Vogel's algebra and its application to the knot invariants

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   (universal Vassiliev invariant) (quantum knot invariants)

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4. Characters of Λ
5. Universality of Lie algebras
6. Kernel of Lie algebras weight systems
7. Implications for knot invariants

Chern-Simons theory

The Chern-Simons action for the vector field $A_\mu = A_\mu^a T^a$ is

$$S(A) = \frac{\kappa}{4\pi}\int_{S^3} d^3x \epsilon^{\mu\nu\rho} \mathrm{tr}\left(A_\mu \partial_\nu A_\rho + \frac{2}{3} A_\mu A_\nu A_\rho\right) \qquad (1)$$

Gauge algebra is $\mathfrak{g}$, $\hbar = \frac{2\pi i}{\kappa+N}$ is a coupling constant. Gauge invariant functions of A_μ are given by Wilson loops,

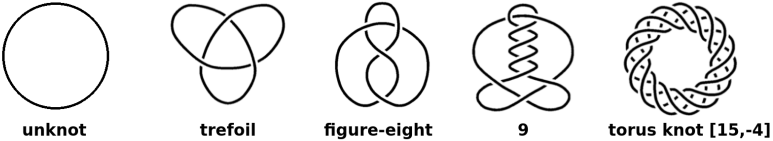
$$\langle W_R(\mathcal{K})\rangle = \frac{1}{Z}\int \mathcal{D}A\ \mathrm{e}^{iS(A)}\ \mathrm{tr}_R\left(\mathrm{Pexp}\ i\oint_C A_\mu dx^\mu\right) \qquad (2)$$

which determine topological knot and link invariants $\mathcal{K}$ **[A.Schwartz'1987; E.Witten'1989]**. Chern-Simons partition function is $Z = \int \mathcal{D}A\ \mathrm{e}^{iS(A)}$.

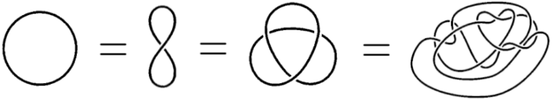
Knot theory

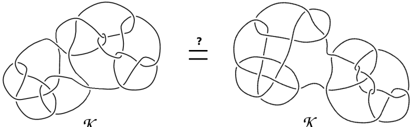
Definition

A knot $\mathcal{K}$ is an embedding of the circle into the three-dimensional Euclidean space $\mathcal{K} : S^1 \hookrightarrow \mathbb{E}^3$.



Ambient isotopy is an equivalence relation on knots.





$\mathcal{K}_1$ $\mathcal{K}_2$

Holomorphic gauge for the CS theory

Following **[Labastida, J. M. F., Perez, E., 1998]**:

1. Wick rotation $A^a_z = A^a_1 - iA^a_2,\ A^a_{\bar z} = A^a_1 + iA^a_2$;
2. fix gauge $A_{\bar z} = 0$;
3. quadratic action $S(A)\big|_{A_{\bar z}=0} = i\displaystyle\int dt d\bar z dz\, \epsilon^{mn} A^a_m \partial_{\bar z} A^a_n$;
4. pair correlator
$$\langle\, A^a_m(t_1,z_1,\bar z_1)\ A^b_n(t_2,z_2,\bar z_2)\,\rangle = \epsilon_{mn}\,\delta^{ab}\,\frac{\hbar}{2\pi i}\,\frac{\delta(t_1-t_2)}{z_1-z_2};$$
5. Wick theorem.

Answer for the vacuum expectation value

$$\langle\, W_R(K)\,\rangle = \sum_{n=0}^{\infty}\frac{\hbar^n}{(2\pi i)^n}\times \int\limits_{o(z_1)<..<o(z_n)} \sum_{p\in P_{2n}} (-1)^{p_\downarrow} \bigwedge_{k=1}^{n} \frac{dz_{i_k}-dz_{j_k}}{z_{i_k}-z_{j_k}} \times$$
$$\times\, \mathrm{tr}_R\Big(T^{a_{\sigma_p(1)}}\, T^{a_{\sigma_p(2)}} \ldots T^{a_{\sigma_p(2n)}}\Big) \quad (3)$$

Vassiliev invariants and group factors

From this expansion we see that the information about the knot and the gauge group $\langle\, W_R(\mathcal{K})\,\rangle$ contributes separately. The information about the embedding of knot into S^3 is encoded in the integrals of the form **[Birman,Lin'1993; D.Bar-Natan'1995]**:

$$v^{\mathcal{K}}_{n,m} \sim \int\limits_{o(z_1)<\ldots<o(z_n)} \sum_{p\in P_{2n}} (-1)^{p_\downarrow} \bigwedge_{k=1}^{n} \frac{dz_{i_k}-dz_{j_k}}{z_{i_k}-z_{j_k}}$$

and the information about the gauge group and representation enter the answer as the group factors:

$$G^{\mathfrak{g},R}_{n,m} \sim \mathrm{tr}_R\Big(T^{a_{\sigma_p(1)}}T^{a_{\sigma_p(2)}}\ldots T^{a_{\sigma_p(2n)}}\Big).$$

Thus, schematically one can rewrite the perturbative series as follows

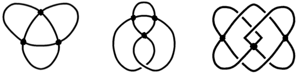
$$\langle\, W^{\mathfrak{g}}_R(\mathcal{K})\,\rangle = \sum_{n=0}^{\infty}\hbar^n\cdot\sum_{m=1}^{\mathcal{N}_n} G^{\mathfrak{g},R}_{n,m}\cdot v^{\mathcal{K}}_{n,m}, \qquad (4)$$

where $\mathcal{N}_n$ is the number of independent group elements of degree n.

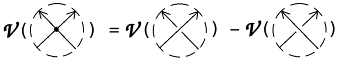
Vassiliev invariants

Definition (**Singular knots**)

A singular knot is an isotopy class of S^1 immersions in $\mathbb{R}^3$ such that all self-intersection points are simple double points with transversal intersections.



The continuation of knot invariants to the set of singular knots is determined by *Vassiliev skein relation*:



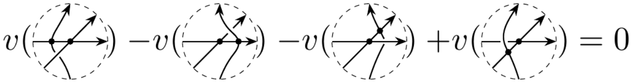
Definition (**Vassiliev invariants, [V.Vassiliev'1990]**)

A knot invariant v is called a Vassiliev invariant of order no more than n if v vanishes at singular knots with $\geq n+1$ double points.

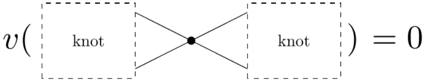
Relations on Vassiliev invariants

Vassiliev invariants satisfy the following relations

4T:



1T:



Theorem

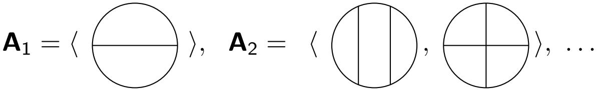
The product of two Vassiliev invariants of orders $\leq n$ and $\leq m$ is a Vassiliev invariants of order $\leq n+m$

$$v_n \cdot v_m = v_{n+m}$$

Chord diagrams

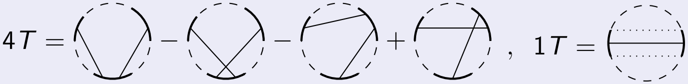
Definition

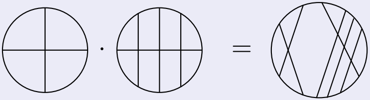
A chord diagram of order n is an oriented circle with n pairs of distinct points. The set of chord diagrams of order n is denoted $\mathbf{A}_n$.



Definition

Algebra of chord diagrams is a vector space $\mathcal{A}_n = \mathbf{A}_n / \langle 4\text{T}, 1\text{T} \rangle$ modulo the four-term and the one-term relations





Knotsevich integral

Definition (Weight systems)

The space of weight systems $\mathcal{W}_n = \mathcal{A}_n^* = \text{Hom}\ (\mathcal{A}_n, \mathbb{R})$ is a space of linear functions on $\mathcal{A}_n$.

Theorem (Vassiliev-Kontsevich theorem)

$$\mathcal{W} = \bigoplus_{n=0}^{\infty} \mathcal{W}_n \cong \bigoplus_{n=0}^{\infty} \mathcal{V}_n / \mathcal{V}_{n-1} = \mathcal{V} \tag{6}$$

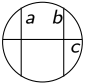
Kontsevich integral is the generating function for Vassiliev invariants

$$I(\mathcal{K}) = \sum_{n=0}^{\infty} \hbar^n \sum_{m=1}^{\dim \tilde{\mathcal{V}}_n} D_{n,m} \cdot v_{n,m}^{\mathcal{K}}, \tag{7}$$

Lie algebra weight systems

Choose a basis $T^1, .., T^r$ of $\mathfrak{g}$ and let $T_1, .., T_r$ be the dual basis with respect to the non-degenerate bilinear form **[D.Bar-Natan'1995]**.

$$\varphi_{\mathfrak{g}}^R : \mathcal{A}_n \to ZU(\mathfrak{g}), \quad D \mapsto \mathrm{tr}_R \left(T^a T^b ... \right) \tag{8}$$

$D_{2,1} =$ 

$$\varphi_{\mathfrak{g}}^R(D_{2,1}) = \mathrm{tr}_R \left(\sum_{a,b,c=1} T^a \, T^b \, T^c \, T_b \, T_a \, T_c \right)$$

$$\varphi_{\mathfrak{g}}^R \left(D_{n,m} \right) = G_{n,m}^{\mathfrak{g},R}$$

$$\varphi_{\mathfrak{g}}^R \left(I(\mathcal{K}) \right) = \sum_{n=0}^{\infty} \hbar^n \sum_{m=1}^{\dim \tilde{\mathcal{V}}_n} \varphi_{\mathfrak{g}}^R \left(D_{n,m} \right) \cdot v_{n,m}^{\mathcal{K}} = \langle \, W_R^{\mathfrak{g}}(\mathcal{K}) \, \rangle \tag{9}$$

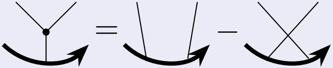
Are all Vassiliev invariants contained in quantum knot invariants for all possible semisimple Lie (super)algebras $\mathfrak{g}$?

Vogel's answer (2011) is no, because $\bigcap_{\mathfrak{g},R} \mathrm{Ker}\, \varphi_{\mathfrak{g}}^R \neq \varnothing$.

Closed Jacobi diagrams

Definition (Jacobi diagrams $\mathcal{C} = \bigoplus_n \mathcal{C}_n$)

The space of Jacobi diagrams $\mathcal{C}_n$ is a linear space generated by connected graphs having $2n$ trivalent vertices, as well as having a distinguished cycle (vertices on this cycle are called external), the remaining (internal) vertices are equipped with a cyclic order of half-edges, modulo the STU relation:



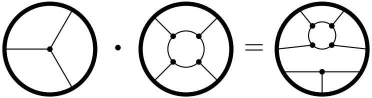


Figure: Product of Jacobi diagrams

Isomorphism of diagrams: $\mathcal{A}_n \cong \mathcal{C}_n/\langle 1T \rangle$

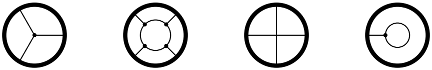


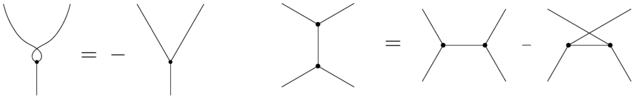
Figure: Examples of Jacobi diagrams

Chord diagrams are a special type of Jacobi diagrams.

Theorem (D.Bar-Natan'1995)

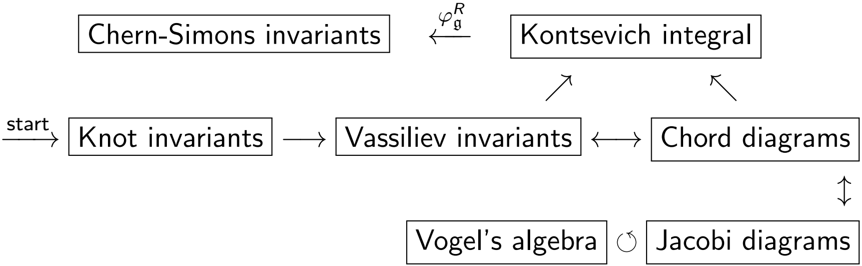
$\mathbf{A}_n/\langle 4T \rangle \cong \mathcal{C}_n$, and $\mathcal{A}_n \cong \mathcal{C}_n/\langle 1T \rangle$.

The space of *primitive* Jacobi diagrams $\mathcal{P}$ is a subspace of $\mathcal{C}$ linearly generated by Jacobi diagrams with a connected internal graph.
The STU relation imposes additional constraints on an internal graph of Jacobi diagrams:



AS IHX

Don't get lost



Vogel's algebra Λ

Definition (Algebra Λ)

Λ is an algebra over $\mathbb{Q}$, generated by 3-legged 3-valent diagrams (legs are numerated) modulo AS and IHX relations, antisymmetric with respect to permutations of legs. Multiplication in Λ is given by insertion of one factor into any vertex of the other factor.

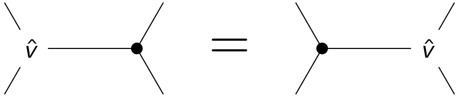


Figure: Well-defined multiplication

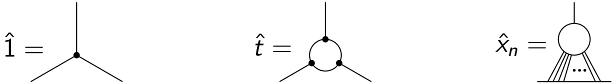
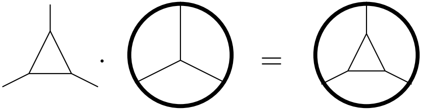


Figure: Some elements of Λ

Jacobi diagrams as Λ-module

Λ-algebra naturally acts on closed Jacobi diagrams (with connected internal graph):



Theorem

Let $\hat{\lambda} \in \Lambda$ and $u \in \mathcal{C}$ is the "Mercedes-Benz". Then the element $\hat{\lambda} \cdot u$ is also non-zero primitive in $\mathcal{C}$.

Λ : generators

- The structure of the algebra Λ is unknown.

- The dimension of Λ is known up to order 10, because Λ-algebra is isomorphic to the algebra of 3-graphs, defined in **[Chmutov-Duzhin-Kaishev'1998]**.

- Λ-algebra is a commutative.

- There exists a unique graded algebra homomorphism

$$\psi : \mathbb{Q}[t] \oplus \omega\mathbb{Q}[t, \sigma, \omega] \to \Lambda,$$

where formal variables $\deg(t) = 1$, $\deg(\sigma) = 2$ and $\deg(\omega) = 3$. ψ is bijective for the degree ≤ 10 and injective for the degree ≤ 15. In degree 16 $\operatorname{Ker}\psi$ is nonzero.

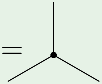
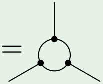
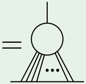
Λ : examples

$$\psi : \mathbb{Q}[t] \oplus \omega\mathbb{Q}[t,\sigma,\omega] \to \Lambda.$$

Example

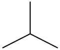
$$\psi(t) = \hat{t} = \tfrac{1}{2}\hat{x}_1, \qquad \psi(t^2) = \hat{t}^2 = \hat{x}_2,$$

$$\psi(4t^3 - \tfrac{3}{2}\omega) = \hat{x}_3, \qquad \psi(12t^5 - \tfrac{17}{2}t^2\omega + \tfrac{3}{2}\sigma\omega) = \hat{x}_5.$$

$\hat{1} =$  $\hat{t} =$  $\hat{x}_n =$ 

Λ : generalized weight system

Simple Lie algebra $\mathfrak{g}$ (over field K) with non-degenerate bilinear form gives a weight system on Λ by the map

$$\Phi_{\mathfrak{g}} : \Lambda \to \mathfrak{g}^{\otimes 3},$$  $$\mapsto \; f_{abc}$$

Then AS means anticommutativity, IHX means Jacobi identity. Any simple Lie algebra $\mathfrak{g}$ has only 1 primitive 3-tensor:

$$\Phi_{\mathfrak{g}}(\hat{x}) = \chi_{\mathfrak{g}}(\hat{x})\, f_{abc},$$

where $\chi_{\mathfrak{g}}(\hat{x}) : \Lambda \to K$ is a well-defined character. It is a polynomial in t, σ, ω.

Characters

Characters $\chi_{\mathfrak{g}}(\hat{x})$ are polynomials in t, σ, ω.

$$
\begin{aligned}
P_{sl} &= -2t^3 + 2\sigma t - \omega \\
P_{osp} &= -24\sigma t^4 + 40\omega t^3 + 29\sigma^2 t^2 - 72\omega\sigma t + 4\sigma^3 + 27\omega^2 \\
P_{exc} &= 40t^3 - 45\sigma t + 27\omega
\end{aligned}
$$

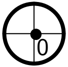
Lie algebra	t	σ	ω
$A_n = \mathfrak{sl}_{n+1}$	$n+1$	$2n^2+4n-2$	$2(n^2-1)(n+3)$
$B_n = \mathfrak{so}_{2n+1}$	$2n-1$	4(n+1)(2n-3)	4(n-1)(2n+3)(2n-3)
$C_n = \mathfrak{sp}_{2n}$	$n+1$	(n+2)(2n-1)	(n-1)(2n+3)(n+2)
$D_n = \mathfrak{so}_{2n}$	$2n-2$	4(2n+1)(n-2)	8(n-2)(2n-3)(n+1)
G_2	4	260/9	880/9
F_4	9	170	1470
E_6	12	308	3600
E_7	18	704	12480
E_8	30	1976	58800

Kernel of sl_n weight system

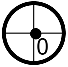
Up to degree 10 calculate from Vogel's algebra **[D.Khudoteplov, E.Lanina, A.S., 2024]**:

n=8: $\widehat{\omega P_{sl}} \cdot$  $\Rightarrow$ $\dim \operatorname{Ker} \varphi_{sl_n}(\mathcal{P}_n) = \mathbf{1}$

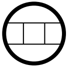
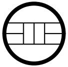
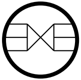
If $\hat{\lambda} \neq 0 \in \Lambda$ and D is the "Mercedes-Benz" diagram in $\mathcal{C}$, then $\hat{\lambda} \cdot D \neq 0$ is also nonzero primitive diagram.

n=9: $\hat{t} \cdot \widehat{\omega P_{sl}} \cdot$ , $\psi(P_{sl}) \cdot$ 

$\Rightarrow$ $\dim \operatorname{Ker} \varphi_{sl_n}(\mathcal{P}_n) = \mathbf{2}$

n=10: $\hat{t}^2 \cdot \widehat{\omega P_{sl}} \cdot$ , $\widehat{\omega\sigma P_{sl}} \cdot$ , $\hat{t} \cdot \psi(P_{sl}) \cdot$ 

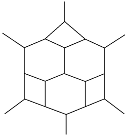
,

$\widehat{\omega P_{sl}} \cdot$ , $\psi(P_{sl}) \cdot$  $+ \hat{t} \cdot$  $\Rightarrow$ $\dim \operatorname{Ker} \varphi_{sl_n}(\mathcal{P}_n) = \mathbf{5}$

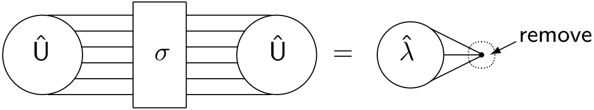
Kernel of all Lie algebras weight system

$$\psi(\omega P_{sl} P_{osp} P_{exc}) = \hat{\lambda}$$

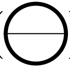
Let $\hat{U}$ be the following 6-legged diagram

$\hat{U} =$ 

then we define $\hat{\lambda} \neq 0 \in \Lambda$ by removing a trivalent vertex from V:

$V = \sum_{\sigma \in S_6} \operatorname{sign}(\sigma) \cdot$ 

$D_{17} := \hat{\lambda} \cdot$  $\neq 0$

$\Phi_{\mathfrak{g}}(D_{17}) = \Phi_{\mathfrak{g}}\left(\hat{\lambda} \cdot \hat{t}\right) \cdot \Phi_{\mathfrak{g}}\Big($  $\Big) = 0$ (10)

Zero divisor: $\hat{t} \cdot \hat{\lambda} = 0, \quad \hat{t} =$ , $(2 \cdot 3 = 0 \bmod 6, \text{ where } 2, 3 \neq 0)$

Conclusion

Vogel's Λ-algebra implies:

1. Universal formulas for various contractions of Lie algebra structure constants (*universal Lie algebra*?)
2. Alternative approach (zero divisor) to the classification of Lie algebras from knot invariants
3. Explicit description for kernels of Lie algebra $\mathfrak{g}$ weight system
4. Infinitely many Vassiliev invariants are absent in the Chern-Simons invariants

Today it cannot be claimed that we can distinguish more knots using Vassiliev invariants than using the Chern-Simons invariants (Jones, HOMFLY-PT, Kauffman, etc. polynomials).

Thank you for your attention!