Regularization and gluing of partition functions

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PMMP — 2025

Content

1) **Introduction**

2) **Problem statement (classical and effective actions)**

3) **Regularization and deformation**

4) **Quasi-locality**

5) **Regularization in context of gluing**

6) **The main statement**

7) **Conclusion, corollaries, and remarks**

Historical background

Quantum field theory (standard) + FQFT (in recent years)

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Perturbative methods + "functional integration"

↑

Divergences

↑

Regularization + renormalization

Why are there so many regularizations?
How do we choose the appropriate one?

Standard QFT:
Bogoliubov, Shirkov (1959)
Faddeev, Slavnov (1978)
Itzykson, Zuber (1980)
Peskin, Schroeder (1995)

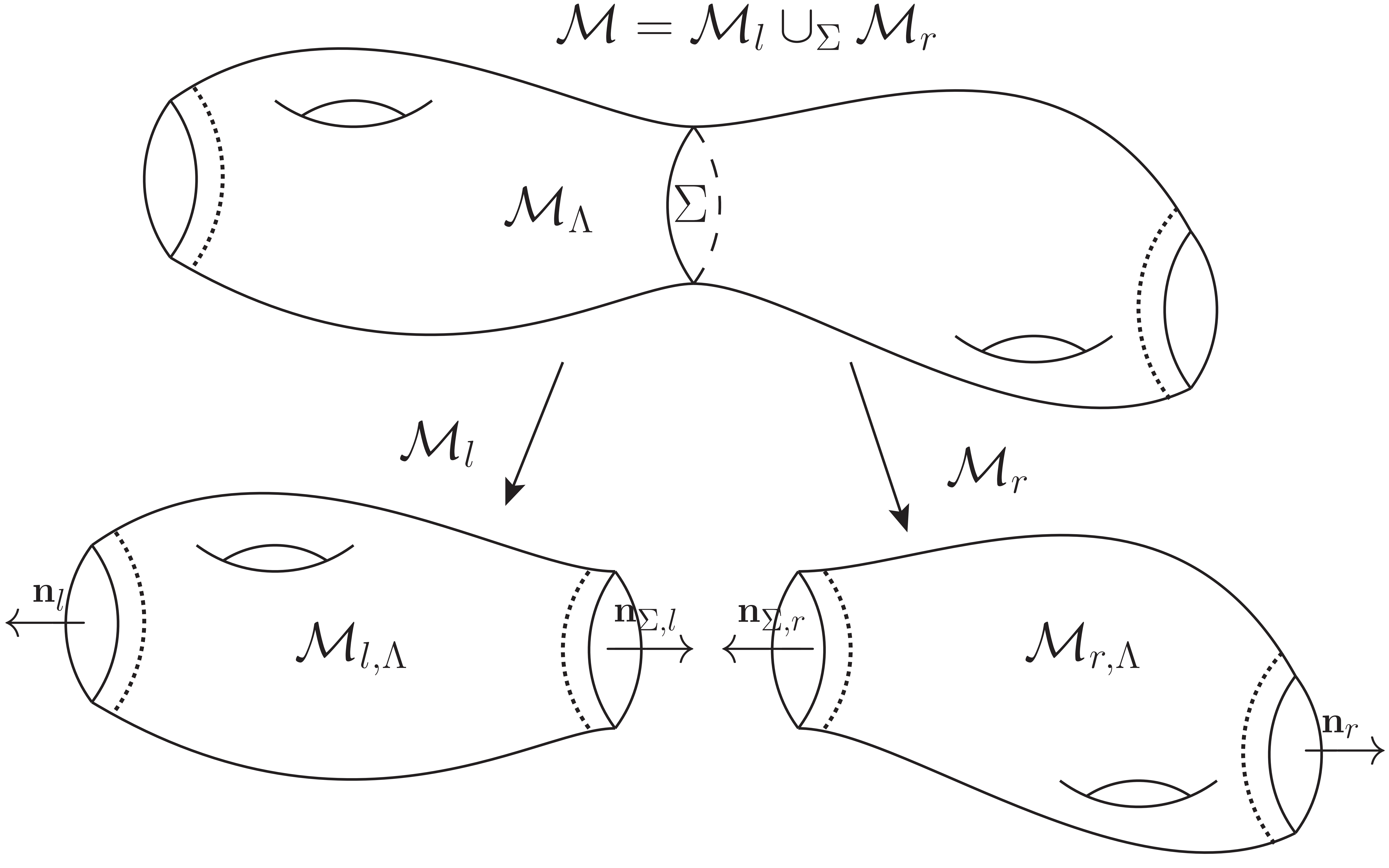
FQFT:
Atiyah (1988)
Segal (2004)
Reshetikhin (2010)
Stolz, Teichner (2011)
Cattaneo, Mnev, Reshetikhin (2018)
Kandel, Mnev, Wernli (2021)

Renormalization:
Faddeev, Slavnov (1978)
Collins (1984)
Zavialov (1990)
Kazakov (2009)

Regularization:
Brizola, Battistel, Sampaio, Nemes (1999)
Cherchiglia and other (2021)
Pauli, Villars (1949)
Bollini, Giambiagi (1972)
't Hooft, Veltman (1972)
Bakeyev, Slavnov (1996)
Stepanyantz (2020)

Cutoff regularization:
Oleszczuk (1994)
Liao (1997)
Bagaev (2008)
Cynolter, Lendvai (2015)
Ivanov, Kharuk (2019—н.в.)

Manifolds


$\mathcal{M} = \mathcal{M}_l \cup_\Sigma \mathcal{M}_r$
$\mathcal{M}_\Lambda$
Σ
$\mathcal{M}_l$
$\mathcal{M}_r$
$\mathbf{n}_l$
$\mathcal{M}_{l,\Lambda}$
$\mathbf{n}_{\Sigma,l}$
$\mathbf{n}_{\Sigma,r}$
$\mathcal{M}_{r,\Lambda}$
$\mathbf{n}_r$

Scalar theory

$$S_0[\phi;\mathscr{M}] = \frac{1}{2}\sum_{\alpha\in I}\int_{V_\alpha} \mathrm{d}^n x\, g_\alpha^{1/2}(x)\left(g_\alpha^{\mu\nu}(x)\left(\partial_{x^\mu}\phi_\alpha(x)\right)\left(\partial_{x^\nu}\phi_\alpha(x)\right) + m^2\phi_\alpha^2(x)\right)\chi_\alpha(\varphi_\alpha^{-1}(x))$$

$$\mathscr{A} = \{(U_\alpha,\varphi_\alpha)\}_{\alpha\in I}$$

$$\varphi_\alpha(U_\alpha) = V_\alpha \subset \mathbb{R}^n$$

$$\sum_{\alpha\in I}\chi_\alpha(p) = 1$$

Examples:
Cubic k=3, n=1,…,6
Quartic k=3,4, n=1,…,4
Sextic k=3,…,6, n=1,…,3

$$S_{\mathrm{int}}[\phi;\mathscr{M}] = \sum_{k=3}^{+\infty}\sum_{\alpha\in I}\int_{V_\alpha} \mathrm{d}^n x\, t_k g_\alpha^{1/2}(x)\phi_\alpha^k(x)\chi_\alpha(\varphi_\alpha^{-1}(x))$$

$S_{\mathrm{cl}}[\,\cdot\,;\mathscr{M}] = S_0[\,\cdot\,,\mathscr{M}] + S_{\mathrm{int}}[\,\cdot\,;\mathscr{M}]$ **is classical action**

Quadratic forms

$i \in \{l, r\}$

$\eta_i \in C^\infty(Y_i, \mathbb{R})$

$\eta_\Sigma \in C^\infty(\Sigma, \mathbb{R})$

$$S_{l,r}(\eta_l, \eta_r; \mathscr{M}) = \int_{Y_l} \mathrm{d}^{n-1}q_1 \int_{Y_r} \mathrm{d}^{n-1}q_2\, \eta_l(q_1)\Big(N_l(q_1)N_r(q_2)G(q_1, q_2)\Big)\eta_r(q_2) = S_{r,l}(\eta_r, \eta_l; \mathscr{M})$$

$$S_{i,\Sigma}(\eta_i, \eta_\Sigma; \mathscr{M}_i) = \int_{Y_i} \mathrm{d}^{n-1}q_1 \int_{\Sigma} \mathrm{d}^{n-1}q_2\, \eta_i(q_1)\Big(N_i(q_1)N_{\Sigma,i}(q_2)G_i(q_1, q_2)\Big)\eta_\Sigma(q_2) = S_{\Sigma,i}(\eta_\Sigma, \eta_i; \mathscr{M}_i)$$

$\eta_i \to \phi^{\eta_i}, \phi_i^{\eta_i}$

$\eta_\Sigma \to \phi_i^{\eta_\Sigma}$

$\downarrow$

$$S_0[\phi + \phi^{\eta_l+\eta_r}, \mathscr{M}] = S_0[\phi, \mathscr{M}] + S_0[\phi^{\eta_l}, \mathscr{M}] + S_0[\phi^{\eta_r}, \mathscr{M}] - S_{l,r}(\eta_l, \eta_r; \mathscr{M})$$

$$S_0[\phi_i + \phi_i^{\eta_i+\eta_\Sigma}, \mathscr{M}_i] = S_0[\phi_i, \mathscr{M}_i] + S_0[\phi_i^{\eta_i}, \mathscr{M}_i] + S_0[\phi_i^{\eta_\Sigma}, \mathscr{M}_i] - S_{i,\Sigma}(\eta_i, \eta_\Sigma; \mathscr{M}_i)$$

Effective action

$$e^{-W_{\text{eff}}[\sqrt{\hbar}\eta;\mathscr{M}]/\hbar} = \mathcal{N}^{-1}(\mathscr{M}) \int_{\mathscr{H}(\sqrt{\hbar}\eta;\mathscr{M})} \mathscr{D}\phi \, e^{-S_{\text{cl}}[\phi;\mathscr{M}]/\hbar}$$

$$W_{\text{eff}}[0;\mathscr{M}]\Big|_{\{t_k=0\}_{k=3}^{+\infty}} = 0$$ **normalization**

$$\phi \to \sqrt{\hbar}\phi^{\eta} + \sqrt{\hbar}\phi$$ **shift**

$$S(\phi+\phi^{\eta};\mathscr{M}) = \exp\Big(-S_{\text{int}}[\sqrt{\hbar}\phi+\sqrt{\hbar}\phi^{\eta};\mathscr{M}]/\hbar\Big) = 1 + \sum_{k\geqslant 3} \hbar^{k/2-1}\left(\prod_{i=1}^{k}\int_{\mathscr{M}} \mathrm{d}^n p_i\right) S_k(p_1,\ldots,p_k)\big(\phi+\phi^{\eta}\big)(p_1)\ldots\big(\phi+\phi^{\eta}\big)(p_k)$$

$$e^{-W_{\text{eff}}[\sqrt{\hbar}\eta;\mathscr{M}]/\hbar} = e^{-S_0[\phi^{\eta};\mathscr{M}]}\left(S(\delta_\psi+\phi^{\eta};\mathscr{M})e^{g(\psi;\mathscr{M})}\right)\Bigg|_{\psi=0}$$

, where $g(\psi;\mathscr{M}) = \frac{1}{2}\int_{\mathscr{M}} \mathrm{d}^n p_1 \int_{\mathscr{M}} \mathrm{d}^n p_1 \psi(p_1) G(p_1,p_2)\psi(p_2)$

Questions

Effective actions contain divergences!

↓

What type of regularization should we use?

We have three effective actions!

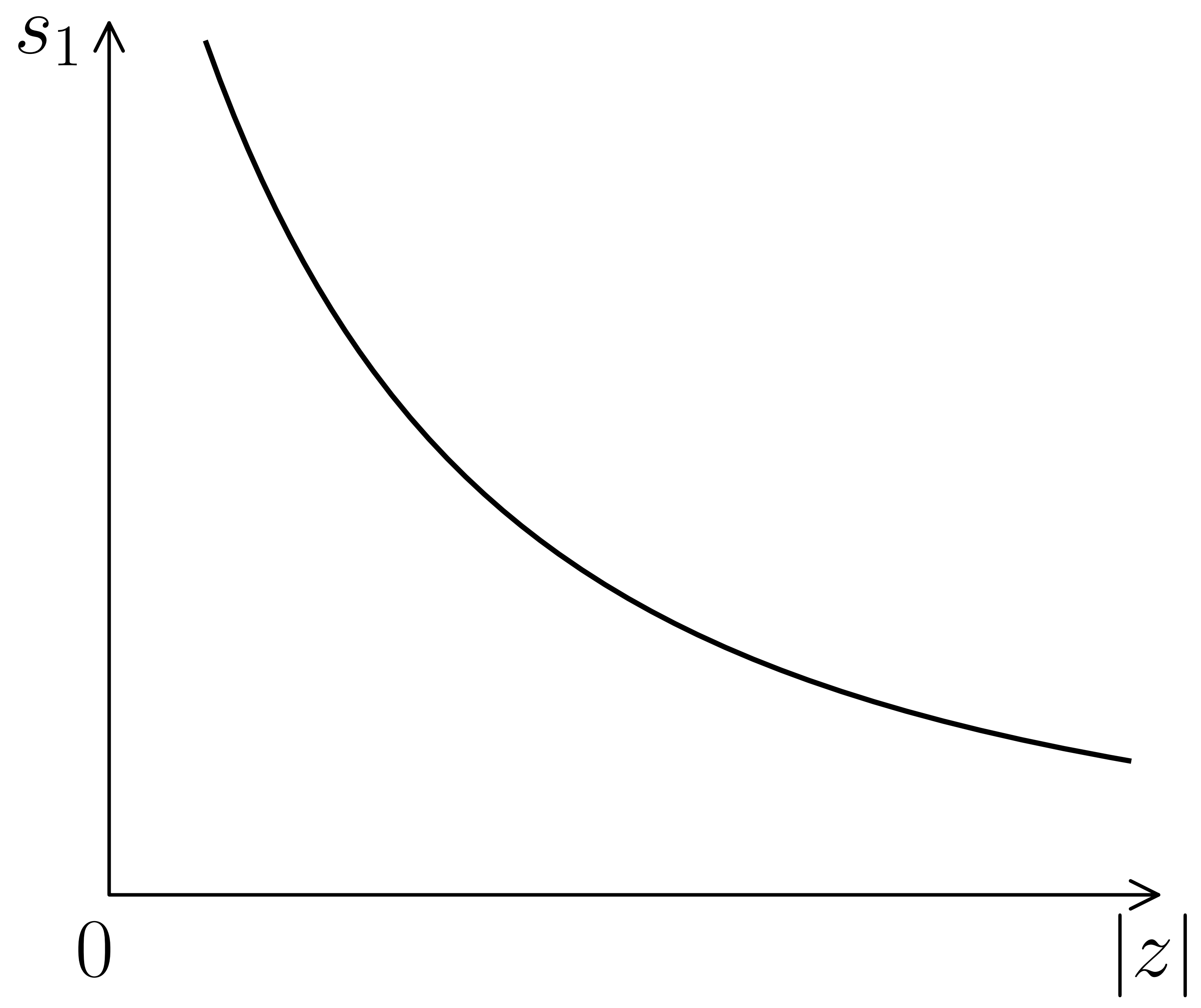
on $\mathscr{M}_l$, $\mathscr{M}_r$, and $\mathscr{M}$

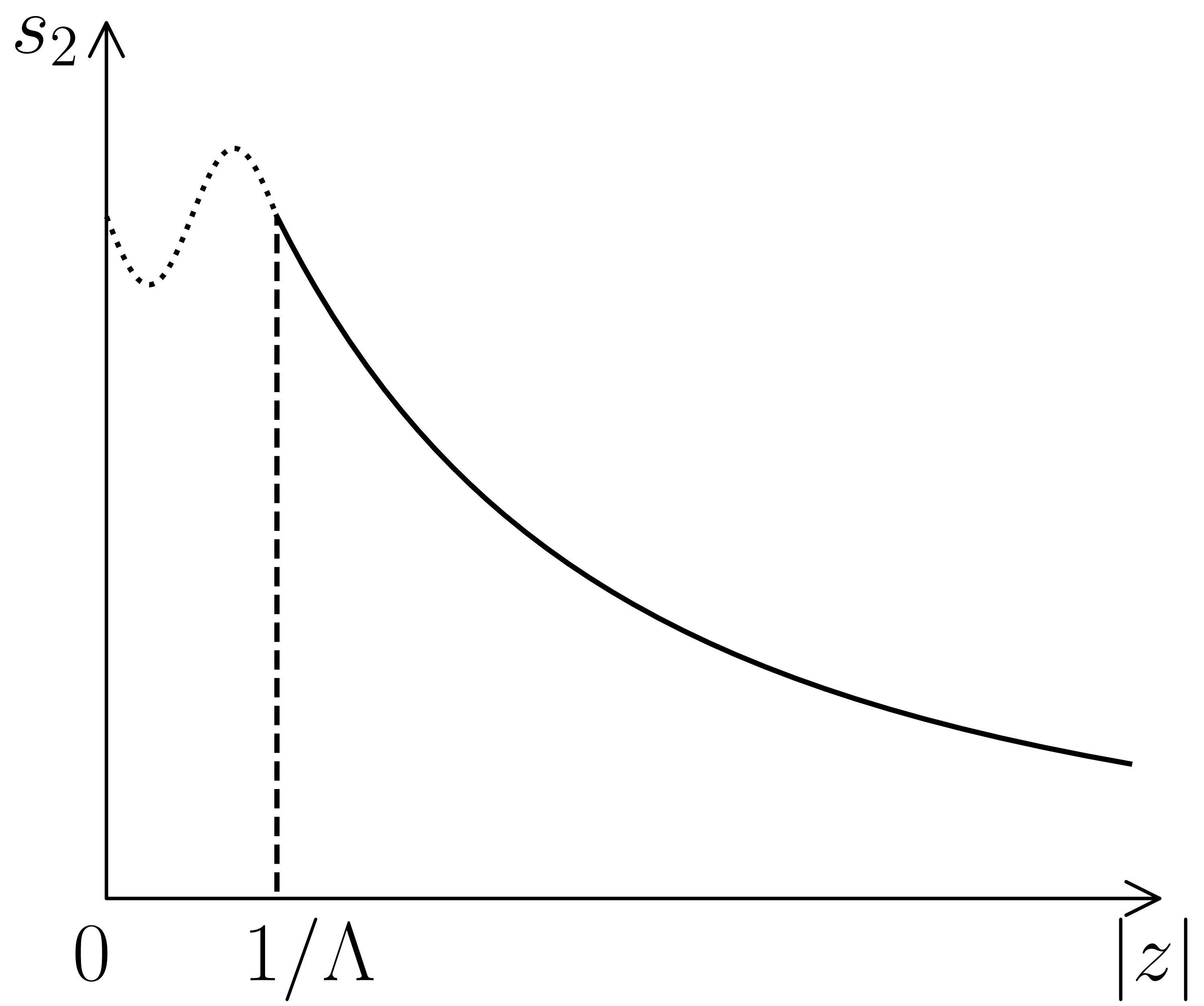
↓

How are they related to each other before and after the regularization process?

Regularization in Euclidean case

$$G(x) = \frac{|x|^{2-n}}{(n-2)S_{n-1}} \longrightarrow G^{\mathbf{f},\Lambda}(x) = \frac{\Lambda^{n-2}}{(n-2)S_{n-1}}\mathbf{f}\left(|x|^2\Lambda^2\right) + \frac{1}{(n-2)S_{n-1}}\begin{cases} \Lambda^{n-2}, & |x| \leqslant 1/\Lambda \\ |x|^{2-n}, & |x| > 1/\Lambda \end{cases}$$





Regularization in Euclidean case

$$\mathrm{H}^{\Lambda}_{\alpha}(\phi)(x) = \int_{\mathbb{S}^{n-1}} \frac{\mathrm{d}^{n-1}\sigma(\hat{x}_k)}{S_{n-1}} \cdots \int_{\mathbb{S}^{n-1}} \frac{\mathrm{d}^{n-1}\sigma(\hat{x}_1)}{S_{n-1}} \phi\left(x + \Lambda^{-1} \sum_{i=1}^{k} \hat{x}_i \alpha_i \right)$$

$$\mathrm{H}^{\Lambda}_{\tilde{\omega},\alpha}(G)(x) = \int_{\mathbb{R}^n} \mathrm{d}^n x_k \, \frac{\omega_k(|x - x_k| \Lambda/\alpha_k)}{(\alpha_k/\Lambda)^n} \cdots \int_{\mathbb{R}^n} \mathrm{d}^n x_1 \, \frac{\omega_1(|x_2 - x_1| \Lambda/\alpha_1)}{(\alpha_1/\Lambda)^n} G(x_1)$$

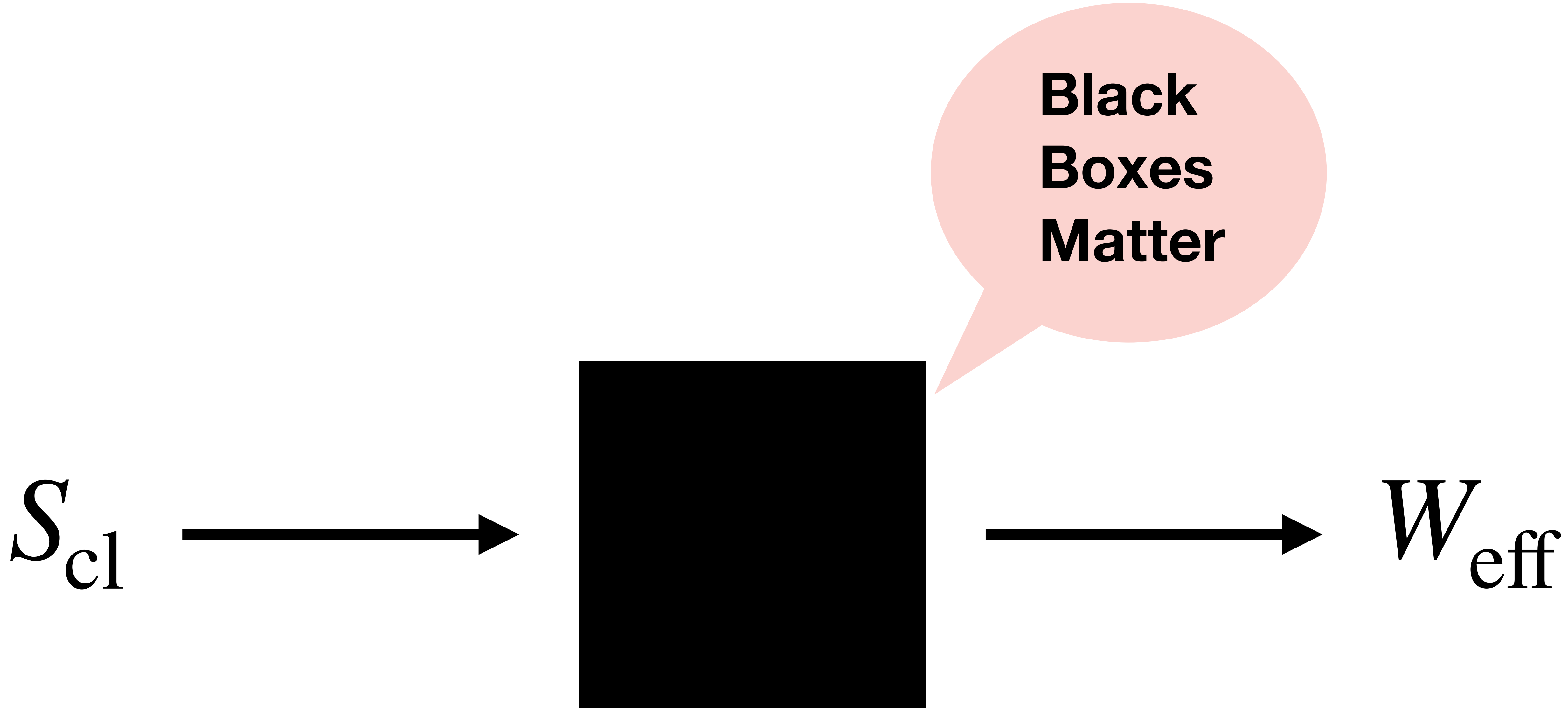
Quasi-locality + **The smoothness is controlled by the choice of the operator kernel!**

Regularization by using the deformation!

$$\phi(x) \longrightarrow \phi^{\Lambda}_{\omega}(x) \equiv \mathrm{H}^{\Lambda}_{\omega}(\phi)(x) = \int_{\mathbb{R}^n} \mathrm{d}^n x_1 \, \Lambda^n \omega(|x - x_1| \Lambda) \phi(x_1)$$

$$\phi(x_1)\phi(x_2) \sim G(x_1 - x_2) \longrightarrow \phi^{\Lambda}_{\omega}(x_1)\phi^{\Lambda}_{\omega}(x_2) \sim \mathrm{H}^{\Lambda}_{\omega}(\mathrm{H}^{\Lambda}_{\omega}(G))(x_1 - x_2)$$

What should we deform?



Rules:
1) Deform the classical action;
2) Do not deform the quadratic form.

Generalization

$$S_{\rm cl}[\,\cdot\,,\mathscr{M}] = S_0[\,\cdot\,,\mathscr{M}] + S_{\rm int}[\,\cdot\,,\mathscr{M}] \quad \longleftrightarrow \quad S^{\Lambda}_{\rm cl}[\,\cdot\,,\mathscr{M}] = S_0[\,\cdot\,,\mathscr{M}] + S_{\rm int}[{\rm H}^{\Lambda}_{\omega}(\,\cdot\,),\mathscr{M}_{\Lambda}]$$

$$\updownarrow \qquad\qquad\qquad\qquad \updownarrow$$

$$W_{\rm eff} \xrightarrow{\text{regularization}} W^{\Lambda}_{\rm eff}$$

$$\phi^{\Lambda}_{\omega}(p) \equiv {\rm H}^{\Lambda}_{\omega}(\phi)(p) = \int_{\mathscr{M}} {\rm d}^n p_1\, \omega(p,p_1;\Lambda)\phi(p_1)$$

It is possible to take such kernels that

$$\omega(p,p_1;\Lambda) = \omega_i(p,p_1;\Lambda) = 0,$$
$$i \in \{l,r\},\ \ p \in \mathscr{M}_{i,\Lambda},\ \ p_1 \in \mathscr{M}_i\,.$$

The support is in a small ball and tends to zero, when the regularization is removed.

Gluing

$$Z(\eta;\mathscr{M}_{i,\Lambda}) = e^{-W_{\text{eff}}[\sqrt{\hbar}\eta;\mathscr{M}_i]/\hbar} = e^{-S_0[\phi_i^\eta;\mathscr{M}_i]}\left(S(\delta_\psi+\phi_i^\eta;\mathscr{M}_i)e^{g_i(\psi_i;\mathscr{M}_i)}\right)\Bigg|_{\psi_i=0}$$

$i = l$

$i = r$

$i\in\{l,r\}$

$\eta=\eta_i+\eta_\Sigma$

$S_0[\phi_i^{\eta_i+\eta_\Sigma},\mathscr{M}_i] = S_0[\phi_i^{\eta_i},\mathscr{M}_i]+S_0[\phi_i^{\eta_\Sigma},\mathscr{M}_i]-S_{i,\Sigma}(\eta_i,\eta_\Sigma;\mathscr{M}_i)$

$$\int_{\mathscr{H}_0(\Sigma)}\mathscr{D}\eta_\Sigma\, Z(\eta_l+\eta_\Sigma;\mathscr{M}_{l,\Lambda})Z(\eta_r+\eta_\Sigma;\mathscr{M}_{r,\Lambda}) =$$

$$= e^{-S_0[\phi_l^{\eta_l};\mathscr{M}_l]-S_0[\phi_r^{\eta_r};\mathscr{M}_r]}e^{S_{l,\Sigma}(\eta_l,\delta_\eta;\mathscr{M}_l)+S_{r,\Sigma}(\eta_r,\delta_\eta;\mathscr{M}_r)}S(\mathrm{H}^\Lambda_\omega(\delta_{\psi_l}+\phi_l^{\delta_\eta}+\phi_l^{\eta_l});\mathscr{M}_{l,\Lambda})\times$$

$$\times S(\mathrm{H}^\Lambda_\omega(\delta_{\psi_r}+\phi_r^{\delta_\eta}+\phi_r^{\eta_r});\mathscr{M}_{r,\Lambda})e^{g_l(\psi_l;\mathscr{M}_l)}e^{g_r(\psi_r;\mathscr{M}_r)}e^{g_0(\eta;\Sigma)}\Bigg|_{\psi_l=0,\psi_r=0,\eta=0}$$

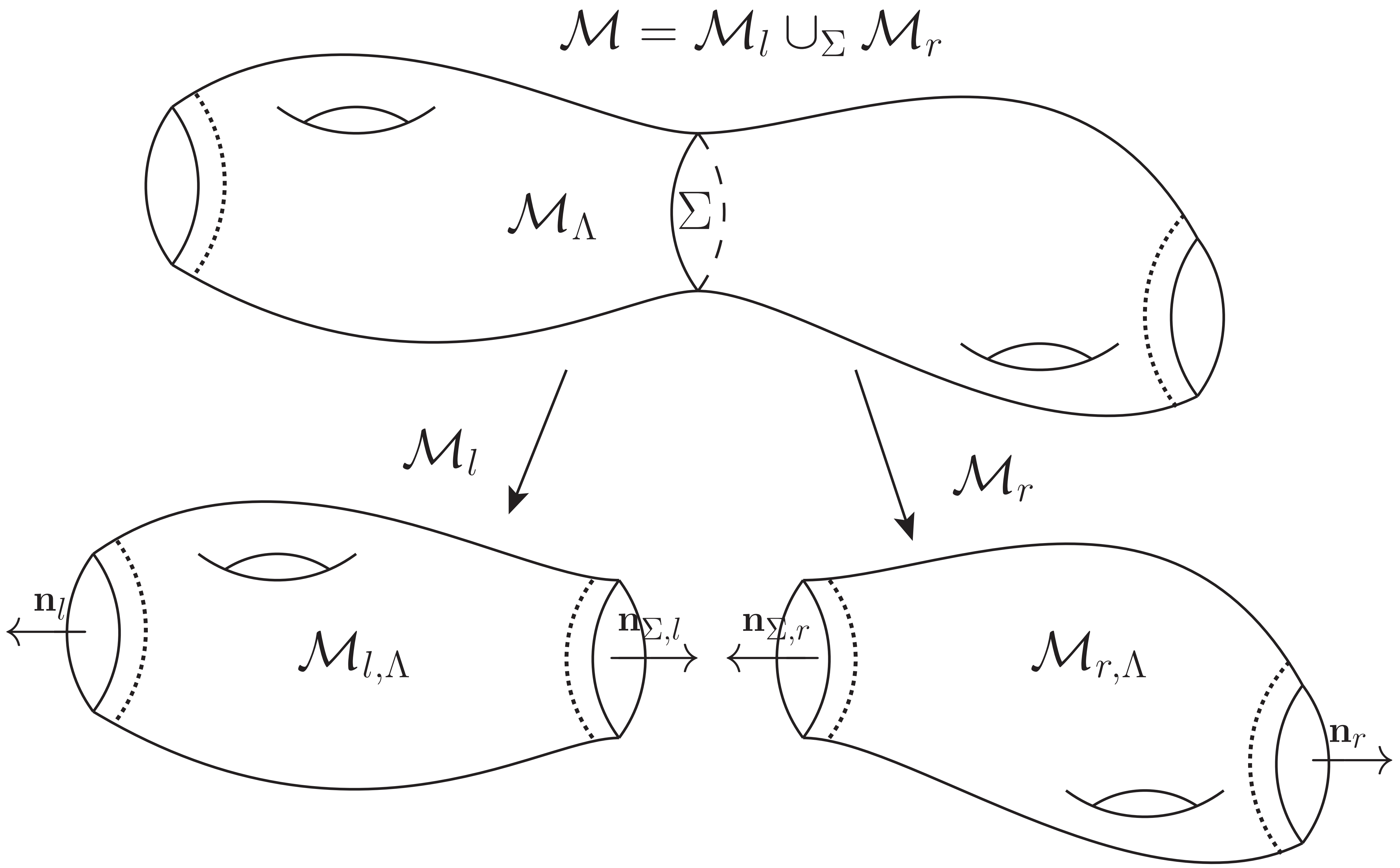
Gluing of partition functions

$$\int_{\mathscr{H}_0(\Sigma)} \mathscr{D}\eta_\Sigma\, Z(\eta_l + \eta_\Sigma; \mathscr{M}_{l,\Lambda}) Z(\eta_r + \eta_\Sigma; \mathscr{M}_{r,\Lambda}) = Z(\eta_l + \eta_r; \mathscr{M}_{l,\Lambda} \cup \mathscr{M}_{r,\Lambda})$$

$$\lim_{\mathscr{M}_{l,\Lambda} \cup \mathscr{M}_{r,\Lambda} \to \mathscr{M}_\Lambda} \int_{\mathscr{H}_0(\Sigma)} \mathscr{D}\eta_\Sigma\, Z(\eta_l + \eta_\Sigma; \mathscr{M}_{l,\Lambda}) Z(\eta_r + \eta_\Sigma; \mathscr{M}_{r,\Lambda}) = Z(\eta_l + \eta_r; \mathscr{M}_\Lambda)$$

It is valid for all dimensions!

It works in the case of non-renormalizable models!



Ivanov (2024)
arXiv: 2411.13857

Applications in scalar models

1) Three-loop calculation in sextic model for n=3

Model:
Lipatov (1976)
Pisarski (1983)
Bardeen, Moshe, Bander (1984)
Gudmundsdottir, Rydnell, Salomonson (1984)
Hager (2002)
Gracey (2020)

Kharuk (2024)
See «ZNS POMI»

2) Three-loop calculation in quartic model for n=4

Model:
Kleinert, Schulte-Frohlinde (2001)
Vasil'ev (2004)

Other regularizations
Jack, Osborn (1982)
Brizola, Battistel, Sampaio (1999)

Ivanov (2024)
arXiv: 2402:14549

3) Three-loop calculation in cubic model for n=5

Ivanov, Kharuk (2024)
arXiv: 2404:07513

An example of the action density for the sextic model: $t_1\phi + \frac{1}{2}\phi\left(A_0 + m^2\right)\phi + \sum_{k=3}^{6}\frac{t_k}{k!}\phi^k$

Many thanks!