Vogel universality and diagrammatic technique

Dmitry Khudoteplov

Institute for Information Transmission Problems,
Moscow Institute of Physics and Technology

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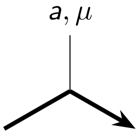
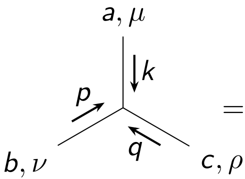
Yang-Mills diagrammatic technique



$$= ig\gamma^{\mu} T^{a}$$



$$= g f^{abc}[g^{\mu\nu}(k-p)^{\rho} + g^{\nu\rho}(p-q)^{\mu} + g^{\rho\mu}(q-k)^{\nu}]$$

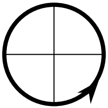
Chern-Simons diagrammatic technique

In Chern-Simons 3D topological theory observables are the Wilson loops:

$$\langle W_R(K) \rangle = \langle \mathrm{tr}_R \mathrm{P} \exp \left(\oint_K A \right) \rangle$$

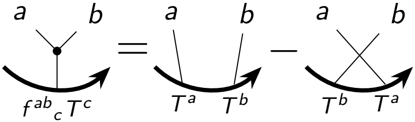
Perturbative expansion of Wilson loop vacuum expectation value yields:

$$\langle W_R(K) \rangle = \sum_m h^m \oint dx_1 \dots dx_m \langle A^{a_1}(x_1) \dots A^{a_m}(x_m) \rangle \mathrm{tr}_R \left(T^{a_1} \dots T^{a_m} \right)$$

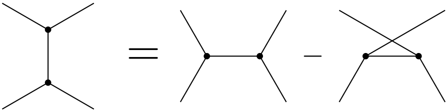


$$\longleftrightarrow \mathrm{tr}_R(T^a T^b T^a T^b)$$

STU, AS and IHX relations

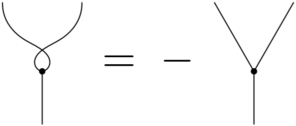


$$\Leftrightarrow f^{ab}{}_c T^c = [T^a, T^b] = T^a T^b - T^b T^a$$



$$\Updownarrow$$

$$f^{ad}{}_e f^{bc}{}_d + f^{bd}{}_e f^{ca}{}_d + f^{cd}{}_e f^{ab}{}_d = 0$$



$$\Updownarrow$$

$$f^{ab}{}_c = -f^{ba}{}_c$$

Vogel universality

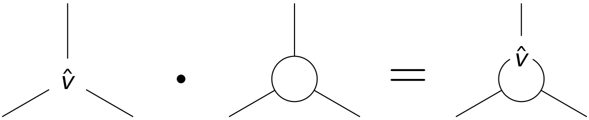
Vogel universality is an idea that some Lie algebra properties can be expressed in a uniform way as a function of the so-called Vogel parameters. Currently there are several features that are known to admit Vogel universalization:

- Decompositions of adjoint representation powers [Vogel, 1999; Isaev, Krivonos, 2024 & others]
- Higher Casimir eigenvalues [Mkrtchyan, Sergeev, Veselov, 2012]
- Quantum dimension [Mkrtchyan, Veselov, 2012]
- Quantum knot invariants [Mironov, Mkrtchyan, Morozov, 2016]

Λ algebra

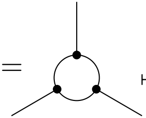
Definition

Λ is an algebra over $\mathbb{Q}$ generated by 3-legged diagrams modulo AS and IHX relations antisymmetric with respect to permutations of legs. Multiplication in Λ is given by insertion of one diagram into any vertex of the other diagram.



Characters on Λ

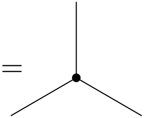
Trivalent diagrams correspond to contractions of structure constants. In case of Λ algebra three free indices remain, therefore the result is proportional to the structure constant.

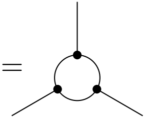
$\hat{t} =$  $\mapsto f^{aik} f^{bji} f^{ckj} = \chi_L(\hat{t}) f^{abc}$

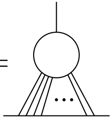
The coefficient of proportionality is called a *character* of this element of Λ. Each Lie algebra induces its own character on Λ. One can say that Lie algebras can by parameterized by these characters.

Structure of Λ

In Λ there exist the following diagrams that happen to be multiplicative generators of the studied sector of Λ:

$\hat{1} =$ 

$\hat{t} =$ 

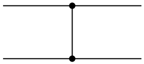
$\hat{x}_n =$ 

However, there are many relations between the x_n diagrams, that allow to rewrite them in terms of polynomials in three parameters t, σ and ω of the form $\mathbb{Q}[t] \oplus \omega\mathbb{Q}[t, \sigma, \omega]$

The most interesting part about the Λ algebra is an existence of zero divisors. They effectively restrict possible values of the Vogel parameters to those that satisfy:

$$t\omega(2t\sigma - \omega - 2t^3)(27\omega - 45t\sigma + 40t^3) \times$$
$$\times(27\omega^2 - 72t\sigma\omega + 40t^3\omega + 4\sigma^3 + 29t^2\sigma^2 - 24t^4\sigma) = 0$$

Adjoint square decomposition

On the tensor square of adjoint representation there exists an operator Ψ, also known as the split Casimir operator: 

It was shown by Vogel that this diagram satisfies a cubic relation on the symmetric square, factorised by a one-dimensional reprsentation:

$$\Psi^3 - t\Psi^2 + (\sigma - 2t^2)\Psi - (\omega - t\sigma)\hat{I} = 0$$

This implies the following decomposition:

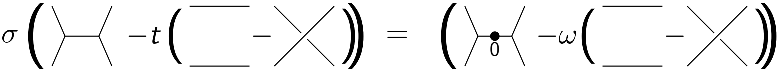
$$S^2\mathrm{adj} = \Omega \oplus Y_\alpha \oplus Y_\beta \oplus Y_\gamma$$

Where α, β and γ are the eigenvalues of Ψ. They are related to other Vogel parameters as follows:

$$\alpha + \beta + \gamma = t, \quad \alpha\beta + \beta\gamma + \alpha\gamma = \sigma - 2t^2, \quad \alpha\beta\gamma = \omega - t\sigma\,.$$

Universal dimension

It Vogel's paper there is following diagrammatic equation:



Taking trace over it yields:

$$2t\sigma \text{dim} - t\sigma(\text{dim} - 1) = -\omega(\text{dim} - 1)$$

Hence, the dimension of Lie algebra can be expressed as:

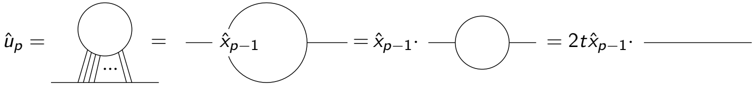
$$\text{dim} = \frac{\omega - 3t\sigma}{\omega - t\sigma} = \frac{(\alpha - 2t)(\beta - 2t)(\gamma - 2t)}{\alpha\beta\gamma}$$

Universal Casimir eigenvalues

Consider the following Casimir operators:

$$C_p = \mathrm{Tr}_{ad}(X^{\mu_1} X^{\mu_2} \cdots X^{\mu_p}) X_{\mu_1} X_{\mu_2} \cdots X_{\mu_p}$$

The correspond to diagrams of the following form:



$C_p = (-1)^p u_p = (-1)^p \cdot 2t x_{p-1}$. x_n is related to α, β and γ as follows [Kneissler, 2001]:

$$x_n = -\frac{\alpha\beta\gamma(2t)^n}{(2t-\alpha)(2t-\beta)(2t-\gamma)} + \frac{t^n}{2} + c_\alpha \alpha^n + c_\beta \beta^n + c_\gamma \gamma^n$$

Same result has been achieved by Mkrtchyan, Sergeev and Veselov for classical Lie algebras, later generalized by Isaev and Provorov for Lie superalgebras.

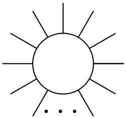
Universal quantum dimension

The formula for the quantum dimension [Mkrtchyan, Veselov, 2012]:

$$r + \sum_{\mu \in R} e^{x(\mu,\rho)} = \frac{\sinh((\alpha-2t)x/4)\sinh((\beta-2t)x/4)\sinh((\gamma-2t)x/4)}{\sinh(\alpha x/4)\sinh(\beta x/4)\sinh(\gamma x/4)}$$

This formula corresponds to the following diagram expansion, known as the Kontsevich integral of unknot:

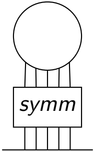
$\mathrm{qdim} = I(\bigcirc) = \exp\left(\sum_{n=1}^{\infty} b_{2n} w_{2n}\right)$, where $b_{2n} = \frac{(-1)^{n+1}\zeta(2n)}{2n(2\pi)^{2n}}$ and

$w_{2n} =$ 

Universal symmetrized Casimir eigenvalues

$$C_n^{\text{symm}} = \text{tr}_{\text{adj}} \left(T^{i_1} T^{i_2} T^{i_3} \dots T^{i_n} \right) \frac{1}{n!} \sum_{\sigma \in S_n} T_{i_{\sigma}(1)} T_{i_{\sigma}(2)} T_{i_{\sigma(3)}} \dots T_{i_{\sigma(n)}}$$

Action of these operators corresponds to the following diagram:



It is unknown how to express this diagram through regular Vogel parameters, but in can be done at low orders:

$$C_4^{\text{symm}} = \frac{20}{3} t^4 - 3t\omega \qquad C_6^{\text{symm}} = \frac{28}{3} t^6 - \frac{42}{5} t^3 \omega + \frac{15}{8} t\sigma\omega$$