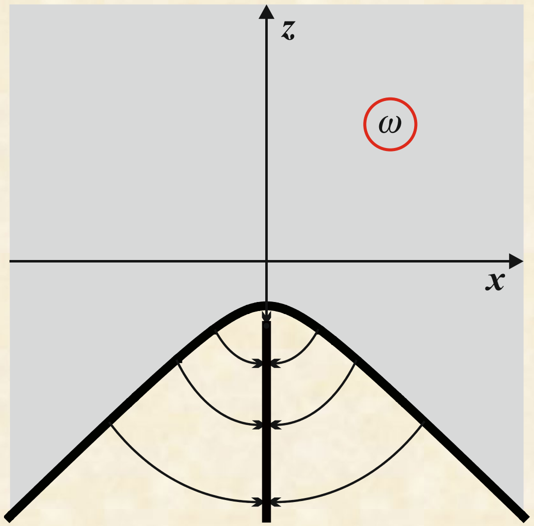
The domain of final coordinates

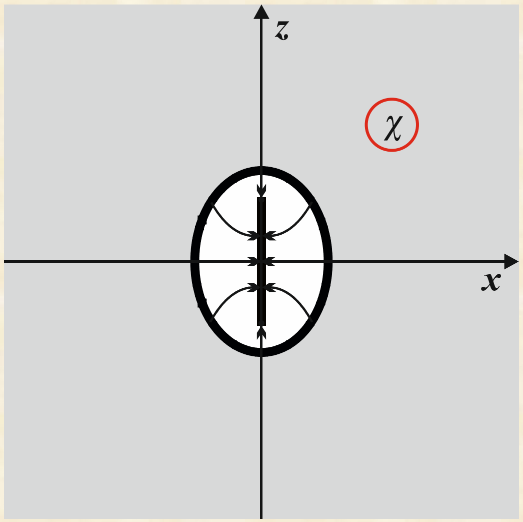
$x^2 \in (0,\infty), \quad x^3 \in (0,\pi), \quad x^1 \in (0,2\pi)$ - to fill the entire $\mathbb{R}^3$

The domain of original coordinates

$u \in \left(\frac{\chi}{2},\infty\right), \quad v \in \left(0,\frac{\omega\pi}{2}\right), \quad \varphi \in \left(0,\frac{2\pi}{E}\right)$

E - wedge dislocation





$$ds_+^2 = \left[\cosh(\omega x^2+\chi)-1\right]\left[1-\cos(\omega x^3)\right]\left(dx^1\right)^2 + \\ +\left[\cosh(\omega x^2+\chi)-\cos(\omega x^3)\right]\left[\left(dx^2\right)^2+\left(dx^3\right)^2\right]$$

$E = A = 1$

Dislocation for g^-

Without loss of generality put $\psi = (2-\omega)\pi / 2$

Introduce new coordinates for g^-: $\varphi := \frac{x^1}{E}, \quad u := \frac{\omega x^2 + \chi}{2}, \quad v := \frac{\omega x^3 + \psi}{2}$

$$ds_-^2 = A^2 \cosh^2 u \, \sin^2 v \, d\varphi^2 + A^2 (\sinh^2 u + \cos^2 v)(du^2 + dv^2)$$

- Euclidean metric in oblate spheroidal coordinates

Denote by $x, y, z \in \mathbb{R}^3$ Cartesian coordinates

Transformation to prolate spheroidal coordinates:

$$x := A \cosh u \sin v \cos\varphi,$$
$$y := A \cosh u \sin v \cos\varphi,$$
$$z := A \sinh u \cos v$$

Dislocation for g^-

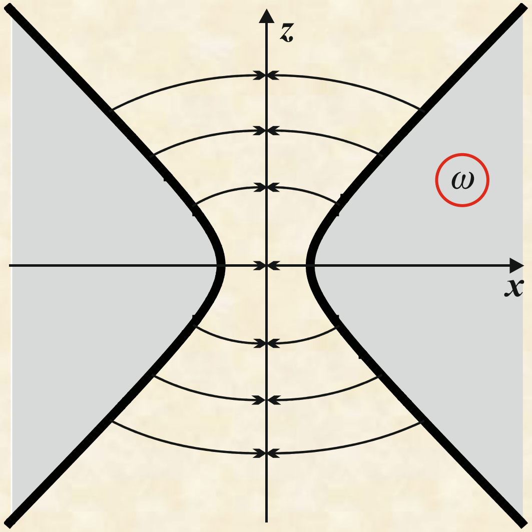
The domain of final coordinates

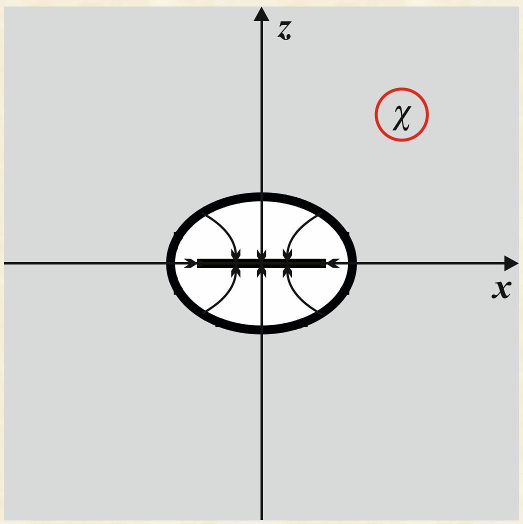
$$x^2 \in (0,\infty), \quad x^3 \in (0,\pi), \quad x^1 \in (0,2\pi)$$ - to fill the entire $\mathbb{R}^3$

The domain of original coordinates

$$u \in \left(\frac{\chi}{2},\infty\right), \quad v \in \left(\frac{(2-\omega)\pi}{4},\frac{(2+\omega)\pi}{4}\right), \quad \varphi \in \left(0,\frac{2\pi}{E}\right)$$

E - wedge dislocation



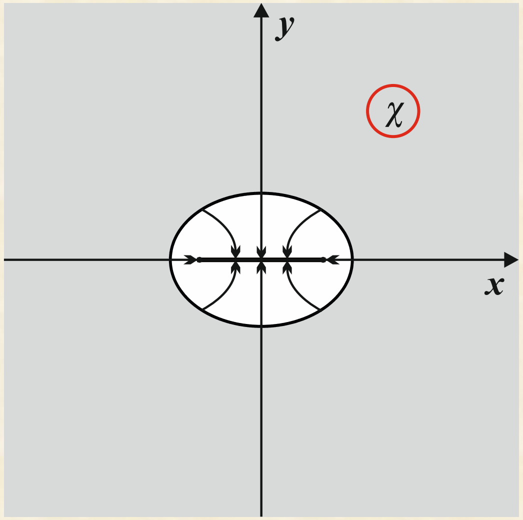


$E = A = 1$

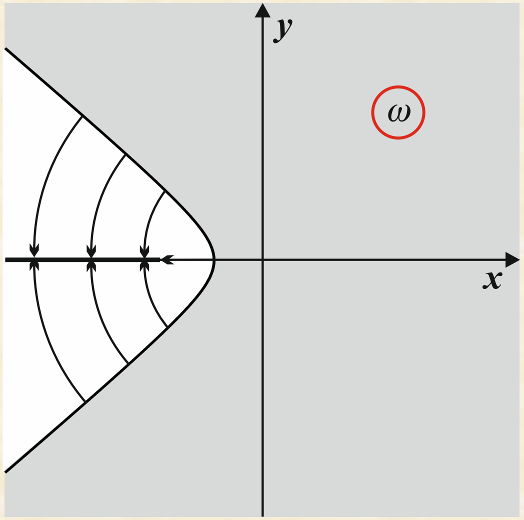
$$ds_+^2 = \left[\cosh(\omega x^2 + \chi) + 1\right]\left[1 + \cos\left(\omega(x^3 - \pi/2)\right)\right]\left(dx^1\right)^2 + \\ + \left[\cosh(\omega x^2 + \chi) - \cos\left(\omega(x^3 - \pi/2)\right)\right]\left[\left(dx^2\right)^2 + \left(dx^3\right)^2\right]$$

Dislocations in degenerate cases

I Elliptic cylinder dislocation



Hyperbolic cylinder dislocation



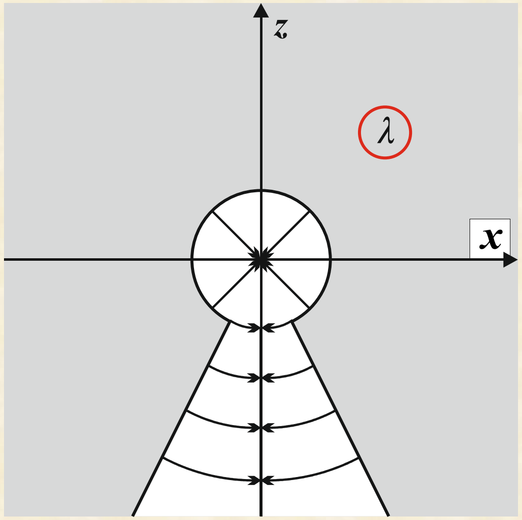
$$ds_{\mathrm{I}}^2 = \left(dy^1\right)^2 + \left[\cosh(\omega y^2 + \chi) + \cos(\omega y^3)\right]\left[\left(dy^2\right)^2 + \left(dy^3\right)^2\right]$$

Here and in what follows, the appearing wedge dislocations are neglected for simplicity.

Dislocations in degenerate cases

IIa

Conical dislocation



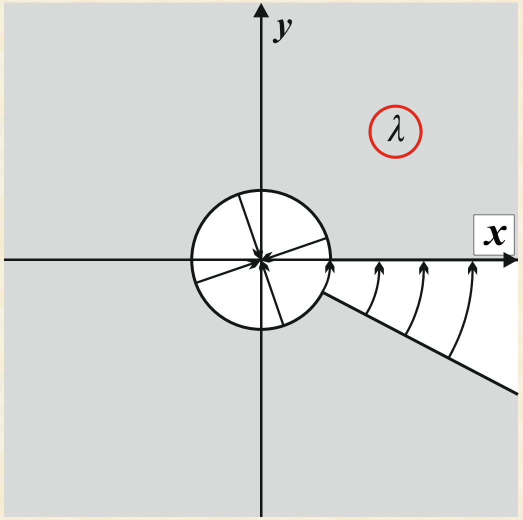
The ball and the round cone
is cut out simultaneously

$$ds^2_{\mathrm{IIa}} = \frac{1}{\lambda^2} \mathrm{e}^{\lambda y^2} \sin^2\left(\frac{\lambda y^3}{2}\right)\left(dy^1\right)^2 + \frac{1}{4}\mathrm{e}^{\lambda x^2}\left[\left(dy^2\right)^2 + \left(dy^3\right)^2\right]$$

Dislocations in degenerate cases

IIb

Wedge dislocation



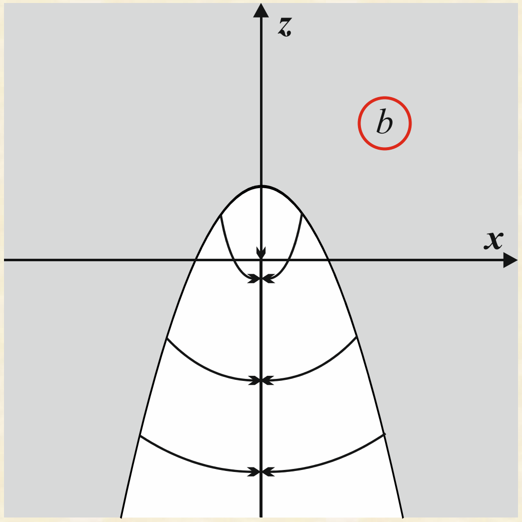
The round cylinder and the wedge is cut out simultaneously

$$ds^2_{\mathrm{IIb}} = \left(dy^1\right)^2 + \mathrm{e}^{\lambda y^2}\left[\left(dy^2\right)^2 + \left(dy^3\right)^2\right]$$

Dislocations in degenerate cases

IIIa

Parabolic dislocation



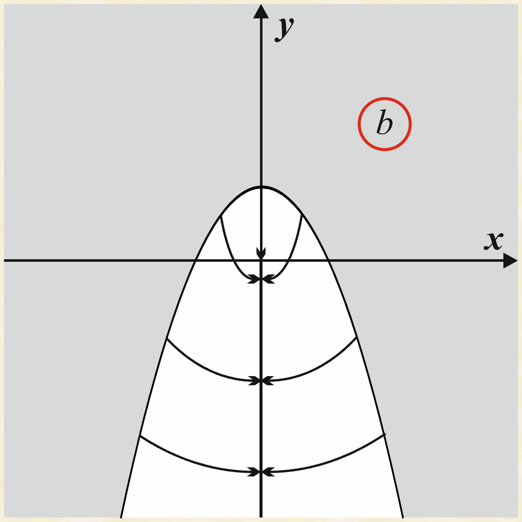
The rotational paraboloid is cut out

$$ds_{\mathrm{IIIa}}^2 = (y^2 - b)^2 \left(y^3 dy^1 \right)^2 + \left[\left(y^2 - b \right)^2 + \left(y^3 \right)^2 \right] \left[\left(dy^2 \right)^2 + \left(dy^3 \right)^2 \right]$$

Dislocations in degenerate cases

IIIb

Parabolic cylinder dislocation



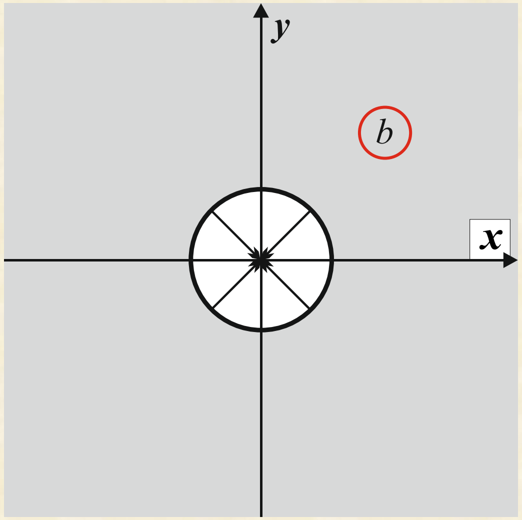
The parabolic cylinder is cut out

$$ds^2_{\mathrm{IIIb}} = \left(dy^1\right)^2 + \left[\left(y^2 - b\right)^2 + \left(y^3\right)^2\right]\left[\left(dy^2\right)^2 + \left(dy^3\right)^2\right]$$

Dislocations in degenerate cases

IV

Cylinder dislocation



The round cylinder is cut out

$$ds_{\mathrm{IV}}^2 = \left(y^2 - b\right)^2 \left(dy^1\right)^2 + \left(dy^2\right)^2 + \left(dy^3\right)^2$$

Conclusion

1) All locally flat separable metrics of class $[1,2,0]$ (one Killing vector and two Killing tensors of second rank are found.
2) These metrics admit complete separation of variables in the geodesic Hamilton-Jacobi and Laplace equations. It means, in particular, that geodesic equations are Liouville integrable.
3) In the geometric theory of defects, these metrics describe the zoo of hyperbolic elliptic and parabolic dislocations along with wedge dislocations.