Ring of invariants in two-higgs doublet model: an application of Gröbner bases to RG studies

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Outline

- Two-Higgs-Doublet Model: scalar sector
  - Basis rotations and basis invariants
  - Ring of basis invariants: generating set and syzygies
  - Counting invariants: Hilbert series, etc
- RG for scalar sector of 2HDM: methods & results
  - Challenges in RG computation
  - RG functions for ring elements at 6 loops
  - One-loop examples
- Conclusions and Outlook

Some motivation

- Predictions in QFT involving fields Φ_i depend on parameters a_j

  entering a Lagrangian $\mathcal{L}(\Phi_i, a_j)$

- Regularization and renormalization introduce aux scale μ

$$a_j \to a_j(\mu)$$

  RG equations for parameters (e.g., in $\overline{\mathrm{MS}}$ scheme):

$$\frac{d}{dt} a_j = \beta_{a_j}(a_k), \quad t = \ln(\mu/\mu_0), \quad a_j(\mu_0) = \tilde{a}_j$$

- Basis redundancies can obscure physical questions, e.g.,

  The SM: complex quark Y_u and $Y_d \to$ single physical CP phase

  Basis-independent combinations of a_j (invariants): [Jarlskog]

- Task: RG equations for basis invariants?

Scalar sector of 2HDM

- 2HDM [Lee'73] is one of the simplest SM extensions (see, e.g., [Branco...'12,Ivanov'17])
- Predicts three more scalar states $(H_0, A_0, H^\pm)$ in the spectrum
- Additional sources for CP (and flavor) violation

$$V_H = m_{11}^2 \Phi_1^\dagger \Phi_1 + m_{22}^2 \Phi_2^\dagger \Phi_2 - \left(m_{12}^2 \Phi_1^\dagger \Phi_2 + \mathrm{h.c.} \right), \quad \Phi_{1,2} \text{ are higgs doublets}$$
$$+ \frac{1}{2}\lambda_1 \left(\Phi_1^\dagger \Phi_1\right)^2 + \frac{1}{2}\lambda_2 \left(\Phi_2^\dagger \Phi_2\right)^2 + \lambda_3 \left(\Phi_1^\dagger \Phi_1\right)\left(\Phi_2^\dagger \Phi_2\right) + \lambda_4 \left(\Phi_1^\dagger \Phi_2\right)\left(\Phi_2^\dagger \Phi_1\right)$$
$$+ \left[\frac{1}{2}\lambda_5 \left(\Phi_1^\dagger \Phi_2\right)^2 + \lambda_6 \left(\Phi_1^\dagger \Phi_1\right)\left(\Phi_1^\dagger \Phi_2\right) + \lambda_7 \left(\Phi_2^\dagger \Phi_2\right)\left(\Phi_1^\dagger \Phi_2\right) + \mathrm{h.c.} \right]$$

m_{11}^2, m_{22}^2, $\lambda_{1,2,3,4}$ are real, m_{12}^2 and $\lambda_{5,6,7}$ can be complex

- Freedom to redefine higgs basis $\Phi_a \to U_{ab}\Phi_b, \quad U \in \mathrm{SU}(2)$ reduces the number of physical parameters in the Higgs sector:

  14 (parameters in V_H) - 3 (broken generators) = 11

Basis rotations and basis invariants

- Convenient (favourite) basis choice can simplify computations
- Reparametrization freedom should not affect observables
- Observables in terms of reparametrization invariants
- How to construct basis invariants? [Botella,Silva'95,Davidson,Haber'05]

One can rewrite V_H in a more general form

$$V_H = \frac{1}{2}\lambda_{ab,cd}(\Phi_a^\dagger\Phi_b)(\Phi_c^\dagger\Phi_d) + m_{ab}^2(\Phi_a^\dagger\Phi_b)$$

$$\lambda_{ab,cd} = \lambda_{dc,ba}^\dagger, \quad m_{ba}^2 = m_{ab}^{\dagger 2},$$

and consider various contractions ("traces") of $\lambda_{ab,cd}$ and m_{ab}^2, together with their products. Natural questions:

- How many invariants? Infinite! At each order of PT? Finite!

Bilinear formalism

[Ivanov'05-07]: $2 \otimes \bar{2} = 3 \oplus 1$

$$\Phi_a \Phi_b^\dagger = \frac{1}{2}\left[\; r_0 \;\right]\delta_{ab} + \frac{1}{2}\left[\; \vec{r} \;\right]\vec{\sigma}_{ab}, \qquad V_H = M_\mu r^\mu + \Lambda_{\mu\nu} r^\mu r^\nu$$

$$r_\mu = \{r_0, \vec{r}\}$$

Λ_{00} $\vec{\Lambda}$

$$\Lambda_{\mu\nu} = \begin{pmatrix} \frac{\lambda_1+\lambda_2}{2} + \lambda_3 & \mathrm{Re}\,(\lambda_6+\lambda_7) & -\mathrm{Im}\,(\lambda_6+\lambda_7) & \frac{\lambda_1-\lambda_2}{2} \\ \mathrm{Re}\,(\lambda_6+\lambda_7) & \lambda_4 + \mathrm{Re}\,(\lambda_5) & -\mathrm{Im}\,(\lambda_5) & \mathrm{Re}\,(\lambda_6-\lambda_7) \\ -\mathrm{Im}\,(\lambda_6+\lambda_7) & -\mathrm{Im}\,(\lambda_5) & \lambda_4 - \mathrm{Re}\,(\lambda_5) & -\mathrm{Im}\,(\lambda_6-\lambda_7) \\ \frac{\lambda_1-\lambda_2}{2} & \mathrm{Re}\,(\lambda_6-\lambda_7) & -\mathrm{Im}\,(\lambda_6-\lambda_7) & \frac{\lambda_1+\lambda_2}{2} - \lambda_3 \end{pmatrix}$$

Λ

M_0 $\vec{M}$

$$M_\mu = \{m_{11}^2 + m_{22}^2, -2\mathrm{Re}\, m_{12}^2, 2\mathrm{Im}\, m_{12}^2, m_{11}^2 - m_{22}^2\}$$

Bilinear formalism: SO(3) covariants

- $U \in \mathrm{SU}(2)$ is mapped to rotations $\mathrm{SO}(3)$, $R_{ij} = 1/2\mathrm{Tr}(U^\dagger \sigma_i U \sigma_j)$:

$$\Lambda_{00} \to \Lambda_{00},\; M_0 \to M_0, \quad \vec{\Lambda} \to R\cdot\vec{\Lambda},\; \vec{M} \to R\cdot\vec{M}, \quad \Lambda \to R\cdot\Lambda\cdot R^T,$$

$$\begin{aligned} \Lambda_{00}, M_0 \text{ [and } \mathrm{tr}\Lambda \text{]} \quad &- \quad \text{singlets,} \\ \vec{\Lambda}\text{, and } \vec{M} \quad &- \quad \text{triplets (vectors),} \\ \tilde{\Lambda} \equiv \Lambda - \tfrac{1}{3}\mathrm{tr}\Lambda \quad &- \quad \text{five-plet (traceless symmetric matrix)} \end{aligned}$$

- Advantage: Cayley-Hamilton theorem for 3x3 matrix

$$\Lambda^3 = \mathrm{tr}\Lambda\Lambda^2 - \frac{1}{2}\left(\mathrm{tr}^2\Lambda - \mathrm{tr}\Lambda^2\right)\Lambda + \frac{1}{3!}\left(\mathrm{tr}^3\Lambda - 3\mathrm{tr}\Lambda\mathrm{tr}\Lambda^2 + 2\mathrm{tr}\Lambda^3\right),$$

- Examples: $\vec{\Lambda}\cdot\vec{\Lambda}$ and $\vec{\Lambda}\cdot[(\tilde{\Lambda}\cdot\vec{\Lambda})\times(\tilde{\Lambda}^2\cdot\vec{\Lambda})]$ are basis invariants.

NB: for Generalized CP-transormation $\Phi_a \to X_{ab}\Phi_b^*$

$X \in \mathrm{SU}(2)$ is mapped to $\tilde{R}_{ij} = 1/2\mathrm{Tr}[X^\dagger\sigma_i X\sigma_j^T]$ with $\det\tilde{R} = -1$

Invariants that involve odd number of triplets are CP-odd [Trautner'18].

Ring of basis invariants $\mathcal{R}$ as graded polynomial ring

- Algebraically independent basis invariants $\{f_i\}$:

$$\forall \mathcal{I} \in \mathcal{R}, \quad \exists P \in \mathbf{k}[x_1, \ldots, x_{N+1}] : \quad P(\mathcal{I}, f_1, \ldots, f_N) = 0, \quad N = 11$$

- Generating set of invariants $\{g_n\}$:

$$\forall \mathcal{I} \in \mathcal{R}, \quad \exists \mathcal{P} \in \mathbf{k}[x_1, \ldots, x_M] : \quad \mathcal{I} = \mathcal{P}(g_1, \ldots, g_M), \quad M = 22$$

- Relations between $\{g_n\}$ ("syzygies") form an ideal (R-module)

$$I_{2HDM} = \langle r_1, \ldots, r_K \rangle, \qquad K = 63$$

in polynomial ring $R \equiv \mathbf{k}[g_1, \ldots, g_M]$ generated by $r_k \in R$

$$r_k(g_1, \ldots, g_M) \xrightarrow{\phi} 0, \quad \phi - \text{substitutes } g_i \text{ by their expressions}$$

- Quotient ring of basis invariants $\mathcal{R} = R/I_{2HDM}$:

$$p_1 \sim p_2 : p_1 - p_2 \in I_{2HDM}$$

- Gröbner basis $G = \langle \mathfrak{g}_1 \ldots \mathfrak{g}_J \rangle$ for I_{2HDM} – membership problem:

$$p \in I_{2HDM} \Longleftrightarrow p = \mathrm{a}_1 \mathfrak{g}_1 + \ldots + \mathrm{a}_J \mathfrak{g}_J, \ \mathrm{a}_i \in R \Longleftrightarrow (p \bmod G) = 0$$

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Ring of basis invariants $\mathcal{R}$ as graded polynomial ring

- (Multi)grading of $\mathcal{R}$ [n - (multi) degree]

$$\mathcal{R} = \bigoplus_{n=0}^{\infty} \mathcal{R}_n = \mathcal{R}_0 \oplus \mathcal{R}_1 \oplus \mathcal{R}_2 \oplus \cdots \qquad \mathcal{R}_i \mathcal{R}_j \subseteq \mathcal{R}_{i+j}$$

  - $n \to$ total number of λ_i and m^2_{ij} entering an invariant
  - $n \to \{\alpha, \beta\}$: homogeneous element $\mathcal{I}_{\alpha\beta} \in \mathcal{R}_{\alpha\beta}$: [convenient of RG]

$$\mathcal{I}_{\alpha\beta} \to x^\alpha y^\beta \cdot \mathcal{I}_{\alpha\beta}, \quad \text{if } \lambda_i \to x \cdot \lambda_i,\ m^2_{ij} \to y \cdot m^2_{ij}$$

  - $n \to [a, b, c]$: homogeneous element $T_{abc} \in \mathcal{R}_{abc}$: [Trautner'18]

$$T_{abc} \to x^a y^b z^c \cdot T_{abc}, \quad \text{if } \tilde{\Lambda} \to x \cdot \tilde{\Lambda},\ \vec{M} \to y \cdot \vec{M},\ \vec{\Lambda} \to z \cdot \vec{\Lambda}$$

- Examples:

$$\Lambda_{00}, \mathrm{tr}\Lambda \in \mathcal{R}_{10} \subset \mathcal{R}_1, \quad M_0 \in \mathcal{R}_{01} \subset \mathcal{R}_1, \quad \vec{\Lambda} \cdot \vec{M} \in \mathcal{R}_{011} \subset \mathcal{R}_{11} \subset \mathcal{R}_2$$

Hilbert Series and Plethystic Logarithm

see [Hanany...'09,Jenkins...'09,Hanany...'10,Lehman...'15]

- Tool to count linear independent elements of $\mathcal{R}_n$:

$$H(t) = \sum_n (\dim \mathcal{R}_n) t^n$$ [Bednyakov'18]

$$H(\lambda, m) = \sum_{\alpha\beta} (\dim \mathcal{R}_{\alpha\beta}) \lambda^\alpha m^\beta$$ [Bednyakov'25]

$$H(q, y, t) = \sum_{abc} (\dim \mathcal{R}_{abc}) q^a y^b t^c$$ [Trautner'18]

$$H(t) = H(t,t), \quad H(\lambda, m) = \frac{1}{\underbrace{(1-\lambda)^2}_{\Lambda_{00}, \mathrm{tr}\,\Lambda} \underbrace{(1-m)}_{M_0}} H(\lambda, m, \lambda)$$

$$H(t) = \frac{1 + t^3 + 4y^4 + 2t^5 + 4t^6 + t^7 + t^{10}}{\underbrace{(1-t)^3(1-t^2)^4(1-t^3)^3(1-t^4)}_{\text{degree and number of } \{f_i\}}} = \frac{7/864}{\underbrace{(1-t)^{11}}_{N=11}} + \dots$$

Hilbert Series and Plethystic Logarithm

see [Hanany…'09,Jenkins…'09,Hanany…'10,Lehman…'15]

- Tool to enumerate generators of the invariants ring and of the syzygy module:

$$PL[H(q,y,t)] \equiv \sum_{k=1}^{\infty} \frac{\mu(k) \ln H[q^k, y^k, t^k]}{k}, \quad \mu(k) - \text{Möbius function}$$

$$PL[H(t)] \equiv \sum_{k=1}^{\infty} \frac{\mu(k) \ln H[t^k]}{k} = \sum_{n=1}^{\infty} d_n t^{a_n}, \qquad \sum_{k|m} \mu(k) = \delta_{m,1}$$

$$= 3t + 4t^2 + 4t^3 + 5t^4 + 2t^5 + [4-1]t^6$$

$$- 3t^7 - 12t^8 - 12t^9 - [17-2]t^{10} - [8-14]t^{11} - [10-38]t^{12} + \dots$$

- First positive terms – ring generators and their degree: $M = 22$
- First negative terms – syzygies and their degree: $K = 63$
- Explicitly construct generators $\{g_n\}$ and syzygies $\{r_k\}$?

Generating set of invariants

- Full set $\{g_n\}$ was constructed for the first time in [Trautner'18]:

$3t$:	$Z_{1_{(1)}}, Z_{1_{(2)}}, Y_1,$	
$4t^2$:	$T_{200}, T_{020}, T_{002}, T_{011},$	expressed
$4t^3$:	$T_{300}, T_{120}, T_{102}, T_{111},$	in terms of
$5t^4$:	$T_{220}, T_{202}, T_{211}, T_{112}, T_{121},$	14 real parameters
$2t^5$:	$T_{212}, T_{221},$	(related to λ_i and m_{ij}^2)
$4t^6$:	$T_{312}, T_{321}, T_{303}, T_{330}$	

- We re-express $\{g_n\}$ in terms of SO(3) covariants, e.g.,

$$8 \cdot T_{200} = \mathrm{tr}\tilde{\Lambda}^2, \; 8 \cdot T_{111} = (\vec{\Lambda} \cdot \tilde{\Lambda} \cdot \vec{M}), \qquad \text{CP-even}$$

$$-16i \cdot T_{112} = \vec{M} \cdot [\vec{\Lambda} \times (\tilde{\Lambda} \cdot \vec{\Lambda})] \qquad \text{CP-odd}$$

- Algebraically independent elements $\{f_i\}$: [Trautner'18,Bento et al'20]

$$\underbrace{Z_{1_{(1)}}, Z_{1_{(2)}}, Y_1,}_{\text{degree 1}} \underbrace{T_{200}, T_{002}, T_{020}, T_{011},}_{\text{degree 2}} \underbrace{T_{300}, T_{102}, T_{120},}_{\text{degree 3}} \underbrace{T_{211}}_{\text{degree 4}} .$$

Syzygies: How to find?

- Can be computed by a linear-algebra method: [Trautner'18]
  - construct a linear combination of all possible power products I_k of g_n at certain degree with unknown coefficients C_k:

$$C_k \cdot I_k(g_n) \stackrel{?}{=} 0$$

  - express all g_n in terms of $x_j \propto \{\lambda_i, m_{ij}^2\}$: $g_n \stackrel{\phi}{\to} g_n(x_j)$
  - set coefficients of monomials constructed from x_j to zero and obtain a system of linear equations on C_k:

$$A_{mk} \cdot C_k = 0 \quad \Rightarrow \quad C_k \cdot I_k(g_n) \in I_{2HDM}$$

  - Disadvantage: At higher degrees it is hard to decide whether an obtained relation is a "new" one or a consequence of ones of lower degree...
- Alternative: use Gröbner-basis techniques to derive syzygies and find minimal generators r_k of the syzygy module...

Syzygies: How to find?

see, e.g., textbook [Cox, Little, O'Shea'2015]

- Consider an auxiliary polynomial ring over a field $\mathbf{k}$

$$\tilde{R} = \mathbf{k}[x_1, \ldots, x_N, g_1, \ldots, g_M]$$

NB: convenient choice of x_j can simplify/speed up calculations

- Consider an ideal

$$\mathcal{I} = \langle g_1 - g_1(x_1, \ldots, x_N), \ldots, g_M - g_M(x_1, \ldots, x_N)\rangle$$

- The syzygy ideal is the elimination ideal

$$I_{2HDM} = \mathcal{I} \cap \mathbf{k}[g_1, \ldots, g_M]$$

- syzygies up to certain degree belonging to I_{2HDM} can be found by constructing a Gröbner basis for $\mathcal{I}$ [Bednyakov'25]
- We use [Macaulay2] package to implement the algorithm and find minimal set of relations (generators r_k of syzygy module)

NB: The form of generators r_k depends on the monomial ordering!

Generators of syzygy module

[Trautner'18,Bento et al'20,Bednyakov'25]

- 3 "Even×Even" syzygies allowing one to get rid of products

$$T_{111}{}^2 \quad T_{111}T_{211} \quad T_{211}{}^2$$
$$^6[222] \quad ^7[322] \quad ^8[422]$$

Example: $[222]$ - syzygy

$$T_{111}{}^2 = 2T_{011}T_{211} + 3T_{102}T_{120} + (T_{002}T_{020} - T_{011}^2)T_{200} - T_{020}T_{202} - T_{002}T_{220}$$

- 24 "Odd×Even" syzygies: get rid, e.g., of products

$$T_{111} \cdot \{T_{112}, T_{212}, T_{312}, T_{303}\}, \quad T_{211} \cdot \{T_{112}, T_{212}, T_{312}, T_{303}\},$$
$$T_{303} \cdot \{T_{011}, T_{020}, T_{120}, T_{220}\},$$

$[acb]$ – syzygy from $[abc]$ one

- 36 "Odd ×Odd" syzygies: any product of two odd invariants

$$\{T_{112}, T_{121}, T_{212}, T_{221}, T_{312}, T_{321}, T_{303}, T_{330}\}$$

can be expressed in terms of even elements

Ring of invariants and RG

- Given $\{g_n\}$ and $\{r_k\}$ we explicitly construct bases* $\{I^{(i)}_{\alpha,\beta}\}$ of $\mathcal{R}_{\alpha\beta}$ needed to find L-loop beta function of an element $I_{\alpha\beta} \in \{g_n\}$

$$\frac{d}{dt} I_{\alpha\beta} = \sum_{l=1}^{L} h^l \cdot \beta^{(l)}_{I_{\alpha\beta}}, \quad \beta^{(l)}_{I_{\alpha\beta}} = \underbrace{\sum_{i=1}^{d_{\alpha+l,\beta}} c^{(i)}_{\alpha,\beta;l} I^{(i)}_{\alpha+l,\beta}}_{\text{element of } \mathcal{R}_{\alpha+l,\beta}}, \qquad h \equiv 1/(16\pi^2)$$

$$d_{\alpha,\beta} = (\dim R_{\alpha\beta}) = d^{even}_{\alpha,\beta} + d^{odd}_{\alpha,\beta} \qquad \text{[from Hilbert Series } H(\lambda, m)\text{]}$$

*We use a Gröbner basis G for I_{2HDM} to test if a monomial $m_{\alpha\beta} = g_1^{\alpha_1} \ldots g_M^{\alpha_M}$ with multidegree $\{\alpha\beta\}$ belongs to basis of $\mathcal{R}_{\alpha\beta}$: normal form $(m_{\alpha\beta} \bmod G) = m_{\alpha\beta}$

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$$\left[\beta^{(l)}_{\lambda_i}\partial_{\lambda_i} + \beta^{(l)}_{m^2_{ij}}\partial_{m^2_{ij}}\right] I_{\alpha\beta}(\lambda_k, m^2_{pq}) = \sum_{i=1}^{d_{\alpha+l,\beta}} c^{(i)}_{\alpha,\beta;l} I^{(i)}_{\alpha+l,\beta}(\lambda_k, m^2_{pq})$$

$d_{\alpha,\beta} = (\dim R_{\alpha\beta}) = d^{even}_{\alpha,\beta} + d^{odd}_{\alpha,\beta}$ [from Hilbert Series $H(\lambda, m)$]

- Numerical coefficients $c^{(i)}_{\alpha,\beta;l}$ are found by linear algebra: we express LHS and RHS in terms of λ_i, and m^2_{ij} and use 6-loop beta functions for the latter derived in [Bednyakov,Pikelner'21]

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- Beta function for an arbitrary invariant $p \in \mathcal{R}$:

$$\beta_p = \left[\sum_{n=1}^{M} \frac{\partial p}{\partial g_n}\beta_{g_n}\right] \mod G$$

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Various dimensions of $\mathcal{R}_{\alpha\beta}$ for $\beta = 0, 1, 2, 3$

	$Z_{1_{(1)}}$	T_{200}	T_{300}								
	$Z_{1_{(2)}}$	T_{002}	T_{102}	T_{202}							
α	2	3	4	5	6	7	8	9	10	11	12
even	**5**	**10**	**19**	**32**	**54**	**84**	**129**	**190**	**275**	386	536
odd	0	0	0	0	1	**2**	**5**	**10**	**19**	**32**	**54**
						T_{303}					

	Y	T_{011}	T_{111}	T_{211}							
α	1	2	3	4	5	6	7	8	9	10	11
even	**3**	**8**	**18**	**36**	**66**	**115**	**189**	**299**	**457**	678	980
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	T_{020}	T_{120}	T_{220}							
α	1	2	3	4	5	6	7	8	9	10
even	**6**	**17**	**38**	**78**	**144**	**254**	**420**	**671**	1030	1539
odd	0	1	**4**	**12**	**28**	**60**	**114**	**205**	**346**	**561**
			T_{121}	T_{221}	T_{321}					

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Results: one-loop RG (examples)

$$
\begin{aligned}
\beta^{(1)}_{Z_{1_{(1)}}} &= 28T_{200} + 48T_{002} + \frac{29}{3}Z^2_{1_{(1)}} + 2Z_{1_{(1)}}Z_{1_{(2)}} + 3Z^2_{1_{(2)}}, & d^{even}_{2,0} &= 5\\
\beta^{(1)}_{T_{303}} &= 12[6Z_{1_{(1)}} - Z_{1_{(2)}}]T_{303}, & d^{odd}_{7,0} &= 2\\
\beta^{(1)}_{Y_1} &= 2[3Z_{1_{(1)}} + Z_{1_{(2)}}]Y_1 + 24T_{011}, & d^{even}_{1,1} &= 3\\
\beta^{(1)}_{T_{112}} &= \frac{2}{3}[65Z_{1_{(1)}} - 9Z_{1_{(2)}}]T_{112} - 28T_{212}, & d^{odd}_{4,1} &= 3\\
\beta^{(1)}_{T_{020}} &= 4[Z_{1_{(1)}} - Z_{1_{(2)}}]T_{020} + 24T_{120} + 12T_{011}Y_1, & d^{even}_{1,2} &= 6\\
\beta^{(1)}_{T_{121}} &= \frac{8}{3}[11Z_{1_{(1)}} - 3Z_{1_{(2)}}]T_{121} - 6T_{112}Y_1 - 4T_{221}, & d^{odd}_{3,2} &= 4\\
\beta^{(1)}_{T_{330}} &= 2(17Z_{1_{(1)}} - 9Z_{1_{(2)}})T_{330} + 24[T_{120}T_{112} - T_{020}T_{212}] & &\\
&\quad - 72T_{011}T_{221} + 6T_{321}Y_1, & d^{odd}_{4,3} &= 21
\end{aligned}
$$

We derived all RG functions up to 6 loops and provided [Macaulay2] routines to compute normal forms in [Bednyakov'25]

Conclusions and outlook

Summary: [Bednyakov'25]

- Considered polynomial ring of basis invariants in 2HDM
- Introduced convenient representation [SO(3) covariants] for all $M = 22$ generators $\{g_i\}$ of the ring
- Used Gröbner bases to find all $K = 63$ non-trivial relations $\{r_k\}$ that generate I_{2HDM} (syzygy module)
- Derived 6-loop beta functions for elements of the ring
- Obtained Hilbert series* via free resolution of syzygy module
- Checked RG invariance* of syzygies

TODO:

- RG for reduced ring (imposed symmetries) [Bento et al'20]
- Gauge and Yukawa interactions: CP non-conservation for real 2HDM potential discussed recently in literature [de Lima,Logan'24]
- Multi-doublet extensions of the SM [Bento'21]

Thank you for attention!

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Backup

Hilbert Series: Molien-Weyl formula

- Plethystic exponent

$$\mathrm{PE}[z,t;\mathrm{r}] = \exp\left[\sum_{i=1}^{\infty} \frac{t^i \chi_{\mathrm{r}}(z^i)}{i}\right]$$

- Characters of [SU(2)] representations $\mathrm{r} = \{3, 5\}$

$$\chi_3(z) = z^2 + 1 + \frac{1}{z^2}, \quad \chi_5(z) = z^4 + z^2 + 1 + \frac{1}{z^2} + \frac{1}{z^4}$$

- (Multi-graded) Hilbert Series (via Molien-Weyl formula)

$$H(q,y,t) = \int d\mu_{\mathrm{SU}(2)}(z) \cdot \underbrace{\mathrm{PE}[z,q,5]}_{\tilde{\Lambda}} \cdot \underbrace{\mathrm{PE}[z,y,3]}_{\vec{M}} \cdot \underbrace{\mathrm{PE}[z,t,3]}_{\vec{\Lambda}}$$

$$\int d\mu_{\mathrm{SU}(2)} = \frac{1}{2\pi i} \oint_{|z|=1} \frac{dz}{z}(1 - z^2)$$

Simple [Mathematica] routine based on [LieART] package

Hilbert Series: MacMahon Omega calculus

see [Bento'21] for application to 2HDM and NHDM

- An operator $\Omega_{=}$ that extracts constant term of a series:

$$\underset{=}{\Omega} \sum_{j_1=-\infty}^{\infty} \cdots \sum_{j_n=-\infty}^{\infty} a_{j_1,\ldots,j_n} \lambda_1^{j_1} \ldots \lambda_n^{j_n} := a_{0,\ldots,0}$$

- Fast evaluation based on

$$H(q, y, t) = \underset{=}{\Omega} \left[(1 - z^2) \cdot \underbrace{\mathrm{PE}[z, q, 5]}_{\tilde{\Lambda}} \cdot \underbrace{\mathrm{PE}[z, y, 3]}_{\vec{M}} \cdot \underbrace{\mathrm{PE}[z, t, 3]}_{\vec{\Lambda}} \right]$$

[Omega] Mathematica package by [Andrews et al'01]

[Maple] code by [G.Xin'04]

Hilbert Series: Free resolution of syzygy module

- Syzygy module for 2HDM ($\mathcal{M}_0$):

$$a_k \in R, k = 1 \ldots 63, \quad \{a_k\} \xrightarrow{\phi_0} f = \sum_{K=0}^{63} a_k \cdot r_k \in R,$$

is not free (not isomorphic to $F_0 = R^{63}$): there are relations (higher syzygies) between generators r_k, i.e., $\ker \phi_0 \neq \emptyset$:

$$\exists b_k^{(s)} \in R : \qquad \sum_{K=0}^{63} b_k^{(s)} \cdot g_k = 0, \quad s = 1, \ldots, 411,$$

so

$$\mathcal{M}_0 = F_0 / \ker \phi_0$$

- The Module $\mathcal{M}_1 = \ker \phi_0 = \mathrm{im}\ \phi_1$ of second syzygies generated by 63-tuples $b^{(s)}$, is again non-free (not isomorphic to R^{411}).
- Repeating this procedure until $\mathrm{im}\ \phi_i = 0$, we get a free resolution of the syzygy module.

Hilbert Series: Free resolution of syzygy module

- Free resolution can be represented by exact sequence

$$0 \xleftarrow{\phi} R \xleftarrow{\phi_0} R^{63} \xleftarrow{\phi_1} R^{411} \longleftarrow R^{1358} \longleftarrow R^{2835} \longleftarrow R^{4038} \leftarrow \cdots \leftarrow R^{63} \leftarrow R \leftarrow 0$$

by means of [Macaulay2] package

- Betti table counts the number and degree of (higher) syzygies

	0	1	2	3	4	5	6	7	8	9	10	11
total	1	63	411	1358	2835	4038	4038	2835	1358	4111	63	1
0	1	0	0	0	0	0	0	0	0	0	0	0
...						...						
5	0	1	0	0	0	0	0	0	0	0	0	0
6	0	3	0	0	0	0	0	0	0	0	0	0
7	0	12	0	0	0	0	0	0	0	0	0	0
8	0	12	2	0	0	0	0	0	0	0	0	0
9	0	17	14	0	0	0	0	0	0	0	0	0
10	0	8	38	0	0	0	0	0	0	0	0	0
11	0	10	63	1	0	0	0	0	0	0	0	0
12	0	0	91	23	0	0	0	0	0	0	0	0
13	0	0	79	60	0	0	0	0	0	0	0	0
14	0	0	72	143	0	0	0	0	0	0	0	0
15	0	0	32	220	16	0	0	0	0	0	0	0
16	0	0	20	274	56	0	0	0	0	0	0	0
17	0	0	0	260	170	0	0	0	0	0	0	0
...						...						
52	0	0	0	0	0	0	0	0	0	0	0	1

an element b_{ij} counts (higher) syzygies of degree $i + j$

Full table in [Bednyakov'25]

Hilbert Series: Free resolution of syzygy module

- Given Betti table elements b_{ij}, one constructs a numerator:

$$\tilde{N}(t) = H(t) \times \left(1 - t^1\right)^3 \qquad [Y_1, Z_{1_{(1)}}, Z_{1_{(2)}}]$$
$$\times \left(1 - t^2\right)^4 \qquad [T_{200}, T_{020}, T_{002}, T_{011}]$$
$$\times \left(1 - t^3\right)^4 \qquad [T_{300}, T_{120}, T_{102}, T_{111}]$$
$$\times \left(1 - t^4\right)^5 \qquad [T_{220}, T_{202}, T_{211}, T_{112}, T_{121}]$$
$$\times \left(1 - t^5\right)^2 \qquad [T_{212}, T_{221}]$$
$$\times \left(1 - t^6\right)^4 . \qquad [T_{312}, T_{321}, T_{303}, T_{330}]$$

via

$$\tilde{N}(t) = \sum_{ij} (-1)^j b_{ij} t^{i+j} = 1 - t^6 - 3t^7 - 12t^8 - 12t^9 \ldots$$
$$+ 12t^{54} + 12t^{55} + 3t^{56} + t^{57} - t^{63}$$

Full result [Bednyakov'25]