A.V.Kotikov, JINR, Dubna

(in collab. with S. Teber (LPTHE, Paris), I.A. Kotikov (MSU, Moscow),
R.N. Lee (IYF, Novosibirsk), A.I. Onishchenko (JINR, Dubna)).

International Workshop Problems of Modern Mathematical Physics, February 10 - 14, Dubna

based on Phys.Rev. D100 (2019) no.10, 105017 [1906.10930 [hep-th]]

work in progress

On hatted zeta-values in Feynman diagrams

OUTLINE

1. Introduction
2. Results
3. Conclusions and Prospects

Abstract

With the help of the Landau-Khalatnikov-Fradkin (LKF) transformation, we derive a non-perturbative identity between massless propagators in two different gauges.
From this identity, we find that the corresponding perturbative series can be exactly expressed in terms of a hatted transcendental basis that eliminates all even ζ-values. Our construction further allows us to derive an exact formula relating hatted and standard ζ-values to all orders of perturbation theory.

From consideration of 4-loop master integrals, we find the perturbative series of a hatted transcendental basis based on more complicated Euler-Zagier (nested) ζ-values, up to the transcendentality weight 14.

0 Hatted ζ-values

A seemingly unrelated topic is focused on the multi-loop structure of propagator-type functions (p-functions).
Following (Baikov, Chetyrkin: 2018) by p-functions we understand ($\overline{\mathrm{MS}}$-renormalized) Euclidean 2-point functions (that can also be obtained from 3-point functions by setting one external momentum to zero with the help of infra-red rearrangement) expressible in terms of massless propagator-type Feynman integrals also known as p-integrals.

About three decades ago, it was noticed that all contributions proportional to $\zeta_4 = \pi^4/90$ mysteriously cancel out in the Adler function at three-loops (Gorishnii, Kataev, Larin: 1990).

Two decades later, it was shown that the four-loop contribution is also π-free and that a similar fact holds for the coefficient function of the Bjorken sum rule (Baikov, Chetyrkin, Kühn: 2010).

There is by now mounting evidence, see, *e.g.*, (Baikov, Chetyrkin, Kühn: 2017), (Chetyrkin, Falcioni, Herzog, Vermaseren: 2017), (Herzog, Ruijl, Ueda, Vermaseren, Vogt: 2017, 2018), (Davies, Vogt: 2018), (Moch, Ruijl, Ueda, Vermaseren, Vogt: 2018), (Vogt, Herzog, Moch, Ruijl, Ueda, Vermaseren: 2018), that various massless Euclidean physical quantities demonstrate striking regularities in terms proportional to even ζ-function values, ζ_{2n}, *e.g.*, to π^{2n} with n being a positive integer.

Such puzzling facts have recently given rise to the "no-π theorem". The latter is based on the observation (Broadhurst: 1999), (Baikov, Chetyrkin: 2010, 2018) that the ε-dependent transformation of the ζ-values:

$$\hat{\zeta}_3 \equiv \zeta_3 + \frac{3\varepsilon}{2}\zeta_4 - \frac{5\varepsilon^3}{2}\zeta_6, \quad \hat{\zeta}_5 \equiv \zeta_5 + \frac{5\varepsilon}{2}\zeta_6, \quad \hat{\zeta}_7 \equiv \zeta_7\,,$$

eliminates even zetas from the expansion of four-loop p-integrals. A generalization to 5- , 6- and 7-loops is available in (Georgoudis, Goncalves, Panzer, Pereira: 2018), (Baikov, Chetyrkin: 2018, 2019), respectively.

Definition:

$$\zeta_a = \sum_{k>1} \frac{1}{k^a}, \quad \zeta_{a,b} = \sum_{k>m\geq 1} \frac{1}{k^a m^b}, \quad \zeta_{a,b,c} = \sum_{k>m>l\geq 1} \frac{1}{k^a m^b l^c}\,.$$

Here we shall use the LKF transformation in order to study general properties of the coefficients of the propagator. We will show how the transformation naturally reveals the existence of the hatted transcendental basis. Moreover, it will allow us to extend the above results to any order in ε.

The appearance of the hatted transcendental basis from the LKF transformation can be naturally understood in the following way.

The LKF transformation produces all-loop results for very restricted objects: the difference of fermion propagators in two gauges. So, at every order of the ε-expansion these all-loop results should contain (at least, a part of) the basic properties of the corresponding master integrals, $i.e.$, the all-loop results should be expressed in the form of (at least, a part of) the corresponding hatted ζ-values.

In a sense, it is not the full set of the hatted ζ-values but only the one-fold ones. This comes from the fact that the results produced by the LKF transformation contain only products of Γ-functions and, thus, their expansions contain only the simple one-fold ζ-values.

1. LKF transformation

In the following, we shall consider QED in an Euclidean space of dimension d ($d = 4 - 2\varepsilon$). The general form of the fermion propagator $S_F(p, \xi)$ in some gauge ξ reads:

$$S_F(p, \xi) = \frac{i}{\hat{p}} P(p, \xi) \,,$$

where the factor $\hat{p}$ containing Dirac γ-matrices, has been extracted. It is also convenient to introduce the x-space representation $S_F(x, \xi)$ of the fermion propagator as:

$$S_F(x, \xi) = \hat{x}\, X(x, \xi) \,.$$

The two representations, $S_F(x,\xi)$ and $S_F(p,\xi)$, are related by the Fourier transform which is defined as:

$$S_F(p,\xi) = \int \frac{\mathrm{d}^d x}{(2\pi)^{d/2}}\, e^{\mathrm{i}px}\, S_F(x,\xi)\,,$$

$$S_F(x,\xi) = \int \frac{\mathrm{d}^d p}{(2\pi)^{d/2}}\, e^{-\mathrm{i}px}\, S_F(p,\xi)\,.$$

The famous LKF transformation connects in a very simple way the fermion propagator in two different gauges, $e.g.$, ξ and η. In dimensional regularization, it reads:

$$S_F(x,\xi) = S_F(x,\eta)\, e^{\mathrm{i}\,(D(x)-D(0))}\,,$$

where

$$D(x) = -\mathrm{i}\,\Delta\, e^2\, \mu^{4-d} \int \frac{\mathrm{d}^d p}{(2\pi)^d}\, \frac{e^{-\mathrm{i}px}}{p^4}, \quad \Delta = \xi - \eta\,.$$

Note that, in dimensional regularization, the term $D(0)$ is proportional to the massless tadpole T_2, the massive counterpart of which is defined as:

$$T_\alpha(m^2) = \int \frac{\mathrm{d}^d p}{(2\pi)^d} \frac{e^{-\mathrm{i}px}}{(p^2+m^2)^\alpha}\,.$$

The tadpole $T_\alpha(m^2) \sim \delta(\alpha - d/2)$ in the massless limit and, thus, $D(0) = 0$ in the framework of dimensional regularization. So, the LKF transformation can be simplified as follows:

$$S_F(x,\xi) = S_F(x,\eta)\, e^{\mathrm{i}D(x)}\,.$$

We may now proceed in calculating $D(x)$ using the Fourier transforms

$$\int \mathrm{d}^d x \, \frac{e^{\mathrm{i}px}}{x^{2\alpha}} = \frac{2^{2\tilde{\alpha}}\pi^{d/2}a(\alpha)}{p^{2\tilde{\alpha}}}, \quad a(\alpha) = \frac{\Gamma(\tilde{\alpha})}{\Gamma(\alpha)}, \quad \tilde{\alpha} = \frac{d}{2} - \alpha \, ,$$
$$\int \mathrm{d}^d p \, \frac{e^{-\mathrm{i}px}}{p^{2\alpha}} = \frac{2^{2\tilde{\alpha}}\pi^{d/2}a(\alpha)}{x^{2\tilde{\alpha}}} \, .$$

This yields:

$$D(x) = -\mathrm{i}\,\Delta\, e^2 \, (\mu^2 x^2)^{2-d/2} \, \frac{\Gamma(d/2-2)}{2^4(\pi)^{d/2}} \, ,$$

or, equivalently, with the parameter ε made explicit:

$$D(x) = \frac{\mathrm{i}\,\Delta\, A}{\varepsilon} \, \Gamma(1-\varepsilon) \, (\pi\mu^2 x^2)^{\varepsilon}, \quad A = \frac{\alpha_{\mathrm{em}}}{4\pi} = \frac{e^2}{(4\pi)^2} \, .$$

We see that $D(x)$ contributes with a common factor ΔA accompanied by the singularity ε^{-1}.

2. LKF transformation in momentum space

Let's assume that, for some gauge fixing parameter η, the fermion propagator $S_F(p, \eta)$ with external momentum p has the form

$$S_F(p, \xi) = \frac{i}{\hat{p}} P(p, \xi) \,, \quad P(p, \eta) = \sum_{m=0}^{\infty} a_m(\eta)\, A^m \left(\frac{\tilde{\mu}^2}{p^2}\right)^{m\varepsilon} .$$

The $a_m(\eta)$ are coefficients of the loop expansion of the propagator and $\tilde{\mu}$ is the renormalization scale:

$$\tilde{\mu}^2 = 4\pi\mu^2 \,,$$

which lies somehow between the MS-scale μ and the $\overline{\mathrm{MS}}$-scale $\overline{\mu}$.

Then, using Fourier transforms , we obtain that:

$$S_F(x,\eta) = \frac{2^{d-1}\,\hat{x}}{(4\pi\,x^2)^{d/2}} \sum_{m=0}^{\infty} b_m(\eta)\,A^m \left(\pi\mu^2 x^2\right)^{m\varepsilon},$$

$$b_m(\eta) = a_m(\eta)\,\frac{\Gamma(d/2 - m\varepsilon)}{\Gamma(1+m\varepsilon)}\,.$$

With the help of an expansion of the LKF exponent, we have

$$S_F(x,\xi) = S_F(x,\eta)e^{D(x)} = \frac{2^{d-1}\,\hat{x}}{(4\pi\,x^2)^{d/2}} \sum_{m=0}^{\infty} b_m(\eta)\,A^m \left(\pi\mu^2 x^2\right)^{m\varepsilon}$$

$$\times \sum_{l=0}^{\infty} \left(-\frac{A^m\,\Delta}{\varepsilon}\right)^l \frac{\Gamma^l(1-\varepsilon)}{l!}\,(\pi\mu^2 x^2)^{l\varepsilon}\,.$$

Factorizing all x-dependence yields:

$$S_F(x,\xi) = \frac{2^{d-1}\,\hat{x}}{(4\pi\,x^2)^{d/2}} \sum_{p=0}^{\infty} b_p(\xi)\,A^m \left(\pi\mu^2 x^2\right)^{p\varepsilon},$$

$$b_p(\xi) = \sum_{m=0}^{p} \frac{b_m(\eta)}{(p-m)!}\left(-\frac{\Delta}{\varepsilon}\right)^{p-m} \Gamma^{p-m}(1-\varepsilon)\,.$$

Hence, taking the correspondence between the results for propagators $P(p,\eta)$ and $S_F(x,\eta)$, respectively, together with the result for $S_F(x,\xi)$, we have for $P(p,\xi)$:

$$P(p,\xi) = \sum_{m=0}^{\infty} a_m(\xi)\, A^m \left(\frac{\tilde{\mu}^2}{p^2}\right)^{m\varepsilon} , \qquad (1)$$

where

$$\begin{aligned} a_m(\xi) &= b_m(\xi)\, \frac{\Gamma(1+m\varepsilon)}{\Gamma(d/2-m\varepsilon)} \\ &= \sum_{l=0}^{m} \frac{a_l(\eta)}{(m-l)!}\, \frac{\Gamma(d/2-l\varepsilon)\Gamma(1+m\varepsilon)}{\Gamma(1+l\varepsilon)\Gamma(d/2-m\varepsilon)} \left(-\frac{\Delta}{\varepsilon}\right)^{m-l} \Gamma^{m-l}(1-\varepsilon)\,. \end{aligned}$$

In this way, we have derived the expression of $a_m(\xi)$ using a simple expansion of the LKF exponent in x-space. From this representation of the LKF transformation, we see that the magnitude $a_m(\xi)$ is determined by $a_l(\eta)$ with $0 \le l \le m$.

The corresponding result for the p- and Δ-dependencies of $\hat{a}_m(\xi, p)$ can be obtained by interchanging the order in the sums in the results for $P(p, \xi)$. So, we have

$$P(p, \xi) = \sum_{m=0}^{\infty} \hat{a}_m(\xi, p)\, A^m \left(\frac{\tilde{\mu}^2}{p^2}\right)^{m\varepsilon} ,$$

where

$$\hat{a}_m(\xi, p) \;=\; a_m(\eta) \sum_{l=0}^{\infty} \frac{\Gamma(d/2 - m\varepsilon)\Gamma(1 + (l+m)\varepsilon}{\Gamma(1 + m\varepsilon)\Gamma(d/2 - (l+m)\varepsilon)} \left(-\frac{A^m\,\Delta}{\varepsilon}\right)^{l} \times \frac{\Gamma^l(1-\varepsilon)}{l!} \left(\frac{\tilde{\mu}^2}{p^2}\right)^{l\varepsilon} .$$

2.1 Scale fixing

In our present study, we consider only the case of the so-called MS-like schemes. In such schemes, we need to fix specific terms coming from the application of dimensional regularization. Such a procedure will be called *scale fixing* and will play a crucial role in our analysis.

Let's first recall that the $\overline{\rm MS}$-scale $\overline{\mu}$ is related to the previously defined scale $\tilde{\mu}$ with the help of:

$$\overline{\mu}^2 = \tilde{\mu}^2 \, e^{-\gamma} (\text{simplest possibility}),$$
$$[\overline{\mu}^{2\varepsilon} = \tilde{\mu}^{2\varepsilon} \, \Gamma(1+\varepsilon), \quad \overline{\mu}^{2l\varepsilon} = \tilde{\mu}^{2l\varepsilon} \, \Gamma(1+l\varepsilon), (\text{other possibilities})]$$

where γ is the Euler constant. An advantage of the $\overline{\rm MS}$-scale is that it subtracts the Euler constant γ from the ε-expansion.

Moreover, it is well known that, in calculations of two-point massless diagrams, the final results do not display any ζ_2. So it is convenient to choose some scale which also subtracts ζ_2 in intermediate steps of the calculation. For this purpose, we shall consider two different scales.

The first one is the popular G-scale (Chetyrkin, Kataev, Tkachov: 1980),

which subtracts the coefficient in factor of the singularity $1/\varepsilon$ in the one-loop scalar p-type integral, $i.e.$,

$$\mu_G^{2\varepsilon} = \tilde{\mu}^{2\varepsilon} \frac{\Gamma^2(1-\varepsilon)\Gamma(1+\varepsilon)}{\Gamma(2-2\varepsilon)} .$$

Following (Broadhurst: 1999),

we shall use a slight modification of this scale that we will refer to as the g-scale and in which an additional factor $1/(1-2\varepsilon)$ is subtracted from the one-loop result, $i.e.$,

$$\mu_g^{2\varepsilon} = \tilde{\mu}^{2\varepsilon} \frac{\Gamma^2(1-\varepsilon)\Gamma(1+\varepsilon)}{\Gamma(1-2\varepsilon)} .$$

The advantage of the g-scale (over the G-scale) will reveal itself in discussions below related to the so-called transcendental weight of various contributions.

We shall also introduce a new scale which is based on old calculations of massless diagrams performed by Vladimirov (Vladimirov: 1980), who added an additional factor $\Gamma(1-\varepsilon)$ to each loop contribution. The latter corresponds to adding the factor $\Gamma^{-1}(1-\varepsilon)$ to the corresponding scale. We shall refer to this scale as the minimal Vladimirov-scale, or MV-scale, and define

$$\mu_{\rm MV}^{2\varepsilon} = \frac{\tilde{\mu}^{2\varepsilon}}{\Gamma(1-\varepsilon)}\,.$$

Notice that this form has been used once to define the $\overline{\rm MS}$ scheme (see Errata to (Kataev, Vardiashvili: 1988).)

As we will show below, the use of the MV-scale leads to simpler results in comparison with the g one. Hence, the MV-scale is more appropriate to our analysis and all our basic results will be given in the MV-scale. After that we will discuss the differences coming from the use of the g-scale.

In both the MV-scale and g-scale, we can rewrite the above result in the following general form:

$$a_m(\xi) = a_m(\eta) \sum_{l=0}^{\infty} \frac{1-(m+1)\varepsilon}{1-(m+l+1)\varepsilon} \, \Phi_p(m,l,\varepsilon) \, \frac{(\Delta\, A)^l}{(-\varepsilon)^l l!} \left(\frac{\mu_p^2}{p^2}\right)^{l\varepsilon} ,$$

where $p =$MV$, g$.

The factor $(1-(m+1)\varepsilon)/(1-(m+l+1)\varepsilon)$ has been specially extracted from $\Phi_p(m,l,\varepsilon)$ in order to insure equal transcendental level, $i.e.$, the same value of s for ζ_s at every order of the ε-expansion of $\Phi_p(m,l,\varepsilon)$ (see below).

Central to the present work, the factors $\Phi_{\rm MV}(m,l,\varepsilon)$ and $\Phi_g(m,l,\varepsilon)$ read:

$$\Phi_{\rm MV}(m,l,\varepsilon) = \frac{\Gamma(1-(m+1)\varepsilon)\Gamma(1+(m+l)\varepsilon)\Gamma^{2l}(1-\varepsilon)}{\Gamma(1+m\varepsilon)\Gamma(1-(m+l+1)\varepsilon)}\,,$$

$$\Phi_g(m,l,\varepsilon) = \Phi_{\rm MV}(m,l,\varepsilon)\,\frac{\Gamma^{l}(1-2\varepsilon)}{\Gamma^{3l}(1-\varepsilon)\Gamma^{l}(1+\varepsilon)}\,,$$

and may be expressed as expansions in ζ_i $(i \geq 3)$.

3. MV-scale

The Γ-function $\Gamma(1+\beta\varepsilon)$ has the following expansion:

$$\Gamma(1+\beta\varepsilon) = \exp\left[-\gamma\beta\varepsilon + \sum_{s=2}^{\infty}(-1)^s \eta_s \beta^s \varepsilon^s\right], \quad \eta_s = \frac{\zeta_s}{s}.$$

that yields for the factor $\Phi_{\mathrm{MV}}(m,l,\varepsilon)$:

$$\Phi_{\mathrm{MV}}(m,l,\varepsilon) = \exp\left[\sum_{s=2}^{\infty} \eta_s\, p_s(m,l)\, \varepsilon^s\right],$$

where

$$p_s(m,l) = \ (m+1)^s - (m+l+1)^s + 2l + (-1)^s\{(m+l)^s - m^s\},$$

and, as expected from the MV-scale, we do have:

$$p_1(m,l) = 0, \quad p_2(m,l) = 0.$$

Moreover $\Phi_{\mathrm{MV}}(m,l,\varepsilon)$ contains ζ_s-function values of a given weight (or transcendental level) s in factor of ε^s.

4. Solution of the recurrence relations

We now focus on the polynomial $p_s(m,l)$ that is conveniently separated in even and odd s values. Then, we see that the following recursion relations hold:

$$p_{2k} = p_{2k-1} + Lp_{2k-2} + p_3, \quad L = l(l+1)\,,$$
$$p_{2k-1} = p_{2k-2} + Lp_{2k-3} + p_3\,.$$

Specific to the MV-scheme, these relations only depend on L which leads to strong simplifications.

Nevertheless, they are difficult to solve for arbitrary k. It is simpler to proceed by explicitly considering the first values of k:

$$p_4 = 2p_3\,,$$
$$p_5 = p_4 + Lp_3 + p_3 = (3 + L)p_3\,,$$
$$p_6 = p_5 + Lp_4 + p_3 = (4 + 3L)p_3\,,$$

showing that p_s takes the form of a polynomial in L in factor of p_3. Then, taking Lp_3 from the second equation and put it to the thirs one, yields:

$$Lp_3 = p_5 - 3p_3, \quad p_6 = 3p_5 - 5p_3\,,$$

which reveals that the even polynomial p_6 can be entirely expressed in terms of the lower order odd ones, p_3 and p_5.

We may automate this procedure for higher values of k. The general expression of p_s is given by:

$$p_s = \sum_{m=0}^{[\frac{s+1}{2}-2]} A_{s,m}\, L^m\, p_3\,.$$

Taking $L^k p_3$ from the equations for p_{2k-1} and substituting them in the equations for p_{2k} yields:

$$p_{2k} = \sum_{s=2}^{k} p_{2s-1}\, C_{2k,2s-1} = \sum_{m=1}^{k-1} p_{2k-2m+1}\, C_{2k,2k-2m+1}\,.$$

From these results, it is possible to determine the exact k-dependence of $C_{2k,2s-1}$, which has the following structure:

$$C_{2k,2k-2m+1} = b_{2m-1}\,\frac{(2k)!}{(2m-1)!\,(2k-2m+1)!}\,,$$

with the first coefficients b_{2m-1} taking the values:

$$b_1 = \frac{1}{2}, \quad b_3 = -\frac{1}{4}, \quad b_5 = \frac{1}{2}, \quad b_7 = -\frac{17}{2}, \quad b_9 = \frac{31}{2}\,,$$
$$b_{11} = -\frac{691}{4}, \quad b_{13} = \frac{5461}{2}, \quad b_{15} = -\frac{929569}{16}\,,$$
$$b_{17} = \frac{3202291}{2}, \quad b_{19} = -\frac{221930581}{4}\,,$$
$$b_{21} = \frac{4722116521}{2}, \quad b_{23} = -\frac{968383680827}{8}\,.$$

Examining the numerators of b_{2m-1}, one can see that they are proportional to the numerators of Bernoulli numbers. Indeed, a closer inspection reveals that, accurate to a sign, the coefficients

b_{2m-1} coincide with the zero values of Euler polynomials $E_n(x)$:

$$b_{2m-1} = -E_{2m-1}(x=0)\,,$$

and therefore to Bernoulli and Genocchi numbers, B_m and G_m, respectively, because

$$E_{2m-1}(x=0) = \frac{G_{2m}}{2m}, \quad G_{2m} = -2\,(2^{2m}-1)\,B_{2m}\,.$$

Hence, the compact formula for the coefficients b_{2m-1}, expressed through the well known Bernoulli numbers B_m, reads:

$$b_{2m-1} = \frac{(2^{2m}-1)}{m}\,B_{2m}\,.$$

4.1 Hatted ζ-values

At this point, it is convenient to represent the argument of the exponential as follows:

$$\sum_{s=3}^{\infty} \eta_s \, p_s \, \varepsilon^s = \sum_{k=2}^{\infty} \eta_{2k} \, p_{2k} \, \varepsilon^{2k} + \sum_{k=2}^{\infty} \eta_{2k-1} \, p_{2k-1} \, \varepsilon^{2k-1} \, .$$

Then

$$\sum_{k=2}^{\infty} \eta_{2k} \, p_{2k} \, \varepsilon^{2k} = \sum_{k=2}^{\infty} \eta_{2k} \, \varepsilon^{2k} \sum_{s=2}^{k} p_{2s-1} \, C_{2k,2s-1}$$
$$= \sum_{s=2}^{\infty} p_{2s-1} \sum_{k=s}^{\infty} \eta_{2k} \, C_{2k,2s-1} \, \varepsilon^{2k} \, .$$

Then, can be written as $\sum_{s=2}^{\infty} \hat{\eta}_{2s-1} \, p_{2s-1} \, \varepsilon^{2s-1}$ where

$$\hat{\eta}_{2s-1} = \eta_{2s-1} + \sum_{k=s}^{\infty} \eta_{2k} \, C_{2k,2s-1} \, \varepsilon^{2(k-s)+1} \, .$$

Thus, we have

$$\Phi_{\rm MV}(m,l,\varepsilon) = \exp\big[\sum_{s=2}^{\infty} \hat{\eta}_{2s-1}\, p_{2s-1}\, \varepsilon^{2s-1}\big] = \exp\big[\sum_{s=2}^{\infty} \frac{\hat{\zeta}_{2s-1}}{2s-1}\, p_{2s-1}\, \varepsilon^{2s-1}\big],$$

where

$$\hat{\zeta}_{2s-1} = \zeta_{2s-1} + \sum_{k=s}^{\infty} \zeta_{2k}\, \hat{C}_{2k,2s-1}\, \varepsilon^{2(k-s)+1}$$

with

$$C_{2k,2s-1} = b_{2k-2s+1}\, \frac{(2k)!}{(2s-1)!\,(2k-2s+1)!}\,,$$

$$\hat{C}_{2k,2s-1} = \frac{2s-1}{2k}\, C_{2k,2s-1} = b_{2k-2s+1}\, \frac{(2k-1)!}{(2s-2)!\,(2k-2s+1)!}\,.$$

So, we provide an exact expression for the hatted ζ-values in terms of the standard ones valid for all ε.

4.2 g-scale

We may proceed in a similar way for the factor $\Phi_g(m,l,\varepsilon)$, which has the form

$$\Phi_g(m,l,\varepsilon) = \exp\left[\sum_{s=2}^{\infty} \eta_s\, p_s^g(m,l)\,\varepsilon^s\right],$$

where the new polynomial $p_s^g(m,l)$ can be expressed in terms of $p_s(m,l)$, as:

$$p_s^g(m,l) = p_s(m,l) + \delta_s(m,l)\,, \quad \delta_s(m,l) = (2^s - 3 - (-1)^s)l\,,$$

where $\delta_s(m,l) = 0$ for $s = 1$ and $s = 2$ and, thus,

$$p_1^g(m,l) = 0, \quad p_2^g(m,l) = 0\,, \tag{2}$$

similarly to the Vladimirov case, considered earlier.

We may then consider the even and odd values of s separately leading to the following recursion relations:

$$\begin{aligned} p^g_{2k} &= p_{2k} + \delta_{2k}, \qquad & \delta_{2k} = 4(2^{2k-2} - 1)l\,, \\ p^g_{2k-1} &= p_{2k-1} + \delta_{2k-1}, \qquad & \delta_{2k-1} = \frac{1}{2}\delta_{2k}\,. \end{aligned}$$

These recurrence relations depend on the variable l but not on the product $L = l(l+1)$ as it was for the MV-scale. So, the g-scale recursion relations are essentially more complicated than the MV-scale ones. Fortunately, it is very simple to see that in the relations:

$$p^g_{2k} = \sum_{s=2}^{k} p^g_{2s-1}\, C_{2k,2s-1}\,,$$

the coefficients $C_{2k,2s-1}$ are exactly the same as earlier because the corrections δ_{2k} and δ_{2k-1} exactly cancel each other. So, the hatted ζ-values for the g-scale are identical to the ones of the MV-scale.

5. Integral representations

Taking the integral representation for ζ-values

$$\zeta_{n+1} = \frac{1}{n!} \int_0^1 \frac{dt}{1-t} \ln^n \left(\frac{1}{t}\right) ,$$

we have

$$\zeta_{2m} = -\frac{1}{(2m-1)!} \int_0^1 \frac{dt}{1-t} \ln^{2m-1} t \, , \quad \zeta_{2m+1} = \frac{1}{(2m)!} \int_0^1 \frac{dt}{1-t} \ln^{2m} t \, .$$

Now it is convenient to rewrite $\hat{\zeta}_{2m+1}$ as

$$\hat{\zeta}_{2m+1} = \zeta_{2m+1} + \sum_{k=m+1}^{\infty} \zeta_{2k} \, \hat{C}_{2k,2m+1} \, \varepsilon^{2(k-m)-1} \, .$$

The integral expression for the first term in the r.h.s. is shown above. The second term in the r.h.s. can be represented as

$$- \sum_{k=m+1}^{\infty} \hat{C}_{2k,2m+1} \frac{\varepsilon^{2(k-m)-1}}{(2k-1)!} \int_0^1 \frac{dt}{1-t} \ln^{2k-1} t \, .$$

The subintegral expression can be rewritten as $(k = m + l)$

$$\sum_{k=m+1}^{\infty} \hat{C}_{2k,2m+1} \frac{\varepsilon^{2(k-m)-1}}{(2k-1)!} \ln^{2k-1} t = \sum_{k=m+1}^{\infty} \frac{b_{2(k-m)-1}\, \varepsilon^{2(k-m)-1}}{(2m)!(2(k-m)-1)!} \ln^{2k-1} t$$

$$= 2 \frac{\ln^m t}{(2m)!} \sum_{l=1}^{\infty} B_{2l} \frac{2^{2l}-1}{(2l)!} (\varepsilon \ln t)^{2l-1} .$$

Now we use the property of the Bernoulli numbers B_n

$$\frac{y}{e^y - 1} = 1 - \frac{y}{2} + \sum_{n=2}^{\infty} \frac{B_n}{n!} y^n = 1 - \frac{y}{2} + \sum_{l=1}^{\infty} \frac{B_{2l}}{(2l)!} y^{2l} ,$$

since $B_{2m+1} = 0$, when $m \geq 1$.

So, we have

$$\sum_{l=1}^{\infty} \frac{2^{2l}-1}{(2l)!} B_{2l}\, y^{2l} = \frac{y}{2} - \frac{y}{e^y + 1}$$

and

$$2 \sum_{l=1}^{\infty} \frac{2^{2l}-1}{(2l)!} B_{2l} (\varepsilon \ln t)^{2l-1} = 1 - \frac{2}{e^{\varepsilon \ln t} + 1} = 1 - \frac{2}{t^{\varepsilon} + 1} .$$

Combining this result with the one for ζ_{2m+1}, we obtain the Lee formula for $\hat{\zeta}_{2m+1}$:

$$\hat{\zeta}_{2m+1} = \frac{1}{(2m)!} \int_0^1 \frac{dt}{(1-t)} \frac{2}{(t^{\varepsilon}+1)} \ln^{2m} t \,,$$

i.e. the transform $\zeta_{2m+1} \to \hat{\zeta}_{2m+1}$ corresponds to the simple transition $1 \to 2/(t^{\varepsilon}+1)$ in their integrands.

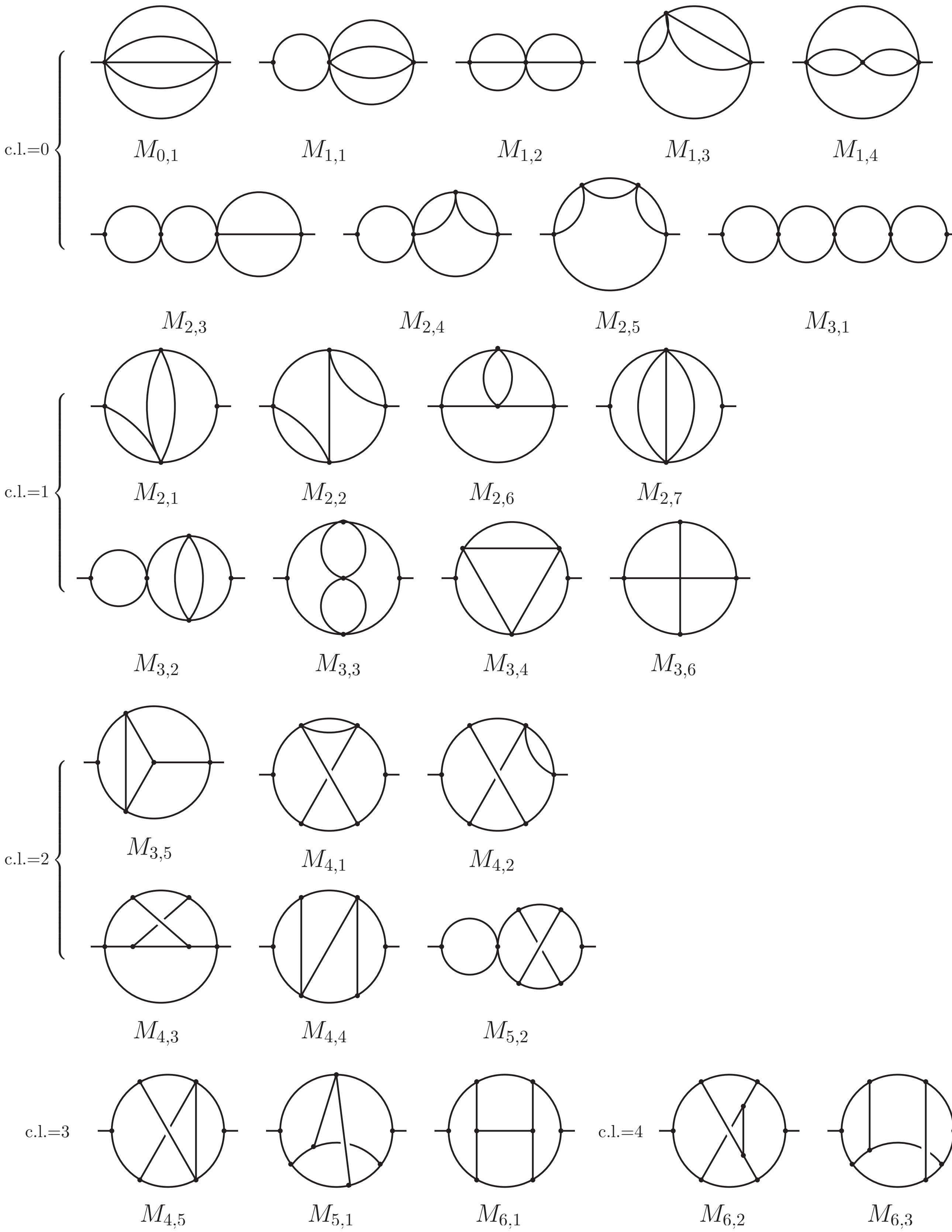

Figure 1: Master diagrams for four-loop massless propagators.

6. More complicated cases: $\zeta_{a,b,\dots}$

From an analysis of master-integrals (Lee, Smirnov, Smirnov: 2012) we have (up to the transcendentality weight 12)

$$\hat{\zeta}_{5,3} = \zeta_{5,3} - \frac{28}{12}\zeta_8 - \frac{15}{2}\zeta_4\zeta_5\varepsilon - \frac{2905}{376}\zeta_{10}\varepsilon^2 + \frac{25}{2}\zeta_6\zeta_5\varepsilon^3 + a_{5,3}\zeta_{12}\varepsilon^4 + O(\varepsilon^5);$$

$$\hat{\zeta}_{7,3} = \zeta_{7,3} - \frac{793}{54}\zeta_{10} - 3\varepsilon(5\zeta_6\zeta_5 + 7\zeta_4\zeta_7) + a_{7,3}\zeta_{12}\varepsilon^2 + O(\varepsilon^3);$$

$$\hat{\zeta}_{5,3,3} = \zeta_{5,3,3} + 45\zeta_2\zeta_9 + 3\zeta_4\zeta_7 - \frac{5}{2}\zeta_6\zeta_5 + \varepsilon(a_{5,5,3}\zeta_{12} - \frac{3}{2}\zeta_4\zeta_{5,3}) + O(\varepsilon^2);$$

$$\hat{\zeta}_{9,3} = \zeta_{9,3} + a_{9,3}\zeta_{12} + O(\varepsilon^1);$$

$$\hat{\zeta}_{6,4,1,1} = \zeta_{6,4,1,1} + a_{6,4,1,1}\zeta_{12} - \frac{3}{2}\zeta_4\zeta_{5,3} + \frac{1}{2}\zeta_4\zeta_5\zeta_3 + \frac{9}{4}\zeta_6\zeta_3^2 - 3\zeta_2\zeta_{7,3}$$
$$-\frac{7}{2}\zeta_2\zeta_5^2 - 10\zeta_2\zeta_7\zeta_3 + O(\varepsilon^1),$$

where $a_{\dots}$ are known but have rather long expressions.

5.1 Integral representations

Introduce

$$\zeta_{n,m} = \frac{1}{\Gamma(n)\Gamma(m)} \int_0^1 \frac{dt}{1-t} \ln^{n-1}\left(\frac{1}{t}\right) \int_0^t \frac{dt_1}{1-t_1} \ln^{m-1}\left(\frac{t}{t_1}\right)$$

and the generalizations

$$\tilde{\zeta}_n = \frac{1}{\Gamma(n)} \int_0^1 \frac{dt}{(1-t)} \frac{2}{(1+t^{\varepsilon})} \ln^{n-1}\left(\frac{1}{t}\right),$$

$$\tilde{\zeta}_{n,m} = \frac{1}{\Gamma(n)\Gamma(m)} \int_0^1 \frac{dt}{(1-t)} \frac{2}{(1+t^{\varepsilon_1})} \ln^{n-1}\left(\frac{1}{t}\right)$$

$$* \int_0^t \frac{dt_1}{(1-t_1)} \frac{2}{(1+(t_1/t)^{\varepsilon_2})} \frac{2}{(1+t_1^{\varepsilon})} \ln^{m-1}\left(\frac{t}{t_1}\right),$$

which can be cansidered as candidates for $\hat{\zeta}_n$ and $\hat{\zeta}_{n,m}$.

From above analyses, we have

$$\frac{2}{t^{\varepsilon}+1} = 1-2\sum_{l=1}^{\infty}\frac{2^{2l}-1}{(2l)!}B_{2l}\,(\varepsilon\ln t)^{2l-1} = 1-3B_2\,\varepsilon\ln t-\frac{5}{4}B_4\,(\varepsilon\ln t)^3$$

$$-\frac{7}{40}B_6\,(\varepsilon\ln t)^5+\ldots$$

$$= 1+\frac{\varepsilon}{2}\ln\frac{1}{t}-\frac{\varepsilon^3}{24}\ln^3\frac{1}{t}+\frac{\varepsilon^5}{240}\ln^5\frac{1}{t}+\ldots,$$

since

$$B_2=\frac{1}{2},\quad B_4=-\frac{1}{30},\quad B_6=\frac{1}{42}\,.$$

5.1.1 $O(\varepsilon)$

Consider $\tilde{\zeta}_{5,3}$ up to $O(\varepsilon_1)$, $O(\varepsilon_2)$ and $O(\varepsilon)$

$$\tilde{\zeta}_{5,3} = \zeta_{5,3} + \frac{5\varepsilon_1}{2}\zeta_{6,3} + \frac{3\varepsilon_2}{2}\zeta_{5,4} + \frac{\varepsilon}{2}(3\zeta_{5,4} + 5\zeta_{6,3}) + \dots .$$

We note that $\zeta_{s,t}$ with $s + t = 2N + 1$ can be represented as combination of the simple ζ-values.

Indeed, for even $s > 0$ and odd $t > 1$ with $s + t = 2N + 1$

$$\zeta_{s,t} = \zeta_s\zeta_t + \frac{1}{2}(C_s^{s+t} - 1)\zeta_{s+t} - \sum_{r=1}^{N-1}[C_{s-1}^{2r} + C_{p-1}^{2r}]\zeta_{2r+1}\zeta_{2N-2r}$$

and other cases can be reproduced from the property:

$$\zeta_{s,t} = \zeta_s\zeta_t - \zeta_{t,s} - \zeta_{s+t}, \quad \left(C_m^n = \frac{n!}{m!(n-m)!}\right)$$

Taking the results, we have for $\tilde{\zeta}_{5,3}$

$$\tilde{\zeta}_{5,3} = \zeta_{5,3} + \frac{5\varepsilon_1}{2}\left(\frac{83}{2}\zeta_9 - 6\zeta_5\zeta_4 - 21\zeta_7\zeta_2\right) - \frac{3\varepsilon_2}{2}\left(\frac{127}{2}\zeta_9 - 5\zeta_5\zeta_4 - 35\zeta_7\zeta_2\right)$$
$$+\frac{\varepsilon}{2}(17\zeta_9 - 15\zeta_5\zeta_4) + \ldots .$$

We can see that for nonzero ε_1 and ε_2 there are the contributions $\sim \zeta_2$, which are absend in exact results above. But they are cancelled for the case $\varepsilon_1 = \varepsilon_2$.

So we have two different possibilities

$$\tilde{\zeta}^{(1)}_{n,m} = \frac{1}{\Gamma(n)\Gamma(m)} \int_0^1 \frac{dt}{1-t} \ln^{n-1}\left(\frac{1}{t}\right) \int_0^t \frac{dt_1}{(1-t_1)} \frac{2}{(1+t_1^{\varepsilon})} \ln^{m-1}\left(\frac{t}{t_1}\right),$$

$$\tilde{\zeta}^{(2)}_{n,m} = \frac{1}{\Gamma(n)\Gamma(m)} \int_0^1 \frac{dt}{(1-t)} \frac{2}{(1+t^{\varepsilon})} \ln^{n-1}\left(\frac{1}{t}\right)$$

$$* \int_0^t \frac{dt_1}{(1-t_1)} \frac{2}{(1+(t/t_1)^{\varepsilon})} \ln^{m-1}\left(\frac{t_1}{t}\right),$$

which can be cansidered as candidates for $\hat{\zeta}_{n,m}$.

We note that they coincide at $O(\varepsilon)$, i.e. $\tilde{\zeta}^{(1)}_{n,m} + O(\varepsilon^3) = \tilde{\zeta}^{(2)}_{n,m} + O(\varepsilon^2) \equiv \tilde{\zeta}_{n,m} + O(\varepsilon^2)$.

So, we have

$$\tilde{\zeta}_{5,3} = \zeta_{5,3} + \frac{\varepsilon}{2}(17\zeta_9 - 15\zeta_5\zeta_4) + O(\varepsilon^3)\,,$$
$$\tilde{\zeta}_{7,3} = \zeta_{7,3} - 3\varepsilon\left(7\zeta_7\zeta_4 + 5\zeta_6\zeta_5 + \frac{155}{12}\zeta_{11}\right) + O(\varepsilon^3)\,,$$
$$\tilde{\zeta}_{9,3} = \zeta_{9,3} - \frac{\varepsilon}{2}\left(27\zeta_9\zeta_4 + 27\zeta_7\zeta_6 + 14\zeta_8\zeta_5 - \frac{139}{2}\zeta_{13}\right) + O(\varepsilon^3)\,,$$

where the products of ζ-values coinside exactly with the exact results obtained above. Moreover, the results contain also the term $\sim \zeta_9$, $\sim \zeta_{11}$ and $\sim \zeta_{13}$, which can be obtained from the corresponding one-fold representation.

5.1.2 Beyond $O(\varepsilon)$

Taking the expansion in $2/(1+t_1^{\varepsilon})$ up to $O(\varepsilon^5)$ we have for $\tilde{\zeta}^{(1)}_{5,3}$ and $\tilde{\zeta}^{(1)}_{7,3}$

$$\tilde{\zeta}^{(1)}_{5,3} = \zeta_{5,3} + \frac{\varepsilon}{2}(17\zeta_9 - 15\zeta_5\zeta_4) - 5\varepsilon^3\left(\zeta_6\zeta_5 - \frac{189}{8}\zeta_{11}\right)$$
$$+\frac{15}{2}\varepsilon^5\left(7\zeta_8\zeta_5 + 5\zeta_6\zeta_7 - 1540\zeta_2\zeta_{11} - \frac{114967}{10}\zeta_{13}\right) + O(\varepsilon^7);$$
$$\tilde{\zeta}^{(1)}_{7,3} = \zeta_{7,3} - 3\varepsilon\left(7\zeta_7\zeta_4 + 5\zeta_6\zeta_5 + \frac{155}{12}\zeta_{11}\right)$$
$$+7\varepsilon^3\left(5\zeta_7\zeta_6 + 6\zeta_8\zeta_5 - \frac{319}{28}\zeta_{13}\right) + O(\varepsilon^5)\,,$$

where the products of ζ-values differ from the exact results obtained above. The situation is similar also for $\tilde{\zeta}^{(2)}_{5,3}$ and $\tilde{\zeta}^{(2)}_{7,3}$.

The construction of the integral representations for $\hat{\zeta}_{a,b,\ldots}$ need additional investigations!!!

5. Summary

From the LKF transformation of the fermion and scalar propagators we have found peculiar recursion relations between even and odd values of the polynomial associated to the uniformly transcendental factor $\Phi_{\mathrm{MV}}(m,l,\varepsilon)$.

These relations are most simple in the new MV-scheme. They relate the even and odd parts in a rather simple way which reveals the possibility to express all results for $\Phi_{\mathrm{MV}}(m,l,\varepsilon)$ in terms of hatted ζ-values.

In the more popular g-scheme, the corresponding recursion relations are slightly more complicated but lead to the same relations between even and odd parts of the polynomial associated to $\Phi_g(m,l,\varepsilon)$ and, correspondingly, to the same hatted ζ-values.

Our careful study of the recursion relations allowed us to derive exact formulas, relating hatted and standard ζ-values to all orders of perturbation theory.
The coefficients of the relations are expressed trough the well-known Bernoulli numbers, B_{2m}. The corresponding integral representations have a very compact form.

For the hatted multi-zeta values $\hat{\zeta}_{a,b,\ldots}$ the situation is more complicate. From ε-expansion of 4-loop master integrals we have the results up to the transcendentality weight 14.
!!! However, the derivation of the corresponding integral representations requires additional research.!!!

Our results for $\hat{\zeta}_a$ and $\hat{\zeta}_{a,b,\ldots}$ provide stringent constraints on the results of multi-loop calculations.