Tyurin parameters and commuting difference operators of rank 2

Andrey E. Mironov

Sobolev Institute of Mathematics, Novosibirsk

International workshop "Problems of the Modern Mathematical Physics – PMMP'25".
February 10–14, 2025

$$L_n = \partial_x^n + u_{n-1}(x)\partial_x^{n-1} + \ldots + u_0(x), \quad L_m = \partial_x^m + v_{m-1}(x)\partial_x^{m-1} + \ldots + v_0(x)$$

Lemma (Burchnall, Chaundy)

If $L_n L_m = L_m L_n$, then there exist a non–trivial polynomial $Q(z,w)$ of two commuting variables such that $Q(L_n, L_m) = 0$.

Example

$$L_2 = \partial_x^2 - \frac{2}{x^2}, \qquad L_3 = \partial_x^3 - \frac{3}{x^2}\partial_x + \frac{3}{x^3},$$
$$L_2^3 = L_3^2, \quad Q(z,w) = z^3 - w^2.$$

Spectral curve

$$\Gamma = \{(z,w) \in \mathbb{C}^2 : Q(z,w) = 0\}.$$

Если $L_n\psi = z\psi, \quad L_m\psi = w\psi$, то $(z,w) \in \Gamma$.

Rank

$$l = \dim\{\psi : L_n\psi = z\psi, \quad L_m\psi = w\psi\}.$$

Example

$$L_4 = (\partial_x^2 + x^3 + \alpha_2 x^2 + \alpha_1 x + \alpha_0)^2 + g(g+1)x, \quad L_{4g+2}$$

$$\Gamma : w^2 = z^{2g+1} + c_{2g} z^{2g} + \ldots + c_0.$$

Baker – Akhiezer function

Spectral data

$$\{\Gamma, q, k^{-1}, \gamma_1, \dots \gamma_g\}$$

There is a unique function $\psi(x, P),\ P \in \Gamma$ with properties

1. $\psi = e^{kx}(1 + \frac{\xi(x)}{k} + \dots)$

2. On $\Gamma \backslash \{q\}$ ψ has simple poles at the points $\gamma_1, \dots \gamma_g$.

$f(P),\ g(P)$ are meromorphic functions on Γ with poles of order n and m in q. Then

$$L_n \psi = f(P)\psi \qquad L_m \psi = g(P)\psi,$$

$$L_n L_m = L_m L_n.$$

Example

$$\Gamma = \mathbb{C}/\{2w\mathbb{Z} + 2w'\mathbb{Z}\}, \quad q = 0,$$

$$\psi = e^{-x\zeta(z)} \frac{\sigma(z+x)}{\sigma(x)\sigma(z)},$$

$$(\partial_x^2 - 2\wp(x))\psi(x,z) = \wp(z)\psi(x,z),$$

$$(\partial_x^3 - 3\wp(x)\partial_x - \frac{3}{2}\wp'(x))\psi(x,z) = \frac{1}{2}\wp'(z)\psi(x,z).$$

Rank $l > 1$

Spectral data (I.M. Krichever, S.P. Novikov)

$$\{\Gamma, q, k^{-1}, \gamma_1, \ldots, \gamma_{lg}, \alpha_1, \ldots, \alpha_{lg}\}$$

$\alpha_i = (\alpha_{1i}, \ldots, \alpha_{il-1})$ is vector

(γ, α) — Tyurin parameters define a semistable vector bundle of rank l of degree lg on Γ with holomorphic sections $\eta_1, \ldots, \eta_l$

$$\eta_l(\gamma_i) = \sum_{j=1}^{l-1} \alpha_{ij} \eta_j(\gamma_i).$$

Baker-Akhiezer function

$\psi(x,P) = (\psi_0(x,P), \dots, \psi_{l-1}(x,P))$:

1. $\psi(x,P) = \left(\sum_{s=0}^{\infty} \xi_s(x) k^{-s}\right) \Psi_0(x,P)$, $\xi_0 = (1,0,\dots,0)$, $\frac{d}{dx}\Psi_0 = A\Psi_0$,

$$A = \begin{pmatrix} 0 & 1 & 0 & \dots & 0 & 0 \\ 0 & 0 & 1 & \dots & 0 & 0 \\ \dots & \dots & \dots & \dots & \dots & \\ 0 & 0 & 0 & \dots & 0 & 1 \\ k+u_0(x) & u_1(x) & u_2(x) & \dots & u_{l-1}(x) & 0 \end{pmatrix}$$

2. On $\Gamma - \{q\}$ ψ is a meromorphic with simple poles at $\gamma_1, \dots, \gamma_{lg}$

3. $\mathrm{Res}_{\gamma_i}\psi_j = \alpha_{ij}\mathrm{Res}_{\gamma_i}\psi_{l-1}$.

If $f(P)$ is a meromorphic function with a pole at q of order n, then there exist $L(f)$ such that

$$L(f)\psi(x,P) = f(P)\psi(x,P), \quad \mathrm{ord} L(f) = ln.$$

Tyurin parameter deformation method (Krichever–Novikov method)

$$\frac{d^l}{dx^l}\psi_j = \chi_{l-1}\frac{d^{l-1}}{dx^{l-1}}\psi_j + \cdots + \chi_0\psi_j$$

χ_s is meromorphic on Γ, χ_s has lg simple poles $P_1(x), \dots, P_{lg}(x)$. In a neighborhood of q the functions χ_s have the form

$$\chi_0(x, P) = k + g_0(x) + O(k^{-1}),$$

$$\chi_j(x, P) = g_j(x) + O(k^{-1}), \quad j < l-1,$$

$$\chi_{l-1}(x, P) = O(k^{-1}).$$

At point $P_i(x)$

$$\chi_j = \frac{c_{ij}(x)}{k - \gamma_i(x)} + d_{ij}(x) + O(k - \gamma_i(x)).$$

Theorem

Parameters $\gamma_i(x), \alpha_{ij}(x) = \frac{c_{ij}(x)}{c_{i,l-1}(x)}$ and $d_{ij}(x), 0 \leq j \leq l-2, 1 \leq i \leq lg$ satisfy the equation

$$c_{i,l-1}(x) = -\gamma_i'(x),$$

$$d_{i0}(x) = \alpha_{i0}(x)\alpha_{i,l-2}(x) + \alpha_{i0}(x)d_{i,l-1}(x) - \alpha_{i0}'(x),$$

$$d_{ij}(x) = \alpha_{ij}(x)\alpha_{i,l-2}(x) - \alpha_{i,j-1}(x) + \alpha_{ij}(x)d_{i,l-1}(x) - \alpha_{ij}'(x), j \geq 1.$$

I.M. Krichever, S.P. Novikov: $g = 1,\ l = 2$

$\Gamma : \mu^2 = P_3(\lambda) = 4\lambda^3 + g_2\lambda + g_3$

$$L_{KN} = \left(\partial_x^2 + u\right)^2 + 2c_x(\wp(\gamma_2) - \wp(\gamma_1))\partial_x + (c_x(\wp(\gamma_2) - \wp(\gamma_1)))_x - \wp(\gamma_2) - \wp(\gamma_1),$$

$$\gamma_1(x) = \gamma_0 + c(x),\ \gamma_2(x) = \gamma_0 - c(x),$$

$$u(x) = -\frac{1}{4c_x^2} + \frac{1}{2}\frac{c_{xx}^2}{c_x^2} + 2\Phi(\gamma_1, \gamma_2)c_x - \frac{c_{xxx}}{2c_x} + c_x^2(\Phi_c(\gamma_0 + c, \gamma_0 - c) - \Phi^2(\gamma_1, \gamma_2)),$$

$$\Phi(\gamma_1, \gamma_2) = \zeta(\gamma_2 - \gamma_1) + \zeta(\gamma_1) - \zeta(\gamma_2).$$

The sixth order operator $\tilde{L}_{KN}$, which commutes with L_{KN} can be found from the equation $\tilde{L}_{KN}^2 = P_3(L_{KN})$.

J. Dixmier: $g = 1,\ l = 2$

$$L_D = \left(\frac{d^2}{dx^2} - x^3 - \alpha\right)^2 - 2x,$$

$$\tilde{L}_D = \left(\frac{d^2}{dx^2} - x^3 - \alpha\right)^3 - \frac{3}{2}\left(x\left(\frac{d^2}{dx^2} - x^3 - \alpha\right) + \left(\frac{d^2}{dx^2} - x^3 - \alpha\right)x\right).$$

Theorem (P.G. Grinevich)

The commuting operators L_{KN} and $\tilde{L}_{KN}$, corresponding to an elliptic curve have rational coefficients if and only if

$$c(x) = \int_{q(x)}^{\infty} \frac{dt}{\sqrt{P_3(t)}},$$

where $q(t)$ is a rational function.

If $\gamma_0 = 0$ и $q(x) = x$, then $L_{KN} = L_D$.

O.I. Mokhov: $g = 1, \ l = 3$

$l = 2,\ g > 1$: self-adjoint case

Let L be a fourth order operator of rank 2, then

$$\Gamma : w^2 = F(z) = z^{2g+1} + c_{2g} z^{2g} + \cdots + c_0,$$

$q = \infty$ is a branch point,

$$\sigma : \Gamma \to \Gamma, \quad \sigma(z, w) = (z, -w).$$

We have

$$L_4 \psi = z\psi, \qquad L_{4g+2} \psi = w\psi,$$

$$\psi'' = \chi_0 \psi + \chi_1 \psi',$$

where $\psi = (\psi_1, \psi_2)$ is the Baker–Akhiezer function.

Theorem (M.)

If L_4 is a self-adjoint operator

$$L_4 = (\partial_x^2 + V(x))^2 + W(x),$$

then

$$\chi_0 = -\frac{1}{2}\frac{Q_{xx}}{Q} + \frac{w}{Q} - V, \qquad \chi_1 = \frac{Q_x}{Q},$$

where

$$Q = z^g + \alpha_{g-1}(x)z^{g-1} + \cdots + \alpha_0(x).$$

The function Q satisfies the equation

$$4F_g(z) = 4(z - W)Q^2 - 4V(Q_x)^2 + (Q_{xx})^2 - 2Q_xQ_{xxx} +$$

$$2Q(2V_xQ_x + 4VQ_{xx} + Q^{(4)}).$$

Corollary

The function Q satisfies the linear equation

$$\partial_x^5 Q + 4VQ_{xxx} + 6V_xQ_{xx} + 2(2z - 2W + V_{xx})Q_x - 2W_xQ = 0.$$

Examples

1. $L_4^\sharp = (\partial_x^2 + \alpha_3 x^3 + \alpha_2 x^2 + \alpha_1 x + \alpha_0)^2 + g(g+1)\alpha_3 x$,

2. $L_4^\checkmark = (\partial_x^2 + \alpha_1 \cos(x) + \alpha_0)^2 - \alpha_1 g(g+1)\cos(x)$,

Commuting discrete operators

$$L_k = \sum_{j=-N_-}^{N_+} u_j(n) T^j, \qquad L_m = \sum_{j=-M_-}^{M_+} v_j(n) T^j,$$

$$Tf(n) = f(n+1), \qquad f : \mathbb{Z} \to \mathbb{C}.$$

Any commutative ring of discrete operators is isomorphic to a ring of meromorphic functions on a spectral curve with m poles. Such operators are said to be m–**point operators**.

Spectral data for two–point operators of rank 1 were found by I. M. Krichever and D. Mumford. Eigenfunctions of two–point operators of rank 1 (Baker–Akhiezer functions) can be found explicitly in terms of the theta function of the spectral curves. Spectral data for one–point operators of rank $l > 1$ were obtained by I. M. Krichever and S. P Novikov.

One point operators of rank two

Let

$$w^2 = F_g(z) = z^{2g+1} + c_{2g} z^{2g} + c_{2g-1} z^{2g-1} + \ldots + c_0,$$

$$L_4 = \sum_{i=-2}^{2} u_i(n) T^i, \qquad L_{4g+2} = \sum_{i=-(2g+1)}^{2g+1} v_i(n) T^i, \qquad u_2 = v_{2g+1} = 1,$$

herewith

$$L_4 \psi = z\psi, \quad L_{4g+2}\psi = w\psi, \quad \psi = \psi(n, P), \quad P = (z, w) \in \Gamma,$$

$$\psi(n+1, P) = \chi_1(n, P)\psi(n-1, P) + \chi_2(n, P)\psi(n, P).$$

Theorem (I.M. Krichever, S.P. Novikov)

The matrix function $\chi(n, P)$ has simple poles on Γ at points $\gamma_j(n)$. There are relations for the residues of matrix elements

$$\alpha_s^j Res_{\gamma_s(n)} \chi_i(n, P) = \alpha_s^i Res_{\gamma_s(n)} \chi_j(n, P).$$

The points $\gamma_s(n+1)$ are the zeros of the determinant of the matrix $\chi(n, P)$, i.e.

$$\det \chi(n, \gamma_s(n+1)) = 0.$$

The vector $\alpha_j(n+1)$ satisfies the equation

$$\alpha_j(n+1)\chi(n, \gamma_j(n+1)) = 0.$$

Example

$$L = L_2^2 - \wp(\gamma_n) - \wp(\gamma_{n-1}),$$

where L_2 is the Schrodinger difference operator

$$L_2 = T + v_n + c_n T^{-1}$$

with coefficients

$$c_n = \frac{1}{4}(s_{n-1}^2 - 1)F(\gamma_n, \gamma_{n-1})F(\gamma_{n-2}, \gamma_{n-1}),$$

$$v_n = \frac{1}{2}\left(s_{n-1}F(\gamma_n, \gamma_{n-1}) - s_n F(\gamma_{n-1}, \gamma_n)\right),$$

$$F(u, v) = \zeta(u + v) - \zeta(u - v) - 2\zeta(v).$$

Here $\wp(u)$, $\zeta(u)$ are Weierstrass functions, s_n, γ_n are functional parameters.

$$\chi_1(n,P)=\chi_1(n,\sigma(P)), \quad L_4=\left(T+U_n+V_nT^{-1}\right)^2+W_n.$$

Theorem (Mauleshova, M.)

A factorization takes place

$$L_4-z=\left(T+A_n+B_nT^{-1}\right)\left(T-\chi_2(n)-\chi_1(n)T^{-1}\right),$$

where

$$A_n=U_n+U_{n+1}+\chi_2(n+1), \qquad B_n=-\frac{V_nV_{n-1}}{\chi_1(n-1)},$$

$$\chi_1(n)=-\frac{V_nQ_{n+1}}{Q_n}, \qquad \chi_2(n)=\frac{S_n}{Q_n}+\frac{w}{Q_n},$$

$$Q_n=-\frac{S_{n-1}+S_n}{U_{n-1}+U_n}, \quad S_n(z)=-U_nz^g+\delta_{g-1}(n)z^{g-1}+\ldots+\delta_0(n),$$

$\delta_j(n)$ are some functions. The functions $U_n, V_n, W_n, S_n(z)$ satisfy the equation

$$F_g(z)=S_n^2+V_nQ_{n-1}Q_{n+1}+V_{n+1}Q_nQ_{n+2}+(z-U_n^2-V_n-V_{n+1}-W_n)Q_nQ_{n+1}.$$

Example 6

Operator

$$L_4 = (T + a + (r_3 n^3 + r_2 n^2 + r_1 n + r_0) T^{-1})^2 + g(g+1) r_3 n, \quad r_3 \neq 0$$

commutes with some difference operator L_{4g+2}.

Theorem (Mauleshova, M.)

Difference operator

$$L_4^\flat = \big(\frac{T_\varepsilon}{\varepsilon^2} + U(x,\varepsilon) + \varepsilon^2 V(x,\varepsilon) T_\varepsilon^{-1}\big)^2 + W(x,\varepsilon),$$

where

$$U(x,\varepsilon) = -\frac{\nu(x,\varepsilon)+\nu(x+\varepsilon,\varepsilon)}{\gamma(x,\varepsilon)-\gamma(x+\varepsilon,\varepsilon)}, \quad W(x,\varepsilon) = -c_2 - \gamma(x,\varepsilon) - \gamma(x+\varepsilon,\varepsilon),$$

$$V(x,\varepsilon) = \frac{\nu^2(x,\varepsilon) - F_1(\gamma(x,\varepsilon))}{(\gamma(x,\varepsilon)-\gamma(x-\varepsilon,\varepsilon))(\gamma(x+\varepsilon,\varepsilon)-\gamma(x,\varepsilon))},$$

$\gamma(x), \ \nu(x)$ be an arbitrary functional parameters, commutes with the operator

$$L_6^\flat = L_4^\flat\big(\frac{T_\varepsilon}{\varepsilon^2} + U(x,\varepsilon) + \varepsilon^2 V(x,\varepsilon) T_\varepsilon^{-1}\big) -$$

$$\gamma(x+2\varepsilon,\varepsilon)\frac{T_\varepsilon}{\varepsilon^2} - (\nu(x,\varepsilon) + U(x,\varepsilon)\gamma(x,\varepsilon)) - \varepsilon^2 V(x,\varepsilon)\gamma(x-\varepsilon,\varepsilon) T_\varepsilon^{-1}.$$

The spectral curve of the operators $L_4^\flat, L_6^\flat$ is given by the equation

$$w^2 = F_1(z) = z^3 + c_2 z^2 + c_1 z + c_0.$$

Let

$$\gamma(x,\varepsilon) = -\frac{1}{2}(c_2 + \mathcal{W}(x)), \qquad \nu(x,\varepsilon) = \frac{\mathcal{W}_x}{2\varepsilon}.$$

Then the expansions take place

$$L_4^\flat = \mathcal{L}_4 + O(\varepsilon), \qquad L_6^\flat = \mathcal{L}_6 + O(\varepsilon),$$

where

$$\mathcal{L}_4 = (\partial_x^2 + \mathcal{U}(x))^2 + \mathcal{W}(x),$$

$$\mathcal{L}_6 = (\partial_x^2 + \mathcal{U})^3 + \frac{1}{2}(c_2 + 3\mathcal{W})(\partial_x^2 + \mathcal{U}) + \frac{3}{2}\mathcal{W}_x\partial_x + \frac{5}{4}\mathcal{W}_{xx},$$

$$\mathcal{U}(x) = \frac{-16F_1(-(c_2 + \mathcal{W})/2) + \mathcal{W}_{xx}^2 - 2\mathcal{W}_x\mathcal{W}_{xxx}}{4\mathcal{W}_x^2}.$$

Herewith, the spectral curves of the pairs $L_4^\flat,\ L_6^\flat$ and $\mathcal{L}_4,\ \mathcal{L}_6$ coincide.