Schwinger-DeWitt and Schwinger-Keldysh techniques: applications in quantum gravity and cosmology

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Euclidean path integral density matrix

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Background (mean) field formalism

Generating functional

$$Z[J] = \int D\varphi \, \exp \frac{1}{\hbar}\Big(- S[\varphi] - \int dx\, \varphi(x) J(x) \Big)$$

Effective action

$$e^{-\Gamma[\phi]/\hbar} = \int D\varphi \, \exp \frac{1}{\hbar}\left(-S[\varphi] + \int dx \,\big(\varphi(x) - \phi(x)\big) \frac{\delta\Gamma[\phi]}{\delta\phi(x)} \right)$$

mean field ($\phi(x)$)

quantum field ($\varphi(x)$)

Loop expansion

$$S[\varphi] \to \Gamma[\phi] = S[\phi] + \hbar\, \Gamma_{1-\mathrm{loop}}[\phi] + \hbar^2 \Gamma_{2-\mathrm{loop}}[\phi] + \ldots$$

Inverse propagator

$$F(\nabla)\, \delta(x,y) = \frac{\delta^2 S[\phi]}{\delta\phi(x)\, \delta\phi(y)},$$

Vertices

$$S^{(n)}(x_1, \ldots x_n) = \frac{\delta^n S[\phi]}{\delta\phi(x_1) \ldots \delta\phi(x_n)}$$

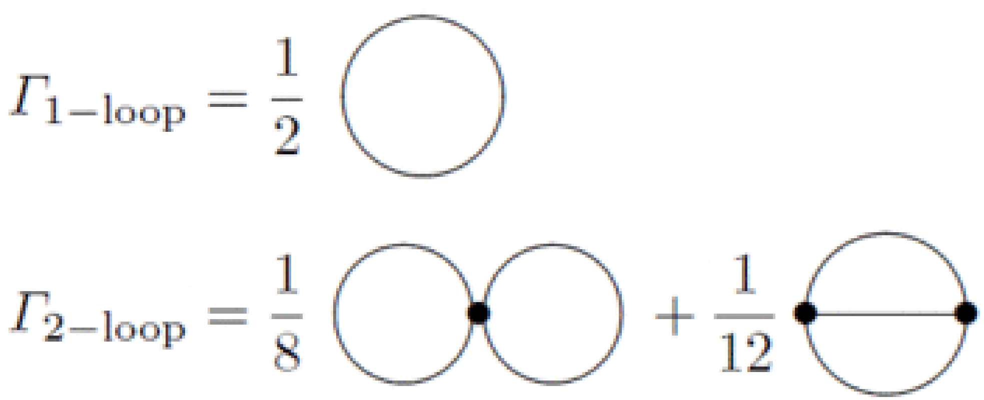
One-loop and two-loop orders

$$\Gamma_{1-\text{loop}} = \frac{1}{2}\,\text{ln}\,\text{Det}\,F(\nabla) = \frac{1}{2}\,\text{Tr}\,\text{ln}\,F(\nabla),$$

$$\Gamma_{2-\text{loop}} = \frac{1}{8}\int dx_1 dx_2 dx_3 dx_4\, G(x_1,x_2)\, S^{(4)}(x_1,x_2,x_3,x_4)\, G(x_3,x_4)$$

$$+\frac{1}{12}\int dx_1 dx_2 dx_3\, dy_1 dy_2 dy_3\, S^{(3)}(x_1,x_2,x_3)$$

$$\times\, G(x_1,y_1)\, G(x_2,y_2)\, G(x_3,y_3)\, S^{(3)}(y_1,y_2,y_3)$$



Propagators and vertices in external background field !

Heat kernel and proper time method

Proper time method

$$G \equiv -F^{-1}(\nabla) = \int_0^\infty ds\, K(s),$$

$$\frac{1}{2}\,\mathrm{Tr}\,\ln F(\nabla) = -\frac{1}{2}\int_0^\infty \frac{ds}{s}\,\mathrm{Tr}\,K(s)$$

Heat kernel and its functional trace

$$\hat{K}(s|\,x,y) = e^{s\hat{F}(\nabla)}\,\delta(x,y),$$

$$\mathrm{Tr}K(s) = \int dx\,\mathrm{tr}\,\hat{K}(s|\,x,x)$$

Operator $\hat{F}(\nabla) = F^A_B(\nabla)$ **acting in the space of fields** $\varphi = \varphi^A(x)$

Generic matrix notation $\hat{X} = X^A_B$

Covariant d'Alembertian

Minimal second order operator in curved spacetime

$$\hat{F}(\nabla) = \Box + \hat{P} - \frac{\hat{1}}{6}R - m^2\hat{1}, \qquad \Box = g^{\mu\nu}\nabla_\mu\nabla_\nu$$

Generalized "curvatures" and covariant derivatives

$$\Re = \hat{P},\ \hat{\mathcal{R}}_{\mu\nu},\ R_{\mu\nu\alpha\beta},$$

$$[\nabla_\mu, \nabla_\nu]\, v^\alpha = R^\alpha_{\ \beta\mu\nu} v^\beta,$$

$$[\nabla_\mu, \nabla_\nu]\,\varphi = \hat{\mathcal{R}}_{\mu\nu}\,\varphi, \qquad [\nabla_\mu, \nabla_\nu]\,\hat{X} = [\hat{\mathcal{R}}_{\mu\nu}, \hat{X}]$$

Heat kernel (Schwinger-DeWitt) expansion for minimal second order operators

Synge world function

$$e^{-s\hat{F}(\nabla)}\delta(x,y) = \frac{\mathcal{D}^{1/2}(x,y)}{(4\pi s)^{d/2}}\, g^{1/2}(y)\, e^{-\frac{\sigma(x,y)}{2s}} \sum_{n=0}^{\infty} s^n\, \hat{a}_n(x,y)$$

Schwinger-DeWitt (Gilkey-Seely) coefficients and their coincidence limits

$$\hat{a}_0(x,x) = \hat{1}, \quad \hat{a}_1(x,x) = \hat{P},$$

$$\hat{a}_2(x,x) = \frac{1}{180}\left(R^2_{\alpha\beta\gamma\delta} - R^2_{\mu\nu} + \Box R\right)\hat{1} + \frac{1}{12}\hat{\boldsymbol{R}}^2_{\mu\nu} + \frac{1}{2}\hat{P}^2 + \frac{1}{6}\Box\hat{P}, \ldots$$

$$\hat{a}_n(x,x) \propto \overbrace{\nabla\cdots\nabla}^{2p}\,\overbrace{\Re\cdots\Re}^{m}, \quad m+p=n$$

One-loop divergences $\quad \Gamma^{\rm div}_{\rm one-loop} = -\frac{1}{32\pi^2\varepsilon}\int d^4x\, g^{1/2}{\rm tr}\,\hat{a}_2(x,x), \quad \varepsilon = 2 - \frac{d}{2} \to 0$

*Extension to **non-minimal and higher-derivative operators** -- the method of universal functional traces (I. Jack and H. Osborn (1984), G.A.Vilkovisky & A.B., Phys. Rept. 119 (1985) 1)*

Idea:

$$\mathrm{Tr}\,\ln\left(\Box^N + P(\nabla)\right) = N\,\mathrm{Tr}\,\ln\Box + \mathrm{Tr}\,\ln\left(1 + P(\nabla)\frac{1}{\Box^N}\right)$$

$$= N\,\mathrm{Tr}\,\ln\Box + \mathrm{Tr}\,P(\nabla)\frac{1}{\Box^N} + \cdots$$

⇓

$$\Gamma^{\mathrm{div}} = \sum_{m,n}\int d^4x\,\mathcal{R}_n^{\mu_1\ldots\mu_m}\nabla_{\mu_1}\ldots\nabla_{\mu_m}\frac{\hat{1}}{\Box^n}\delta(x,y)\,\Big|_{y=x}^{\mathrm{div}}$$

↑ ***universal functional traces***

$$\nabla\ldots\nabla\frac{\hat{1}}{\Box^n}\delta(x,y)\,\Big|_{y=x}^{\mathrm{div}} = \frac{(-1)^n}{\Gamma(n)}\nabla\ldots\nabla\int_0^\infty ds\,s^{n-1}\,e^{s\Box}\,\hat{\delta}(x,y)\,\Big|_{y=x}^{\mathrm{div}}$$

↑ *Schwinger-DeWitt expansion*

Several universal functional traces with $\Box \to \hat{F}(\nabla)$

$$\frac{\hat{1}}{F(\nabla)}\,\delta(x,y)\Big|_{y=x}^{\rm div} = -\frac{1}{16\pi^2\varepsilon}\,g^{1/2}\hat{P},$$

$$\nabla_\mu\frac{\hat{1}}{F(\nabla)}\,\delta(x,y)\Big|_{y=x}^{\rm div} = -\frac{1}{16\pi^2\varepsilon}\,g^{1/2}\Big(\frac{1}{2}\nabla_\mu\hat{P} - \frac{1}{6}\,\nabla^\nu\hat{\mathcal{R}}_{\nu\mu}\Big),$$

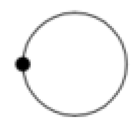
$$\frac{\hat{1}}{F^2(\nabla)}\,\delta(x,y)\Big|_{y=x}^{\rm div} = \frac{1}{16\pi^2\varepsilon}\,g^{1/2}\hat{1},$$

$$\nabla_\mu\frac{\hat{1}}{F^2(\nabla)}\,\delta(x,y)\Big|_{y=x}^{\rm div} = 0,$$

$$\nabla_\mu\nabla_\nu\frac{\hat{1}}{F^2(\nabla)}\,\delta(x,y)\Big|_{y=x}^{\rm div} = \frac{1}{16\pi^2\varepsilon}\,g^{1/2}\Big(\frac{1}{6}\,R_{\mu\nu}\hat{1} - \frac{1}{2}\,g_{\mu\nu}\hat{P} + \frac{1}{2}\,\hat{\mathcal{R}}_{\mu\nu}\Big)$$

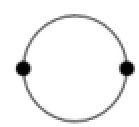
Diagrammatically – tadpoles in external background field:

$$\nabla_x\ldots\nabla_x G(x,y)\Big|_{y=x} =$$



Functional differentiation yields polarization (self-energy) operators:

$$\Pi(x,y) \equiv \frac{\delta}{\delta P(y)}\nabla_x\ldots\nabla_x G(x,y)\Big|_{y=x} = \nabla_x\ldots\nabla_x G(x,y)G(y,x) =$$



Commutator technique for off-diagonal heat kernel of minimal higher-derivative operators (*A.O.B., A.V.Kurov, W.N.Wachowski, Phys. Rev. D 110, (2024) 085023, arXiv:2408.10990***)**

Split of the 2M-th order operator into the M-th order of the minimal operator H and the lower-derivative part W

$$\hat{F}(\nabla) = \hat{H}^M(\nabla) + \hat{W}(\nabla), \quad \hat{W}(\nabla) = \sum_{k=0}^{2M-1} \hat{W}_k^{a_1\ldots a_k}(x)\nabla_{a_1}\cdots\nabla_{a_k}$$

$$\mathcal{E}_{\nu,\alpha}(z) = \frac{1}{\nu}\sum_{k=0}^{\infty} \frac{\Gamma\left(\frac{\alpha+k}{\nu}\right)}{\Gamma(\alpha+k)}\frac{z^k}{k!}$$

Generalized exponential functions

$$\hat{K}_F(\tau|x,x') = \frac{1}{(4\pi\tau^{1/M})^{d/2}}\sum_{m=0}^{\infty}\tau^{\frac{m}{M}}\sum_{k=0}^{\infty}\mathcal{E}_{M,\frac{d}{2}+Mk-m}\left(-\frac{\sigma(x,x')}{2\,\tau^{1/M}}\right)\hat{a}_{m,k}(F|x,x')$$
$$+\frac{1}{(4\pi\tau^{1/M})^{d/2}}\sum_{m=1}^{\infty}\tau^{-\frac{m}{M}}\sum_{k\geq\frac{m}{M-1}}^{\infty}\mathcal{E}_{M,\frac{d}{2}+Mk+m}\left(-\frac{\sigma(x,x')}{2\,\tau^{1/M}}\right)\hat{a}_{-m,k}(F|x,x')$$

negative powers !

$$\hat{a}_{m,k}(F|x,x') = \sum_{l\geq L_{m,k}}^{m+(M-1)k}\left\{\hat{U}_k(\nabla_x)\right\}_{Mk+l-m}\Delta^{1/2}(x,x')\,\hat{a}_l(H|x,x'),$$

$$L_{m,k} = \max\left\{0, m-Mk\right\}, \quad -\infty < m < \infty$$

SDW coefficients of the operator H

Three steps for the construction of $\left\{\hat{U}_k(\nabla)\right\}_n$

1) Multiple nested commutators:

$$\hat{V}_0(\nabla)=\hat{W}, \quad \hat{V}_k(\nabla)=\Big[\dots\Big[\big[\hat{W},\underbrace{\hat{H}^M\big],\hat{H}^M\Big],\dots,\hat{H}^M}_{k}\Big], \quad k>0$$

2) Local differential operators:

$$\hat{U}_k(\nabla)=\hat{1},$$

$$\hat{U}_k(\nabla)=\sum_{\substack{n,k_1,\cdots k_n\\ n+|\mathbf{k}|=k}}\frac{(-1)^n\,\hat{V}_{k_1}(\nabla)\cdots\hat{V}_{k_n}(\nabla)}{k_1!\cdots k_n!\,(k_1+1)(k_1+k_2+2)\cdots(k_1+\cdots+k_n+n)}, \; k>0$$

3) n-th order **SYNGIFICATION** of $\hat{U}_k(\nabla)$, $\quad \hat{U}_k(\nabla)\to\left\{\hat{U}_k(\nabla)\right\}_{\mathbf{n}}$:

Replacement of **n** covariant derivatives in each of their monomials by Synge world function vectors $\nabla_a\to-\nabla_a\sigma(x,x')/2$

$$\left\{\nabla_{a_1}...\nabla_{a_j}\right\}_n=\frac{1}{2^n n!}\frac{\partial^n}{\partial k^n}\Big(\nabla_{a_1}-k\sigma_{a_1}\Big)...\Big(\nabla_{a_j}-k\sigma_{a_j}\Big)\Big|_{k=0}$$

Projectable Horava gravity – single example of local, unitary, renormalizable quantum gravity *(A.B., D.Blas, M.Herrero-Valea, S.M.Sibiryakov and C.F.Steinwachs, Phys. Rev. D 93, 064022 (2016), arXiv:1512.02250)*

Lorentz symmetry violation $O(3,1) \to O(3)$

Foliation preserving diffeomorphisms $x^i \mapsto \tilde{x}^i(x,t)\ , \quad t \mapsto \tilde{t}(t)$

ADM metric decomposition $ds^2 = N^2 dt^2 + \gamma_{ij}(dx^i + N^i dt)(dx^j + N^j dt)\ , \quad i,j = 1,2,3$

Anisotropic scaling invariance: $x^i \to \lambda^{-1} x^i, \quad t \to \lambda^{-3} t, \quad N^i \to \lambda^2 N^i, \quad \gamma_{ij} \to \gamma_{ij}$

kinetic term -- unitarity

$$S = \frac{1}{2G}\int dt\, d^3x \sqrt{\gamma}\Big(K_{ij}K^{ij} - \lambda K^2 - \mathcal{V}(\gamma)\Big), \qquad K_{ij} = \frac{1}{2N}(\dot{\gamma}_{ij} - \nabla_i N_j - \nabla_j N_i)$$

extrinsic curvature

Potential term

relevant

$$\mathcal{V}(\gamma) = 2\Lambda - \eta R + \mu_1 R^2 + \mu_2 R_{ij}R^{ij}$$

$$+\nu_1 R^3 + \nu_2 R R_{ij} R^{ij} + \nu_3 R^i_j R^j_k R^k_i + \nu_4 \nabla_i R \nabla^i R + \nu_5 \nabla_i R_{jk} \nabla^i R^{jk}$$

marginal (dimension 6)

Is it asymptotically free? Need beta-functions!

However, in Horava gravity operators are nonminimal

Structure of operators on the static background with generic 3-metric

$$\hat{F}(\nabla) = -\hat{1}\,\partial_\tau^2 + \hat{\mathbb{F}}(\nabla) \qquad \hat{\mathbb{F}} = \left\{ \mathbf{D}_{ij}{}^{kl}, \mathbf{B}^k_i \right\} \sim \nabla^6 + R\nabla^4 + R^2\nabla^2 + R^3$$

space parts of metric and vector (shifts and ghosts) operators:

$$S_{\rm gf} = \frac{\sigma}{2G} \int dt\, d^3x\, \sqrt{g}\, F_i\, \mathcal{O}^{ij} F_i$$

$$F^i = \dot{n}^i + \frac{1}{2\sigma} (\mathcal{O}^{-1})^{ij} \big(D_k h^k_j - \lambda D_j h \big)\,,$$

$$\mathcal{O}_{ij} = \Big[g^{ij} \Delta^2 - \xi D^i (-\Delta) D^j \Big]^{-1}$$

Quadratic parts of the total action in quasi-relativistic σ, ξ -family of gauges on a 3-metric background with generic 3-metric $\gamma_{ij}(\mathbf{x})$:

$$S_h = \frac{1}{2G} \int d\tau d^3x \sqrt{g}\, h_{mn} G^{mn,ij} \left[-\delta_{ij}{}^{kl} \partial_\tau^2 + \mathbf{D}_{ij}{}^{kl}(\nabla) \right] h_{kl},$$

$$S_{\rm gh} = \frac{1}{G} \int d\tau d^3x \sqrt{g}\, \bar{c}_i \left[-\delta^i_j \partial_\tau^2 + \mathbf{B}^i{}_j(\nabla) \right] c^j.$$

3D covariant Laplacian

Tensor sector operator

$$\mathbf{D}_{ij}{}^{kl}(\nabla) = -\left[\nu_5 \delta_{ij}{}^{kl} + \frac{4\nu_4(1-\lambda)+\nu_5}{1-3\lambda} g_{ij} g^{kl}\right]\Delta^3 + \left[2\nu_5 - \frac{1}{\sigma}\right]\delta_{(i}^{(k}\nabla_{j)}\nabla^{l)}\Delta^2$$
$$+\frac{4\nu_4(1-\lambda)+\nu_5}{1-3\lambda} g_{ij}\nabla^{(k}\nabla^{l)}\Delta^2 + \left[4\nu_4 + \nu_5 + \frac{l(1+\xi)}{\sigma}\right]\nabla_{(i}\nabla_{j)} g^{kl}\Delta^2$$
$$-\left[4\nu_4 + 2\nu_5 + \frac{\xi}{\sigma}\right]\nabla_{(i}\nabla_{j)}\nabla^{(k}\nabla^{l)}\Delta + \left[\sim \textbf{200 lower derivative terms}\right],$$

Shifts and ghost sector vector operator

$$\mathbf{B}^i{}_j(\nabla) = -\frac{1}{2\sigma}\delta^i_j\Delta^3 - \frac{1}{2\sigma}\Delta^2\nabla_j\nabla^i - \frac{\xi}{2\sigma}\nabla^i\Delta\nabla^k\nabla_j\nabla_k$$
$$-\frac{\xi}{2\sigma}\nabla^i\Delta\nabla_j\Delta + \frac{\lambda}{\sigma}\Delta^2\nabla^i\nabla_j + \frac{\lambda\xi}{\sigma}\nabla^i\Delta^2\nabla_j, \quad \Delta = \gamma^{ij}\nabla_i\nabla_j$$

Dimensional reduction *method on a static background with a generic 3-metric*

$$\mathrm{Tr}_4 \ln\left(-\hat{1}\partial_t^2 + \hat{F}(\nabla)\right) = \int dt\, \mathrm{Tr}_3 \sqrt{\hat{F}(\nabla)}$$

3D functional trace

How to proceed with the square root of the 6-th order differential operator?

$$\mathbb{F} = \sum_{a=0}^{6} \mathcal{R}_{(a)} \sum_{6\geq 2k\geq a} \alpha_{a,k} \nabla_1 ... \nabla_{2k-a} (-\Delta)^{3-k}, \quad \mathcal{R}_{(a)} = O\left(\frac{1}{l^a}\right)$$

⇩

Pseudodifferential operator – infinite series in curvature invariants $\mathcal{R}_{(a)}$

$$\sqrt{\mathbb{F}} = \sum_{a=0}^{\infty} \mathcal{R}_{(a)} \sum_{k\geq a/2}^{K_a} \tilde{\alpha}_{a,k} \nabla_1 ... \nabla_{2k-a} \frac{1}{(-\Delta)^{k-3/2}}$$

$$\Downarrow$$

$$\mathrm{Tr}_3\sqrt{\mathbb{F}}\,\Big|^{\rm div} = \sum_{a=2}^{6}\sum_{k}\tilde{\alpha}_{a,k}\int d^3x\,\mathcal{R}_{(a)}(\mathbf{x})\nabla_1\ldots\nabla_{2k-a}\frac{1}{(-\Delta)^{k-3/2}}\delta(\mathbf{x},\mathbf{x}')\,\Big|^{\rm div}_{\mathbf{x}=\mathbf{x}'}$$

Examples:

$$g^{ij}(-\Delta)^{1/2}\delta_{ij}{}^{kl}(x,y)\,\Big|^{\rm div}_{y=x} = -\frac{1}{16\pi^2\varepsilon}\sqrt{g}\,g^{kl}\frac{1}{30}\left(\frac{1}{2}R_{ij}^2+\frac{1}{4}R^2+\Delta R\right)$$

$$\int d^3x\,\delta_{kl}{}^{ij}(-\Delta)^{3/2}\delta_{ij}{}^{kl}(x,y)\,\Big|^{\rm div}_{y=x} = \frac{3}{32\pi^2\varepsilon}\int d^3x\,\sqrt{g}\,\delta_{kl}{}^{ij}\,a_{3ij}{}^{kl}(x,x)$$

The third Schwinger-DeWitt coefficient

$$= \frac{3}{32\pi^2\varepsilon}\int d^3x\,\sqrt{g}\,\left(\frac{31}{45}R^i_jR^j_kR^k_i - \frac{233}{210}R_{ij}^2R + \frac{673}{2520}R^3 + \frac{5}{84}R\Delta R - \frac{67}{420}R_{ij}\Delta R^{ij}\right)$$

Beta functions of (3+1)-dimensional Horava *gravity (A.O. Barvinsky, A.V. Kurov, S. M. Sibiryakov, Beta functions of (3+1)-dimensional projectable Horava gravity, Phys.Rev.D 105 (2022) 044009, arXiv:2110.14688)*

$$S = \frac{1}{2G}\int dt\, d^dx \sqrt{\gamma} N \Big(K_{ij}K^{ij} - \lambda K^2 - \mathcal{V}(\gamma) \Big)$$

$$\mathcal{V}(\gamma) = \nu_1 R^3 + \nu_2 R R_{ij} R^{ij} + \nu_3 R^i_j R^j_k R^k_i + \nu_4 \nabla_i R \nabla^i R + \nu_5 \nabla_i R_{jk} \nabla^i R^{jk}$$

Six essential coupling constants $\mathcal{G},\ \lambda$ and $\chi = (u_s, v_1, v_2, v_3)$

$$\mathcal{G} = \frac{G}{\sqrt{\nu_5}}, \quad \lambda, \quad u_s = \sqrt{\frac{(1-\lambda)(8\nu_4 + 3\nu_5)}{(1-3\lambda)\nu_5}}, \quad v_a = \frac{\nu_a}{\nu_5}, \quad a = 1,2,3$$

Use of Mathematica package xAct

Check of the results: independence of essential beta functions of the choice of gauge (σ, ξ - family of gauge conditions) and spectral sum method in dimensional and zeta-functional regularization.

⇩

Fixed points of (3+1)-dimensional projectable HG *(A.O.B., A.V. Kurov, S.M. Sibiryakov, Phys.Rev.D 108 (2023) 12, L121503, arXiv:2310.07841)*

$\mathcal{G} \to 0$ ***asymptotic freedom***

Fixed points at finite λ

λ	u_s	v_1	v_2	v_3	$\beta_{\mathcal{G}}/\mathcal{G}^2$	AF?	UV attractive along λ?
0.1787	60.57	-928.4	-6.206	-1.711	-0.1416	yes	no
0.2773	390.6	-19.88	-12.45	2.341	-0.2180	yes	no
0.3288	54533	3.798×10^8	-48.66	4.736	-0.8484	yes	no
0.3289	57317	-4.125×10^8	-49.17	4.734	-0.8784	yes	no

(cosmology implication, A.E.Gumrukcuoglu, S.Mukohyama, 1104.2087)

Special limit: $\lambda \to \pm\infty$

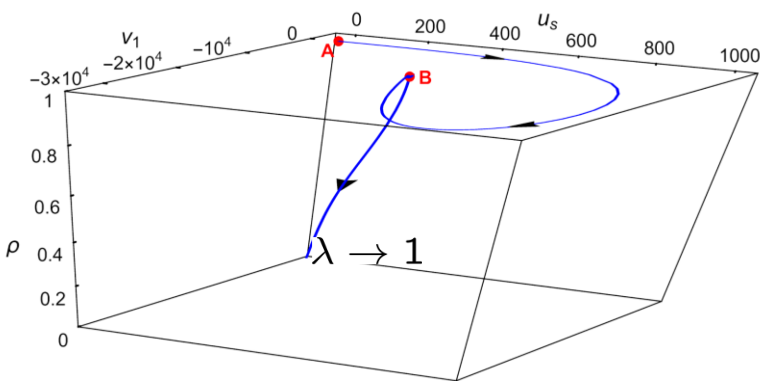
J. I. Radkovski, S. M. Sibiryakov, Scattering amplitudes in high-energy limit of projectable Horava gravity, Phys. Rev. D 108, 046017 (2023), arXiv:2306.00102

Fixed points at $\lambda \to \infty$

	u_s	v_1	v_2	v_3	$\beta_{\mathcal{G}}/\mathcal{G}^2$	asymptotically free?	UV attractive along λ?
	0.01950	0.4994	-2.498	2.999	-0.2004	yes	no
	0.04180	-0.01237	-0.4204	1.321	-1.144	yes	no
	0.05530	-0.2266	0.4136	0.7177	-1.079	yes	no
	12.28	-215.1	-6.007	-2.210	-0.1267	yes	yes
A	21.60	-17.22	-11.43	1.855	-0.1936	yes	yes
B	440.4	-13566	-2.467	2.967	0.05822	no	yes
	571.9	-9.401	13.50	-18.25	-0.07454	yes	yes
	950.6	-61.35	11.86	3.064	0.4237	no	yes

Special fixed points:
A *is AF,* ***B*** *is not AF*
– yield AF RG flow from UV to IR

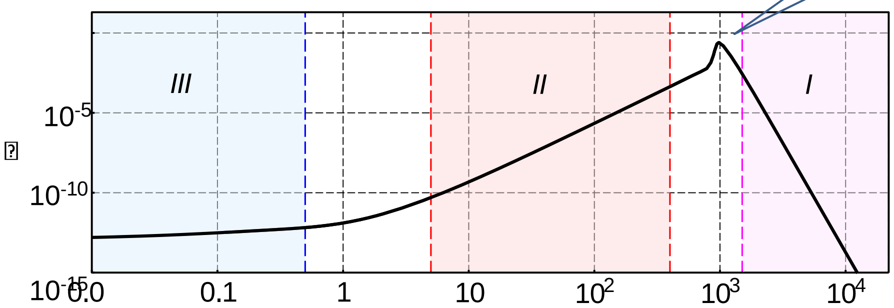
Fixed point B as a transient of RG flow from AF UV point A to IR limit with $\lambda \to 1$ (quasi-GR limit) *(A.O.B., A.V.Kurov and S.M.Sibiryakov, Renormalization group flow of projectable Horava gravity in (3+1) dimensions, Phys. Rev. D 111, (2025) 024030, arXiv:2411.13574)*



$\lambda \to \infty$ *hyperplane*

Behavior of G on the flow: $\mathcal{G} \to 0$ *in UV and IR*

$\mathcal{G}_{\max}$



Behavior of $\mathcal{G}$ as a function of $(\lambda - 1)$ along an RG trajectory connecting the point A to $\lambda \to 1^+$. In regions I, II and III the dependence is well described by the power law $\mathcal{G} \propto (\lambda - 1)^\kappa$ with $\kappa_I = -13.69$, $\kappa_{II} = 3.84$, $\kappa_{III} \approx 0.37$.

Riddles of higher derivative gravity models

1. Renormalizability and AF of nonprojectable HG? It has healthy IR limit fitting GR (D. Blas, O. Pujolas, S.Sibiryakov, JHEP04(2011)018).There are indications that it is renormalizable.

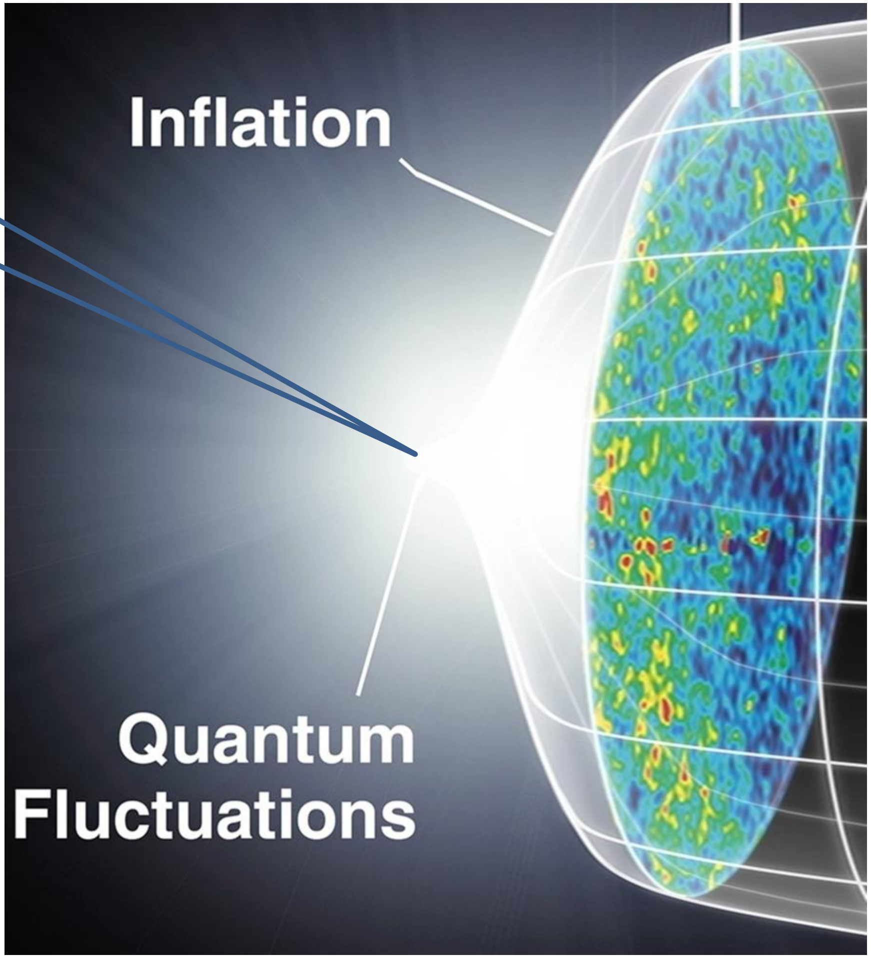
J. Bellorın, C. Borquez, B. Droguett,
Phys. Rev D 106, 044055 (2022), 2207.08938
arXiv:2405.04708

2. Complexity of operator dimensions – eigenvalues of stability matrices. Violation of LI or lack of gauge invariant operators ?

3. Tadpoles, IR divergences and modification of beta functions in quadratic gravity (D.Buccio, J.Donoghue, G.Menezes, R.Percacci, arXiv:2403.02397)

Schwinger-Keldysh formalism and cosmological initial conditions

What was at the beginning?



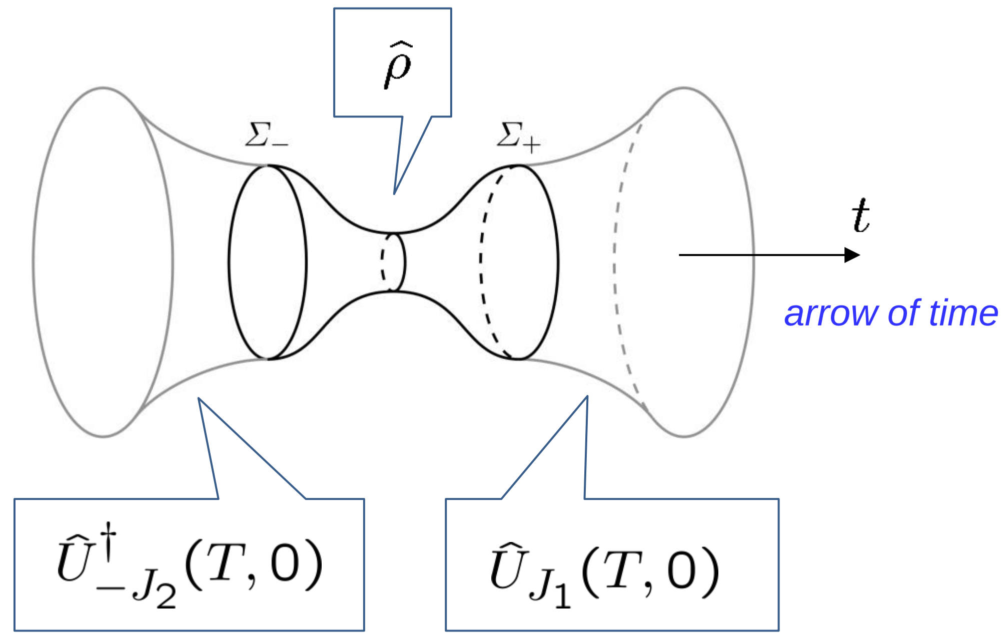
Cosmological context of Schwinger-Keldysh formalism

Closed cosmology with spatial slices $\Sigma \sim S^3$ *of constant time,* $S^3 \times R^1$*--topology*

Initial state density matrix evolving in time:

$$\hat{U}_{J_1}(T,0)\,\hat{\rho}\,\hat{U}^{\dagger}_{-J_2}(T,0)$$

J_1, J_2 *-- sources for quantum fields*



Unitary evolution in real Lorentzian time

$$\hat{U}_J(T,0) = \mathrm{T}\, e^{-i\int_0^T dt\,\left(\hat{H}(t) - J(t)\hat{\phi}\right)}$$

"Euclidean evolution" in imaginary time

$$\hat{\rho} = \frac{1}{Z}\,\mathrm{T}e^{-\int_{\tau_-}^{\tau_+} d\tau\,\hat{H}(\tau)}$$

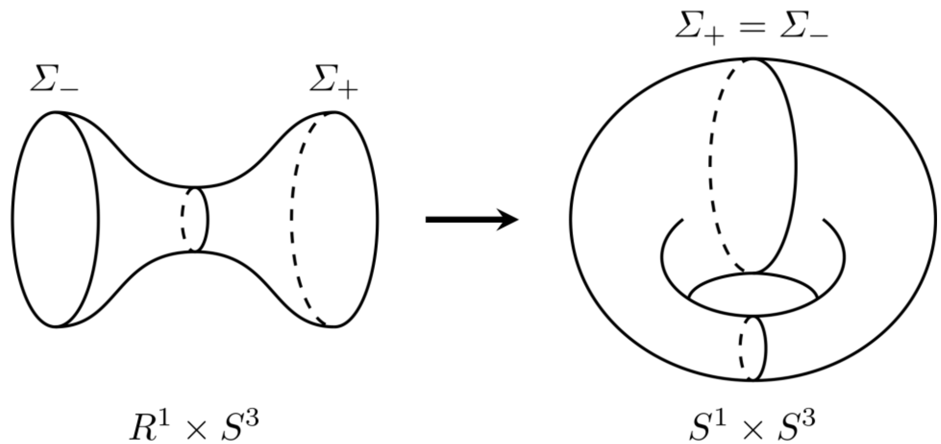
Generating functional of correlation functions

$$Z[J_1, J_2] = \mathrm{tr}\left[\hat{U}_{J_1}(T,0)\,\hat{\rho}\,\hat{U}^{\dagger}_{-J_2}(T,0)\right].$$

Transition to ***statistical sum***

Unitarity with switched off sources

$$\hat{U}_0^{\dagger}(T,0)\,\hat{U}_0(T,0) = 1$$



Euclidean action

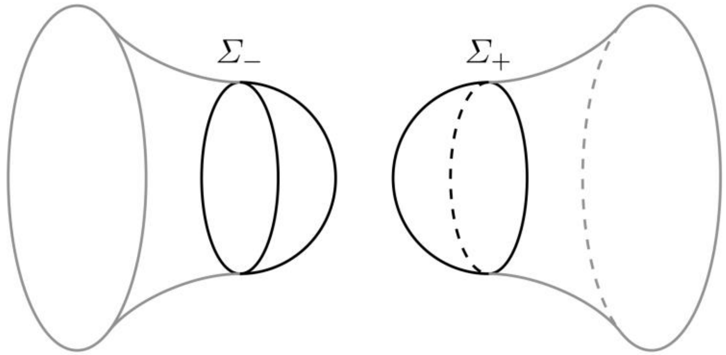
$$\mathrm{tr}\hat{\rho} = 1 \;\rightarrow\; Z = \mathrm{tr}\,\mathrm{T}\exp\left[-\int_{\tau_-}^{\tau_+} d\tau\,\hat{H}(\tau)\right] = \int D\phi\, e^{-S_E[\,\phi\,]}\Big|_{\phi(\tau_+)=\phi(\tau_-)}$$

Euclidean path integral
over periodic fields

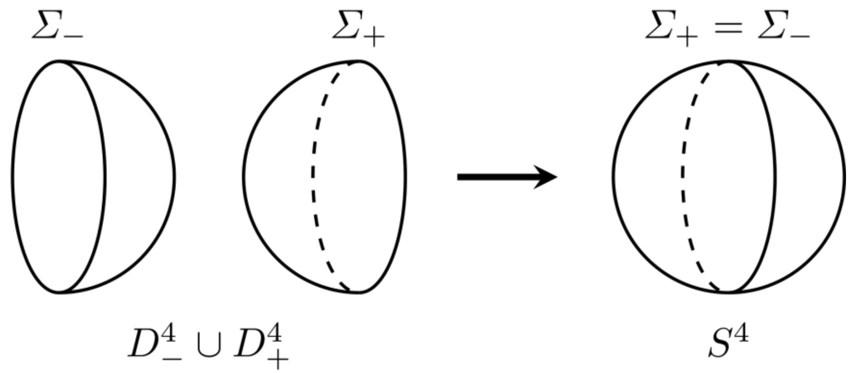
The case of no-boundary Hartle-Hawking wavefunction ***– factorizable*** *density matrix of a* ***pure vacuum*** *state*

$$\hat{\rho}_{HH} = |\,\Psi\,\rangle\langle\,\Psi\,|$$

$$\Psi[\,{}^3\gamma\,] = \int D\,{}^4g\; e^{-S_E[\,{}^4g\,]}$$



statistical sum Z



Parameterizarion of the density matrix elements – Gaussian state

transposition

$$\langle\varphi_+|\,\hat\rho\,|\,\varphi_-\rangle = \text{const}\times\exp\left\{-\frac{1}{2}\boldsymbol{\varphi}^T\boldsymbol{\Omega}\,\boldsymbol{\varphi} + \boldsymbol{j}^T\boldsymbol{\varphi}\right\},$$

$$\boldsymbol{\varphi} = \begin{bmatrix}\varphi_+\\ \varphi_-\end{bmatrix},\quad \boldsymbol{j} = \begin{bmatrix} j_+\\ j_-\end{bmatrix},$$

auxiliary sources

$$\hat\rho^\dagger = \hat\rho \;\rightarrow\; \boldsymbol{\Omega} = \begin{bmatrix} R & S\\ S^* & R^*\end{bmatrix},\quad R = R^T,\quad S = S^\dagger.$$

Gaussian path integral for the generating functional with ***sources*** $\boldsymbol{J} = \begin{bmatrix} J_1(t)\\ J_2(t)\end{bmatrix}$ ***dual*** *to* $\begin{bmatrix}\phi_1(t)\\ \phi_2(t)\end{bmatrix}$

$$Z[\boldsymbol{J},\boldsymbol{j}\,] = \text{tr}\left[\hat U_{J_1}(T,0)\,\hat\rho\,\hat U^\dagger_{-J_2}(T,0)\right].$$

$$\hat U_J(T,0) = \int D\phi\, e^{iS[\phi]+i\int_0^T dt\, J(t)\phi(t))}$$

$$\rightarrow\int d\varphi_+\,d\varphi_-\,D\phi_1\,D\phi_2\Big(\dots\Big)\Bigg|_{\substack{\phi_1(0)=\varphi_+,\;\phi_2(0)=\varphi_-,\\ \phi_1(T)=\phi_2(T)}}$$

double path integral

Matching time derivatives at the saddle point of the Gaussian integral at t=0 and t=T

$$Z[\,\boldsymbol{J},\boldsymbol{j}\,] = \text{const} \times \exp\left\{-\frac{i}{2}\int_0^T dt\,dt'\,\boldsymbol{J}^T(t)\boldsymbol{G}(t,t')\boldsymbol{J}(t)\right.$$

$$\left. - \int_0^T dt\,\boldsymbol{J}^T(t)\boldsymbol{G}(t,0)\,\boldsymbol{j} + \frac{i}{2}\boldsymbol{j}^T\boldsymbol{G}(0,0)\,\boldsymbol{j}\right\},$$

chronological

Wightman

2X2 block structure Green's function

$$\boldsymbol{G}(t,t') = \begin{bmatrix} G_{\top}(t,t') & G_{<}(t,t') \\ G_{<}^{T}(t',t) & G_{\top}^{*}(t,t') \end{bmatrix},$$

anti-chronological

$$FG_{\top}(t,t') = \delta(t-t'), \qquad FG_{>}(t,t') = 0,$$

Hessian of classical action:

$$F\delta(t-t') = \frac{\delta^2 S[\,\phi\,]}{\delta\phi(t)\,\delta\phi(t')}$$

Particle interpretation (*A.B., N.Kolganov, Phys.Rev.D 109 (2024) 2, 025004, arXiv:2309.03687*)

Wronskian operator in Wronskian relation:

$$\phi_2^T F\phi_1 - (F\phi_2)^T\phi_1 = -\frac{d}{dt}\Big[\phi_2^T W\phi_1 - (W\phi_2)^T\phi_1\Big]$$

For simple models with canonically normalized fields: $W = d/dt$

Special choice of basis functions for :

$$Fv(t) = 0,$$
$$(iW-\omega)v(t)\Big|_{t=0} = 0, \quad (iW+\omega^*)v^*(t)\Big|_{t=0} = 0$$

$\omega = ?$

$$\hat\phi(t) = v(t)\,\hat a + v^*(t)\,\hat a^\dagger, \quad \mathrm{tr}\big[\hat\rho\,\hat a\,\hat a\big] = 0$$

Creation-annihilation operators $\hat a^\dagger, \hat a$

From $\Omega = \begin{bmatrix} R & S \\ S^* & R^* \end{bmatrix}$

$$\omega = R^{1/2}\sqrt{I-\sigma^2}R^{1/2}, \quad \sigma = R^{-1/2}SR^{-1/2}$$

Occupation number matrix

$$\nu \equiv \mathrm{tr}\left[\hat{\rho}\,\hat{a}^\dagger\hat{a}\right] = \frac{1}{2}\varkappa\left(\sqrt{\frac{I-\sigma}{I+\sigma}}-1\right)\varkappa^T,$$

$$\varkappa = \left[\omega^{1/2}R^{-1}\omega^{1/2}\right]^{1/2}\omega^{-1/2}R^{1/2} = \left(\varkappa^T\right)^{-1}$$

$$iG_{\mathsf{T}}(t,t') = v(t)\,v^\dagger(t')\,\theta(t-t') + v^*(t)\,v^T(t')\,\theta(t'-t) + v(t)\,\nu\,v^\dagger(t') + v^*(t)\,\nu\,v^T(t'),$$

$$iG_{>}(t,t') = v(t)\,v^\dagger(t') + v(t)\,\nu\,v^\dagger(t') + v^*(t)\,\nu\,v^T(t')$$

$$v(t) \equiv v_{\mathbf{p}}(t,\mathbf{x}) = \frac{1}{\sqrt{2\omega_{\mathbf{p}}}}e^{-i\omega_{\mathbf{p}}t+i\mathbf{p}\mathbf{x}}$$

$$\nu \equiv \nu(\mathbf{p},\mathbf{p}') = \frac{1}{e^{\beta\omega_{\mathbf{p}}}-1}\delta(\mathbf{p}-\mathbf{p}'), \quad \omega_{\mathbf{p}} = \sqrt{\mathbf{p}^2+m^2}$$

$$v(t)\,\nu\,v^\dagger(t') = \int\frac{d^3\mathbf{p}}{(2\pi)^3}\,\frac{1}{e^{\beta\omega_{\mathbf{p}}}-1}\,\frac{e^{-i\omega_{\mathbf{p}}(t-t')+i\mathbf{p}(\mathbf{x}-\mathbf{x}')}}{2\omega_{\mathbf{p}}}$$

Euclidean path integral density matrix

$$\rho_E(\varphi_-,\varphi_+;J_E] = \frac{1}{Z}\int\limits_{\phi(\tau_\pm)=\varphi_\pm} D\phi\,\exp\left\{-S_E[\phi] - \int\limits_{\tau_-}^{\tau_+} d\tau\,J_E(\tau)\phi(\tau)\right\}$$

Analytical continuation of the action to Euclidean time

$$iS[\phi(t)]\Big|_{t=-i\tau} = -S_E[\phi_E(\tau)], \quad \phi(t)|_{t=-i\tau} = \phi_E(\tau)$$

Triple sources Euclidean-Lorentzian generating functional

$$Z[J_1,J_2,J_E] \equiv \mathrm{tr}\left[\hat{U}_{J_1}(T,0)\,\hat{\rho}_E[J_E]\,\hat{U}^\dagger_{-J_2}(T,0)\right]$$

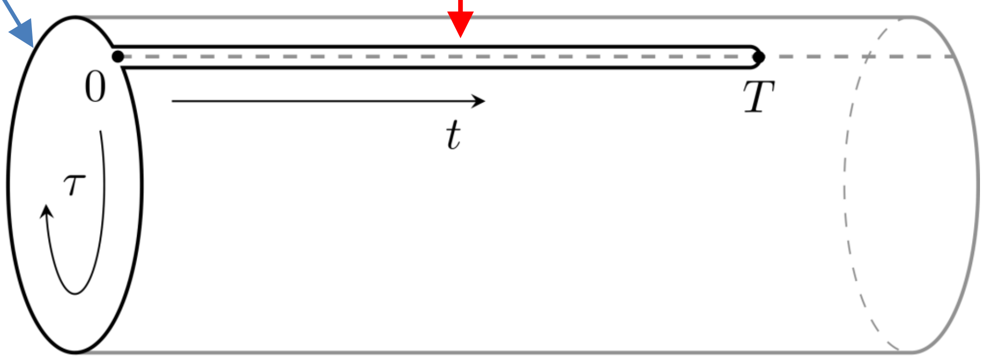
$$\mathbb{J}(z) = \begin{bmatrix} \boldsymbol{J}(t) \\ \\ J_E(\tau) \end{bmatrix} = \begin{bmatrix} J_1(t) \\ J_2(t) \\ J_E(\tau) \end{bmatrix}$$

$$Z[\mathbb{J}] = \text{const} \times \exp\left\{ \frac{1}{2} \int_{\mathbb{C}} dz\, dz'\, \mathbb{J}^T(z)\, \mathbb{G}(z,z')\, \mathbb{J}(z') \right\}$$

Euclidean-Lorentzian contour

$$z = t - i\tau \in \mathbb{C} = [0 \le \tau \le \beta, t = 0] \cup [0 \le t \le T, \tau = 0]$$

doubled



Block structure Green's function on the contour $\mathbb{C}$

$$\mathbb{G}(z,z') = \begin{bmatrix} -i\boldsymbol{G}(t,t') & \boldsymbol{G}^{<}_{LE}(t,\tau') \\ \boldsymbol{G}^{>}_{LE}(\tau,t') & G_E(\tau,\tau') \end{bmatrix}$$

Euclidean block

$$G_E(\tau,\tau') = G^{>}_E(\tau,\tau')\,\theta(\tau-\tau') + G^{<}_E(\tau,\tau')\,\theta(\tau'-\tau),$$

Euclidean-Lorentzian block

$$\boldsymbol{G}^{<}_{LE}(t,\tau) = \begin{bmatrix} I \\ I \end{bmatrix} G^{<}_{LE}(t,\tau), \qquad \boldsymbol{G}^{>}_{LE}(\tau,t) = \left[\boldsymbol{G}^{<}_{LE}(t,\tau)\right]^T$$

Euclidean and Lorentzian basis functions $u_\pm(\tau); v(t), v^*(t)$

$$G^{>}_E(\tau,\tau') = u_+(\tau)(\nu+I)u^T_-(\tau') + u_-(\tau)\,\nu\,u^T_+(\tau'),$$
$$G^{<}_E(\tau,\tau') = \left[G^{>}_E(\tau',\tau)\right]^T,$$
$$G^{<}_{LE}(t,\tau) = v(t)\,(\nu+I)\,u^T_-(\tau) + v^*(t)\,\nu\,u^T_+(\tau)$$

Boundary conditions and quasi-periodicity of Euclidean basis functions

$$(W_E + \omega)u_+\Big|_{\tau=0,\,\beta} = 0, \qquad (W_E - \omega)u_-\Big|_{\tau=0,\,\beta} = 0$$

$$u_-(\tau+\beta) = u_-(\tau)\frac{\nu + I}{\nu}, \qquad u_+(\tau+\beta) = u_+(\tau)\frac{\nu}{\nu + I}$$

$$u_+(\tau) = u_-(-\tau)$$

Relation between Lorentzian and Euclidean basis functions

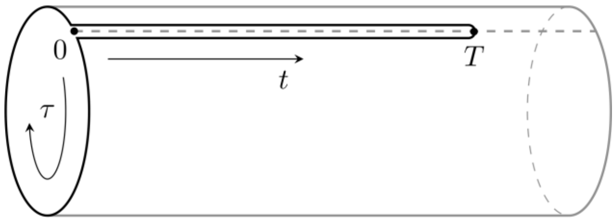
$$v(t) = u_+(it)$$

Analyticity properties on Riemannian surface of the tubular spacetime (A.O.B., N.Kolganov, Phys.Rev.D 109 (2024) 2, 025004, arXiv:2309.03687)

$$F \equiv F(t, d/dt) \to F_{\mathbb{C}} \equiv F(z, d/dz), \quad t \to z = t - i\tau,$$

$$v(t) \to V(z), \quad F_{\mathbb{C}} V(z) = 0,$$

$$\Big(iW_{\mathbb{C}} - \omega\Big) V(z)\Big|_{z=0} = 0$$



$$[V(z)]^* = V(-z^*),$$

$$v(t) = V(z)\Big|_{z=t}, \quad u_\pm(\tau) = V(z)\Big|_{z=\mp i\tau}, \quad V(z) = u_+(iz)$$

Quasi-periodicity of Lorentzian basis functions

$$v(t - i\beta) = v(t)\frac{\nu + I}{\nu}, \quad v^*(t - i\beta) = v^*(t)\frac{\nu}{\nu + I}$$

⇓

KMS condition

$$G_>(t - i\beta, t') = G_<(t, t')$$

Applies to nonstationary, nonequilibrium systems!

Microcanonical density matrix of the Universe *(A.B., Phys. Rev. Lett. 99, 071301 (2007))*

$$\hat{\rho} = \sum_{\text{all } |\boldsymbol{\Psi}\rangle} w_{\Psi} |\,\boldsymbol{\Psi}\,\rangle\langle\,\boldsymbol{\Psi}\,|, \qquad w_{\Psi} = 1$$

sum over "everything" that satisfies
the Wheeler-DeWitt equation $\widehat{H}_{\mu}\,|\boldsymbol{\Psi}\rangle = 0$

Projector onto the subspace of quantum gravitational constraints

$$\hat{\rho} = \frac{1}{Z}\prod_{\mu}\delta(\widehat{H}_{\mu}), \qquad Z = \mathrm{Tr}\prod_{\mu}\delta(\widehat{H}_{\mu})$$

$$\mu = \big(\bot\mathbf{x},\, i\mathbf{x}\big)$$

$$\widehat{H}_{\mu} \equiv \widehat{H}_{\bot\mathbf{x}},\, \widehat{H}_{i\,\mathbf{x}}$$

local operators of Wheeler-DeWitt equations

An ultimate equipartition in the full set of states of the theory --- "Sum over Everything".

Promotion of classical delta function to the quantum level as the path integral on the segment of "time":

$$\prod_\mu \delta(H_\mu) = \int dN\, \exp(-iN^\mu H_\mu),$$

$$\Updownarrow$$

$$\langle\, q_+ \,|\prod_\mu \delta(\hat{H}_\mu)|\, q_- \,\rangle = \int_{q(t_\pm)=q_\pm} DqDpDN\, \exp\Big\{ i \underbrace{\int_{t_-}^{t_+} dt\, \big(p_i\dot{q}^i - N^\mu H_\mu\big)}_{\text{classical ADM action}} \Big\} \times \underbrace{\int D[\,\text{ghosts}\,]\,(...)}_{\text{FP quantum measure}}$$

Origin of time t *in the action entirely as the operator ordering parameter of the noncommutative algebra of quantum constraints*

Transition to Lagrangian path integral $\int DQ\, DP \to \int Dg_{\mu\nu}\, D\Phi$

$$\boldsymbol{\rho}(\varphi_+,\varphi_-) = \frac{1}{Z}\int D[\, g_{\mu\nu},\Phi\,]\; e^{iS[\, g_{\mu\nu},\Phi\,]}\Big|_{\gamma_{ij}(t_\pm)=\gamma^\pm_{ij},\;\; \Phi(t_\pm)=\Phi_\pm}$$

$$\varphi_\pm = (\gamma^\pm_{ij}, \Phi_\pm)$$

Lorentzian

$$Z = \int d\varphi\, \boldsymbol{\rho}(\varphi,\varphi) = \int_{\text{periodic}} D[\, g_{\mu\nu},\Phi\,]\; e^{iS[\, g_{\mu\nu},\Phi\,]}$$

$$\Downarrow$$

Absence of periodic Lorentzian histories and rotation of integration contours over fields and time

Euclidean path integral and its saddle points

$$Z = \int_{\text{periodic}} D[\, g_{\mu\nu},\Phi\,]\; e^{-S[\, g_{\mu\nu},\Phi\,]}$$

Inflationary model driven by the trace anomaly of Weyl invariant fields --- CFT driven cosmology

$$S[\,g_{\mu\nu},\Phi\,]=-\frac{M_P^2}{2}\int d^4x\,g^{1/2}\,(R-2\Lambda)+S_{CFT}[\,g_{\mu\nu},\Phi\,]$$

Λ -- primordial cosmological constant

$$e^{-\Gamma_{CFT}[\,g_{\mu\nu}]}=\int D\Phi\,e^{-S_{CFT}[\,g_{\mu\nu},\Phi]}$$

$$S_{\rm eff}[g_{\mu\nu}]=-\frac{M_P^2}{2}\int d^4x\,g^{1/2}(R-2\Lambda)+\Gamma_{CFT}[g_{\mu\nu}]$$

Recovery of Γ_{CFT} *from the conformal anomaly on a static Einstein Universe (anomaly, Casimir energy and free energy contributions)*

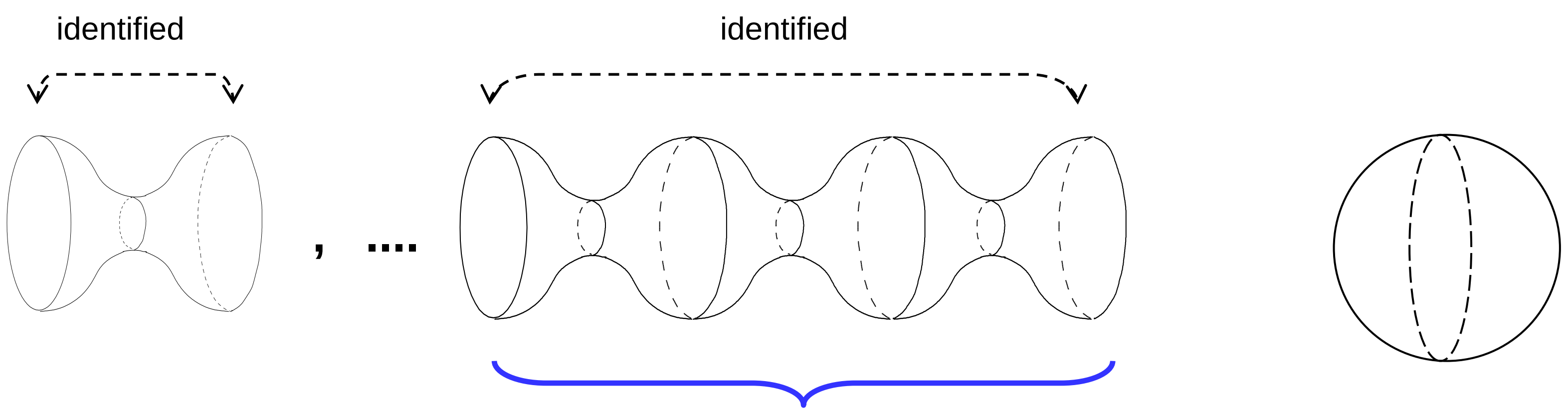
$$g_{\mu\nu}\frac{\delta\Gamma_{CFT}}{\delta g_{\mu\nu}}=\frac{1}{64\pi^2}g^{1/2}\left(\beta E+\alpha\Box R+\gamma C^2_{\mu\nu\alpha\beta}\right)$$

Weyl

β *-- critically important parameter (overall coefficient of Gauss-Bonnet term in conformal anomaly), contributed by CFT fields of various spins*

Effective Friedmann equation for the saddle point of the path integral over the metric

Saddle point solutions --- set of periodic (thermal) garland-type instantons with oscillating scale factor ($S^1 \times S^3$) and the vacuum Hartle-Hawking instantons (S^4)



1- fold, k=1

k- folded garland, k=1,2,3,…

S^4

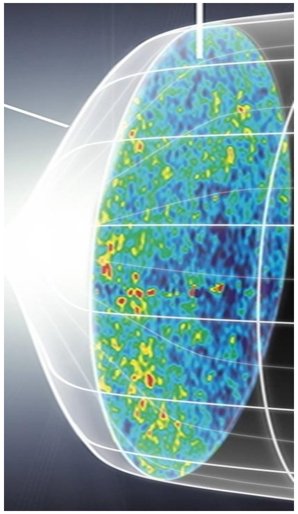
does not contribute: ruled out by infinite positive Euclidean action (effect of conformal anomaly)

UV bounded cosmological constant range:

$$\Lambda_{min} < \Lambda < \Lambda_{max} = \frac{12\pi^2}{\beta} M_P^2$$

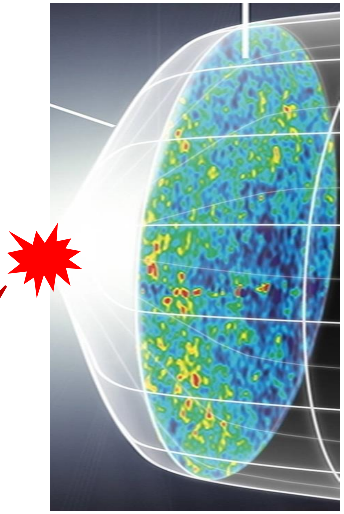
Known inflation paradigm retracted the BB concept by replacing it with the initial vacuum state.
Density matrix scenario “SOME LIKE IT HOT” (SLIH) recovers a new incarnation of Hot Big Bang -- it incorporates effectively thermal state at the onset of the cosmological evolution.

Standard inflation scenario *versus* Density matrix scenario:



Vacuum, absolute zero temperature

Inflation, hot big-bang → relic radiation



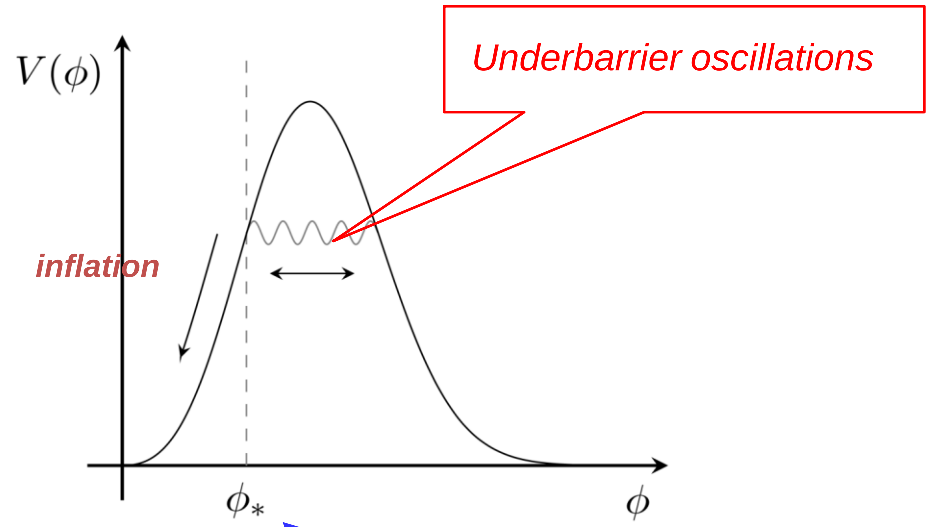
Thermal state, primordial temperature $\boldsymbol{T_{prim}}$

Inflation, $\boldsymbol{T_{prim} \to 0}$*, hot big-bang → relic radiation*

From a toy model with a constant Λ to slow rolling inflaton: $\Lambda \rightarrow V(\phi)$

Dynamical selection of inflaton potential maxima

New type of hill-top inflation:



nucleation point of Lorentzian signature cosmology from Euclidean spacetime

Slow roll parameters typical of Starobinsky R^2 inflation and Higgs inflation (F.Bezrukov, M.Shaposhnikov (2008), A.Kamenshchik, A.Starobinsky & A.B (2008))

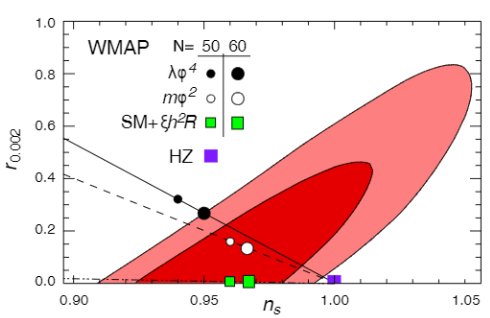


Fig. 2. The allowed WMAP region for inflationary parameters (r, n). The green boxes are our predictions supposing 50 and 60 e–foldings of inflation. Black and white dots are predictions of usual chaotic inflation with $\lambda\phi^4$ and $m^2\phi^2$ potentials, HZ is the Harrison-Zeldovich spectrum.

CMB $\frac{\Delta T}{T} \sim 10^{-5},\ n_s \simeq 0.96,\ r \simeq 0.003 \ll 1$ **?**

The need for Schwinger-Keldysh technique

Conclusions

Schwinger-DeWitt technique for higher derivative and nonminimal operators

UV asymptotic freedom and RG flows in projectable Horava gravity – *single example of renormalizable, local and unitary quantum gravity*

Schwinger-Keldysh formalism for nonstationary density matrix states – *generating functionals, particle interpretation, analyticity properties and KMS condition*

Cosmological initial conditions: *microcanonical density matrix of the Universe*

CFT driven cosmology: *suppression of no-boundary instantons; quasi-thermal stage preceding inflation and UV bounded range of its energy scale -- origin of the Universe is the subplanckian phenomenon, new type of hill-top inflation*