Problems of the Modern Mathematical Physics — PMMP'25

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All-loop renormalization group invariants in supersymmetric theories with multiple gauge couplings

Quantum corrections as a tool for understanding nature

Investigating of quantum corrections can shed a light to the structure of the surrounding world. For instance, the very precise agreement of the theoretical prediction of the electron anomalous magnetic moment with the experimental data tells us that the nature is described by quantum field theory.

The unification of running couplings and absence of divergent quantum corrections to the Higgs boson mass can be considered indirect indications to the existence of supersymmetry and Grand Unification.

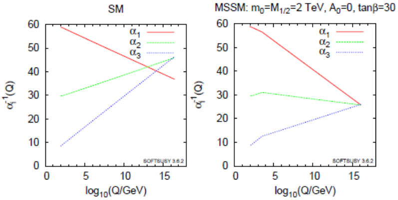

Figure 94.1: Running couplings in SM and MSSM using two-loop RG evolution. The SUSY threshold at 2 TeV is clearly visible on the MSSM side. (We thank Ben Allanach for providing the plots created using SOFTSUSY [61].)

Some important information about new physics can be obtained from the detailed analysis of quantum corrections to (light) Higgs boson in supersymmetric theories, anomalous magnetic moment of muon, etc.

Renormalization group invariants

The renormalization group invariants (RGI) are the scale independent values. Some of them are approximate, but sometimes it is possible to construct the expressions that are RGI in all orders.

For instance, in the MSSM it is possible to construct the approximate RGI from the masses of down quarks and charged leptons

$$\frac{d}{d\ln\mu}\Big(\frac{m_e m_s}{m_d m_\mu}\Big)\approx 0;\qquad \frac{m_e m_s}{m_d m_\mu}\approx\frac{1}{9}\cdot 0.866\approx\frac{1}{9}.$$

This expression is almost protected from quantum corrections and, therefore, at the unification scale it is impossible to reconcile this result with the prediction of the simplest $SU(5)$ GUT

$$m_d = m_e;\qquad m_s = m_\mu;\qquad m_b = m_\tau.$$

A way to solve this problem is to consider more complicated models leading, for example, to the Georgi and Jarlskog textures

H. Georgi, C. Jarlskog, Phys. Lett. B **86** (1979), 297.

RGIs and Georgi–Jarlskog textures

If the Yukawa matrices for the down quarks and charged leptons are chosen in the form

$$Y_d = \begin{pmatrix} 0 & B & 0 \\ B & A & 0 \\ 0 & 0 & C \end{pmatrix}; \qquad Y_e = \begin{pmatrix} 0 & B & 0 \\ B & -3A & 0 \\ 0 & 0 & C \end{pmatrix},$$

then for $B \ll A$ we obtain

$$\frac{m_e m_s}{m_d m_\mu} \approx \frac{1}{9}.$$

The factor -3 can be obtained either from from the Higgs superfield coming from the representation 45_h of the group $SU(5)$

$$5 \times \overline{10} \times 45_h,$$

or with the help of the nonrenormalizable interaction

$$\frac{1}{M} \cdot 5 \times \overline{10} \times 5_h \times 75_H,$$

where 75_H acquires vev breaking $SU(5)$ down to $SU(3) \times SU(2) \times U(1)$, see

S. Raby, Lect. Notes Phys. **939** (2017), 1-308 Springer, 2017.

RGIs also exist in certain supersymmetric theories. In $\mathcal{N}=1$ superspace they are described by the action

$$S=\frac{1}{2e_0^2}\,\mbox{Re}\,\mbox{tr}\int d^4x\,d^2\theta\,W^a W_a+\frac{1}{4}\int d^4x\,d^4\theta\,\phi^{*i}(e^{2V})_i{}^j\phi_j$$
$$+\Big\{\int d^4x\,d^2\theta\,\Big(\frac{1}{4}m_0^{ij}\phi_i\phi_j+\frac{1}{6}\lambda_0^{ijk}\phi_i\phi_j\phi_k\Big)+\mbox{c.c.}\Big\}.$$

Here V is the gauge superfield, ϕ_i are the chiral matter superfields in the representation R of the gauge group G, and

$$W_a=\frac{1}{8}\bar{D}^2\Big(e^{-2V}D_a e^{2V}\Big)$$

is the supersymmetric gauge field strength.

The gauge invariant theory is obtained if the Yukawa couplings and masses satisfy the constraints

$$m_0^{im}(T^A)_m{}^j+m_0^{mj}(T^A)_m{}^i=0;$$
$$\lambda_0^{ijm}(T^A)_m{}^k+\lambda_0^{imk}(T^A)_m{}^j+\lambda_0^{mjk}(T^A)_m{}^i=0,$$

where $(T^A)_i{}^j$ are the generators of the gauge group G in the representation R.

RGIs and the reduction of couplings

The so-called $P = \frac{1}{3}Q$ theories by definition satisfy the constraint

$$\lambda^*_{imn}\lambda^{jmn} - 4\pi\alpha C(R)_i{}^j = \frac{2\pi\alpha}{3} Q\delta_i^j,$$

where $Q \equiv T(R) - 3C_2$. It was demonstrated

I.Jack, D.R.T.Jones, C.G.North, Nucl. Phys. B **473** (1996), 308

that in these theories in the first two orders of the perturbation theory the ratio of the Yukawa couplings to the gauge coupling is RG invariant,

$$\frac{d}{d\ln\mu}\Big(\frac{\lambda^{ijk}}{e}\Big) = 0.$$

similarly to $\mathcal{N} = 2$ supersymmetric theories. If this relation was exact, then it would presumably allow to reduce a number of couplings if we set $\lambda^{ijk} = ec^{ijk}$, where c^{ijk} are certain constants, see

S. Heinemeyer, M. Mondragon, N. Tracas, G. Zoupanos, Phys. Rept. **814** (2019) 1

for more details. However, in the three-loop approximation the above relation is not valid.

NSVZ β-function for $\mathcal{N}=1$ supersymmetric theories

Nevertheless, in supersymmetric theories it is possible to construct RGIs using the exact Novikov, Shifman, Vainshtein, and Zakharov (NSVZ) β-function

V. A. Novikov, M. A. Shifman, A. I. Vainshtein and V. I. Zakharov, Nucl. Phys. B **229** (1983), 381; Phys. Lett. **166B**(1986), 329;
D. R. T. Jones, Phys. Lett. **123B** (1983), 45;
M. A. Shifman and A. I. Vainshtein, Nucl. Phys. B **277** (1986), 456

which relates the β-function and the anomalous dimension of the matter superfields in $\mathcal{N}=1$ supersymmetric gauge theories.

For a general $\mathcal{N}=1$ supersymmetric gauge theory with a single gauge coupling it can be written in the form

$$\beta(\alpha,\lambda) = -\frac{\alpha^2\Big(3C_2 - T(R) + C(R)_i{}^j(\gamma_\phi)_j{}^i(\alpha,\lambda)/r\Big)}{2\pi(1-C_2\alpha/2\pi)}.$$

Here α and λ are the gauge and Yukawa coupling constants, respectively, and we use the notation

$$\text{tr}\,(T^A T^B) \equiv T(R)\,\delta^{AB}; \qquad (T^A)_i{}^k (T^A)_k{}^j \equiv C(R)_i{}^j;$$
$$f^{ACD} f^{BCD} \equiv C_2\delta^{AB}; \qquad r \equiv \delta_{AA} = \dim G.$$

For the pure $\mathcal{N}=1$ SYM theory

$$S = \frac{1}{2e_0^2}\,\mbox{Re}\,\mbox{tr}\int d^4x\, d^2\theta\, W^a W_a$$

from the NSVZ β-function we obtain the equation

$$\frac{1}{\alpha^2}\Big(1-\frac{C_2\alpha}{2\pi}\Big)\frac{d\alpha}{d\ln\mu} = -\frac{3C_2}{2\pi}.$$

Integrating it we obtain the all-loop RGI

$$\Big(\frac{\mu^3}{\alpha}\Big)^{C_2}\exp\Big(-\frac{2\pi}{\alpha}\Big) = \mbox{RGI}$$

Such expressions appear in calculating the instanton contributions to the effective action, and the NSVZ β-function was first obtained by requiring their renormalization group invariance.

V. A. Novikov, M. A. Shifman, A. I. Vainshtein and V. I. Zakharov, Nucl. Phys. B **229** (1983), 381.

For theories with chiral matter superfields the analogous invariants contain the renormalization constants for the matter superfields or masses.

It is known that the renormalization of soft term in theories with softly broken supersymmetry can be related to the renormalization of the rigid theory

J. Hisano, M. A. Shifman, Phys. Rev. D **56** (1997), 5475;
I. Jack, D. R. T. Jones, Phys. Lett. B **415** (1997) 383;
L. V. Avdeev, D. I. Kazakov, I. N. Kondrashuk, Nucl. Phys. B **510** (1998) 289.

For instance, the renormalization of the gaugino mass in the softly broken $\mathcal{N}=1$ SYM theory

$$S = \frac{1}{2e_0^2}\,\mbox{Re}\,\mbox{tr}\int d^4x\,d^2\theta\,(1+2m\theta^2)W^a W_a$$

is described by the RGI

$$\frac{\alpha m}{\beta(\alpha)} = \mbox{RGI}.$$

Differentiating this equation with respect to $\ln\mu$ and substituting the NSVZ expression for the β-function it is possible to obtain the all-order expression for the anomalous dimension of the gaugino mass.

The generalizations for theories containing the chiral matter superfields are also possible.

Scheme dependence of the NSVZ equation

Nevertheless, it is necessary to remember that the NSVZ equation is valid only for certain renormalization prescriptions. Therefore, the all-loop renormalization group invariance of the above expressions does not hold for a general subtraction scheme.

Note that in the $\overline{\text{DR}}$-scheme the NSVZ equation is not valid starting from the order $O(\alpha^4)$ (the three-loop approximation for the β-function and the two-loop approximation for the anomalous dimension)

I. Jack, D. R. T. Jones and C. G. North, Phys.Lett. B **386** (1996) 138;
Nucl.Phys. B **486** (1997) 479;
R. V. Harlander, D. R. T. Jones, P. Kant, L. Mihaila and M. Steinhauser, JHEP **0612** (2006) 024.

However, in this case it is possible to make a special redefinition of the coupling constant which restores the NSVZ relation.

The all-loop NSVZ schemes have been constructed with the help of the higher covariant derivative regularization

A. A. Slavnov, Nucl. Phys. B **31** (1971), 301;
Theor. Math. Phys. **13** (1972), 1064; **33** (1977), 977.

The HD+MSL scheme

In the supersymmetric case the higher covariant derivative regularization can be formulated in terms of superfields and, therefore, does not break supersymmetry

V. K. Krivoshchekov, Theor. Math. Phys. **36** (1978), 745;
P. C. West, Nucl. Phys. B **268** (1986), 113.

In this case (logarithmic) divergences are given by powers of $\ln\Lambda/\mu$, where Λ is a dimensionful regularization parameter, and μ is the renormalization point.

The NSVZ β-function is valid in all loops if a supersymmetric theory is regularized by Higher covariant Derivatives and the renormalization is made by Minimal Subtraction of Logarithms (the so-called HD+MSL scheme), see

K.S., Eur. Phys. J. C **80** (2020) no.10, 911

and references therein.

The whole class of the NSVZ renormalization prescriptions can be obtained from the HD+MSL scheme by making finite renormalizations which satisfy a special constraint

I. O. Goriachuk, A. L. Kataev and K.S., Phys. Lett. B **785** (2018), 561;
I. O. Goriachuk and A. L. Kataev, JETP Lett. **111** (2020) no.12, 663.

The NSVZ relation for $\mathcal{N}=1$ SQED

Let us demonstrate the scheme dependence of the NSVZ equation in the simplest case of $\mathcal{N}=1$ SQED with N_f flavors

$$S=\frac{1}{4e^2}\mbox{Re}\int d^4x\,d^2\theta\,W^aW_a+\sum_{\alpha=1}^{N_f}\frac{1}{4}\int d^4x\,d^4\theta\,\Big(\phi_\alpha^*e^{2V}\phi_\alpha+\widetilde{\phi}_\alpha^*e^{-2V}\widetilde{\phi}_\alpha\Big).$$

For this theory the NSVZ β-function takes the form

$$\beta(\alpha)=\frac{\alpha^2N_f}{\pi}\Big(1-\gamma(\alpha)\Big).$$

M. A. Shifman, A. I. Vainshtein and V. I. Zakharov,
JETP Lett. **42** (1985) 224; Phys. Lett. B **166** (1986) 334.

Expressions for the three-loop β-function and the two-loop anomalous dimension of the matter superfields for $\mathcal{N}=1$ SQED can be found in

A. L. Kataev and K.S., Phys. Lett. B **730** (2014) 184;
Theor. Math. Phys. **181** (2014) 1531.

The HD+MSL-scheme

$$\widetilde{\gamma}_{\text{HD+MSL}}(\alpha) = -\frac{\alpha}{\pi} + \frac{\alpha^2}{\pi^2}\Big(\frac{1}{2} + N_f \ln a + N_f + \frac{N_f A}{2}\Big) + O(\alpha^3);$$

$$\widetilde{\beta}_{\text{HD+MSL}}(\alpha) = \frac{\alpha^2 N_f}{\pi}\Big(1 + \frac{\alpha}{\pi} - \frac{\alpha^2}{\pi^2}\Big(\frac{1}{2} + N_f \ln a + N_f + \frac{N_f A}{2}\Big) + O(\alpha^3)\Big).$$

The MOM-scheme (The result is the same for dimensional reduction and for the higher derivative regularization.)

$$\widetilde{\gamma}_{\text{MOM}}(\alpha) = -\frac{\alpha}{\pi} + \frac{\alpha^2(1+N_f)}{2\pi^2} + O(\alpha^3);$$

$$\widetilde{\beta}_{\text{MOM}}(\alpha) = \frac{\alpha^2 N_f}{\pi}\Big(1 + \frac{\alpha}{\pi} - \frac{\alpha^2}{2\pi^2}\Big(1 + 3N_f\,(1-\zeta(3))\Big) + O(\alpha^3)\Big).$$

The $\overline{\text{DR}}$-scheme

I. Jack, D.R.T. Jones and C.G. North, Phys. Lett. **B386** (1996) 138.

$$\widetilde{\gamma}_{\overline{\text{DR}}}(\alpha) = -\frac{\alpha}{\pi} + \frac{\alpha^2(2+2N_f)}{4\pi^2} + O(\alpha^3);$$

$$\widetilde{\beta}_{\overline{\text{DR}}}(\alpha) = \frac{\alpha^2 N_f}{\pi}\Big(1 + \frac{\alpha}{\pi} - \frac{\alpha^2(2+3N_f)}{4\pi^2} + O(\alpha^3)\Big).$$

Gauge theories with multiple gauge couplings

Let us investigate a possibility of constructing RGIs for some gauge theories with multiple gauge couplings. In this case the gauge group is a direct product

$$G = G_1 \times G_2 \times \ldots \times G_n,$$

where any G_i is either a simple group or $U(1)$. In this case there are n gauge coupling constants α_1, α_2, ..., α_n.
Such theories can be interesting for phenomenology because they include

- QCD+QED
- The Standard Model
- The MSSM
- Some Grand Unified Theories, e.g., the flipped $SU(5)$ theory.

Following

A. L. Kataev, K.S., arXiv:2410.12070 [hep-th];
D. Rystsov, K.S., Phys. Rev. D **111** (2025) no.1, 016012,

we argue that in some $\mathcal{N}=1$ supersymmetric theories with multiple gauge couplings one can construct all-loop RGIs from the gauge and Yukawa couplings. We will also discuss under what renormalization prescriptions the renormalization group invariance is valid in all orders.

QCD+QED

The simplest example of a theory with two gauge coupling constants $\alpha_s \equiv g^2/4\pi$ and $\alpha = e^2/4\pi$ is QCD+QED. In the massless limit this theory is described by the Lagrangian

$$\mathcal{L} = \frac{1}{2g^2}\mbox{tr}\, F_{\mu\nu}^2 - \frac{1}{4e^2}\boldsymbol{F}_{\mu\nu}^2 + \sum_{\mathsf{a}=1}^{N_f} i\bar\psi_{\mathsf{a}}\gamma^\mu \mathcal{D}_\mu \psi_{\mathsf{a}},$$

which is invariant under the transformations of the gauge group $G\times U(1)$. The Dirac spinors ψ_{a} (where the subscript a numerates flavors) lie in a certain irreducible representation R of the group G and have the electromagnetic charges q_{a}. In this case the covariant derivatives are written in the form

$$\mathcal{D}_\mu\psi_{\mathsf{a}} = \partial_\mu\psi_{\mathsf{a}} + A_\mu\psi_{\mathsf{a}} + iq_{\mathsf{a}}\boldsymbol{A}_\mu\psi_{\mathsf{a}},$$

where A_μ and $\boldsymbol{A}_\mu$ are the non-Abelian and Abelian gauge fields, respectively. The corresponding gauge field strengths are given by the expressions

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu + [A_\mu, A_\nu]; \qquad \boldsymbol{F}_{\mu\nu} = \partial_\mu \boldsymbol{A}_\nu - \partial_\nu \boldsymbol{A}_\mu.$$

In quantum field theory the couplings α_s and α depend on scale,

$$\frac{d\alpha}{d\ln\mu} = \beta(\alpha,\alpha_s); \qquad \frac{d\alpha_s}{d\ln\mu} = \beta_s(\alpha_s,\alpha).$$

It is convenient to formulate the supersymmetric version of the above model in terms of superfields

$$S = \frac{1}{2g^2}\,\mbox{Re}\,\mbox{tr}\int d^4x\,d^2\theta\,W^a W_a + \frac{1}{4e^2}\,\mbox{Re}\int d^4x\,d^2\theta\,\boldsymbol{W}^a \boldsymbol{W}_a$$
$$+\sum_{\mathsf{a}=1}^{N_f}\frac{1}{4}\int d^4x\,d^4\theta\,\Big(\phi_{\mathsf{a}}^+ e^{2V+2q_{\mathsf{a}}\boldsymbol{V}}\phi_{\mathsf{a}} + \widetilde\phi_{\mathsf{a}}^+ e^{-2V^T-2q_{\mathsf{a}}\boldsymbol{V}}\widetilde\phi_{\mathsf{a}}\Big),$$

because in this case $\mathcal{N}=1$ supersymmetry is manifest.

Here V and $\boldsymbol{V}$ are the gauge superfields corresponding to the subgroups G and $U(1)$, respectively. The chiral matter superfields ϕ_{a} and $\widetilde\phi_{\mathsf{a}}$ belong to the (conjugated) representations R and $\bar R$, respectively, and have opposite $U(1)$ charges.

Two supersymmetric gauge superfield strengths are written in the form

$$W_a = \frac{1}{8}\bar D^2\Big(e^{-2V}D_a e^{2V}\Big);\qquad \boldsymbol{W}_a = \frac{1}{4}\bar D^2 D_a\boldsymbol{V}.$$

Is it possible to relate running of two gauge coupling constants in this model?

The NSVZ equations can also be written for theories with multiple gauge couplings,

M. A. Shifman, Int. J. Mod. Phys. A **11** (1996), 5761;
D. Korneev, D. Plotnikov, K.S. and N. Tereshina, JHEP **10** (2021), 046.

In the particular case $q_{\rm a}=1$ for $\mathcal{N}=1$ SQCD+SQED they take the form

$$\frac{\beta_s(\alpha_s,\alpha)}{\alpha_s^2} = -\frac{1}{2\pi(1-C_2\alpha_s/2\pi)}\Big[3C_2 - 2T(R)N_f\Big(1-\gamma(\alpha_s,\alpha)\Big)\Big];$$

$$\frac{\beta(\alpha,\alpha_s)}{\alpha^2} = \frac{1}{\pi}\,\mbox{dim}\,R\,N_f\Big(1-\gamma(\alpha_s,\alpha)\Big),$$

where we took into account that if the representation for the matter superfields is irreducible, then in the case under consideration

$$\gamma(\alpha_s,\alpha)_i{}^j = \gamma(\alpha_s,\alpha)\cdot\delta_i^j,$$

where i and j include both the indices numerating chiral matter superfields $\phi_{\rm a}$ and $\widetilde\phi_{\rm a}$ and the indices corresponding to the representation R (or $\overline{R}$).

Comparing the above expressions for the β-functions we see that the anomalous dimension of the matter superfeilds can be eliminated.

The RGI for $\mathcal{N}=1$ SQCD+SQED

After eliminating the anomalous dimension of the matter superfields we obtain that the β-functions satisfy the all-order exact equation

$$\Big(1-\frac{C_2\alpha_s}{2\pi}\Big)\frac{\beta_s(\alpha_s,\alpha)}{\alpha_s^2} = -\frac{3C_2}{2\pi} + \frac{T(R)}{\mbox{dim}\, R}\cdot\frac{\beta(\alpha,\alpha_s)}{\alpha^2}.$$

Evidently, this equation is valid in the HD+MSL scheme, because the original NSVZ equations are satisfied for this renormalization prescription.

Taking into account the boundary conditions for the HD+MSL scheme it is possible to integrate the relation between the β-functions over μ. Then we obtain the equation which relates running of the strong and electromagnetic couplings in the theory under consideration.

$$\frac{1}{\alpha_s} - \frac{1}{\alpha_{s0}} + \frac{C_2}{2\pi}\ln\frac{\alpha_s}{\alpha_{0s}} = -\frac{3C_2}{2\pi}\ln\frac{\Lambda}{\mu} + \frac{T(R)}{\mbox{dim}\, R}\Big(\frac{1}{\alpha}-\frac{1}{\alpha_0}\Big).$$

This in particular implies that the expression

$$\Big(\frac{\alpha_s}{\mu^3}\Big)^{C_2}\exp\Big(\frac{2\pi}{\alpha_s} - \frac{T(R)}{\mbox{dim}\, R}\cdot\frac{2\pi}{\alpha}\Big) = \mbox{RGI}$$

is the renormalization group invariant, i.e. the expression which vanishes afer differentiating with respect to $\ln\mu$.

$\mathcal{N}=1$ SQCD+SQED with different $U(1)$ changes

Let us again consider the theory in which the matter superfields have different $U(1)$ charges q_{a},

$$S=\frac{1}{2g^2}\,\mbox{Re}\,\mbox{tr}\int d^4x\,d^2\theta\,W^a W_a+\frac{1}{4e^2}\,\mbox{Re}\int d^4x\,d^2\theta\,\boldsymbol{W}^a\boldsymbol{W}_a$$
$$+\sum_{\mathsf{a}=1}^{N_f}\frac{1}{4}\int d^4x\,d^4\theta\,\Big(\phi_{\mathsf{a}}^+ e^{2V+2q_{\mathsf{a}}\boldsymbol{V}}\phi_{\mathsf{a}}+\widetilde\phi_{\mathsf{a}}^+ e^{-2V-2q_{\mathsf{a}}\boldsymbol{V}}\widetilde\phi_{\mathsf{a}}\Big)$$

and investigate the limit $\alpha=e^2/4\pi\to 0$. In this case the renormalization group running of the strong coupling constant α_s is exactly the same as in usual $\mathcal{N}=1$ SQCD with the gauge group G and N_f flavors. The running of the electromagnetic coupling constant is described by the Adler D-function

S. L. Adler, Phys. Rev. D **10** (1974), 3714,

which is related to the β-function for the coupling constant α in the limit $\alpha\to 0$,

$$D(\alpha_s)=\frac{3\pi}{2}\lim_{\alpha\to 0}\frac{\beta(\alpha_s,\alpha)}{\alpha^2}.$$

In the limit $\alpha \to 0$ the anomalous dimensions of the matter superfields do not depend on α and, therefore, on q_{a}. This implies that in this case all anomalous dimensions of chiral matter superfields are the same,

$$\lim_{\alpha\to 0} \gamma_{\mathsf{a}}(\alpha_s,\alpha) = \gamma(\alpha_s).$$

Then the NSVZ β-function for $\mathcal{N}=1$ SQCD takes the form

$$\frac{\beta_s(\alpha_s)}{\alpha_s^2} = -\frac{1}{2\pi(1-C_2\alpha_s/2\pi)}\Big[3C_2 - 2T(R)N_f\Big(1-\gamma(\alpha_s)\Big)\Big].$$

The exact NSVZ-like expression for the Adler D-function in the theory under consideration has been derived in

M. Shifman and K.S., Phys. Rev. Lett. **114** (2015) 051601; Phys. Rev. D **91** (2015), 105008.

$$D(\alpha_s) = \frac{3}{2}\,\mbox{dim}\,R\sum_{\mathsf{a}=1}^{N_f}(q_{\mathsf{a}})^2\Big(1-\gamma(\alpha_s)\Big) \equiv \frac{3}{2}\,\boldsymbol{q^2}\,\mbox{dim}\,R\Big(1-\gamma(\alpha_s)\Big),$$

where $\boldsymbol{q^2} \equiv \sum\limits_{\mathsf{a}=1}^{N_f}(q_{\mathsf{a}})^2$.

$\mathcal{N}=1$ SQCD+SQED with different $U(1)$ changes

Therefore, the β-function of $\mathcal{N}=1$ SQCD can be related to the Adler D-function by the all-loop equation

$$\beta_s(\alpha_s) = -\frac{\alpha_s^2}{2\pi(1-C_2\alpha_s/2\pi)}\Big[3C_2 - \frac{4\,T(R)N_f D(\alpha_s)}{3\,\boldsymbol{q^2}\,\mbox{dim}\,R}\Big],$$

which connects the renormalization group running of the strong and electromagnetic coupling constants in the limit $\alpha\to 0$. Evidently, this equation is valid in the HD+MSL scheme in all orders.

Thus, from the NSVZ equation we see that

1. If all $U(1)$ charges $q_{\rm a}$ are the same, then in the $\mathcal{N}=1$ SQCD+SQED, which is a theory with two gauge couplings, it is possible to relate their running.

2. If the charges $q_{\rm a}$ are different, then it is possible to relate the β-function of $\mathcal{N}=1$ SQCD to the Adler D-function. Actually, in this case the exact relation exists only in the limit $\alpha\to 0$.

The MSSM is the simplest supersymmetric extension of the Standard Model. It is a gauge theory with the group $SU_3 \times SU_2 \times U_1$ and softly broken supersymmetry. Consequently, there are 3 gauge coupling constants e_3, e_2, and e_1 in the MSSM (their number is equal to the number of factors in the gauge group). Quarks, leptons, and Higgs fields are components of the chiral matter superfields:

Superfield	SU_3	SU_2	U_1 (Y)	Superfield	SU_3	SU_2	U_1 (Y)
$3 \times Q$	$\bar{3}$	2	$-1/6$	$3 \times N$	1	1	0
$3 \times U$	3	1	$2/3$	$3 \times E$	1	1	-1
$3 \times D$	3	1	$-1/3$	H_d	1	2	$1/2$
$3 \times L$	1	2	$1/2$	H_u	1	2	$-1/2$

where for the superfields which include left quarks and leptons we use the brief notations

$$Q = \begin{pmatrix} \widetilde{U} \\ \widetilde{D} \end{pmatrix}; \qquad L = \begin{pmatrix} \widetilde{N} \\ \widetilde{E} \end{pmatrix}.$$

The MSSM superpotential

The MSSM contains three gauge couplings

$$\alpha_3 = \frac{e_3^2}{4\pi}; \qquad \alpha_2 = \frac{e_2^2}{4\pi}; \qquad \alpha_1 = \frac{5}{3}\cdot\frac{e_1^2}{4\pi}$$

corresponding to the subgroups $SU(3)$, $SU(2)$, and $U(1)$, respectively. (The factor $5/3$ in the coupling constant α_1 is introduced in order that the unification of couplings has the form $\alpha_1 = \alpha_2 = \alpha_3$.) There are also dimensionless Yukawa couplings $(Y_U)_{IJ}$, $(Y_D)_{IJ}$, and $(Y_E)_{IJ}$ (which are 3×3 matrices) inside the superpotential

$$\begin{aligned} W = (Y_U)_{IJ}\left(\widetilde{U}\ \widetilde{D}\right)^a_I \begin{pmatrix} 0 & 1\\ -1 & 0 \end{pmatrix} \begin{pmatrix} H_{u1}\\ H_{u2} \end{pmatrix} U_{aJ} \\ +(Y_D)_{IJ}\left(\widetilde{U}\ \widetilde{D}\right)^a_I \begin{pmatrix} 0 & 1\\ -1 & 0 \end{pmatrix} \begin{pmatrix} H_{d1}\\ H_{d2} \end{pmatrix} D_{aJ} \\ +(Y_E)_{IJ}\left(\widetilde{N}\ \widetilde{E}\right)_I \begin{pmatrix} 0 & 1\\ -1 & 0 \end{pmatrix} \begin{pmatrix} H_{d1}\\ H_{d2} \end{pmatrix} E_J \\ +\boldsymbol{\mu}\left(H_{u1}\ H_{u2}\right) \begin{pmatrix} 0 & 1\\ -1 & 0 \end{pmatrix} \begin{pmatrix} H_{d1}\\ H_{d2} \end{pmatrix}. \end{aligned}$$

Moreover, the superpotential includes a term with the parameter $\boldsymbol{\mu}$, which has the dimension of mass.

The renormalization group running of the gauge couplings in the MSSM can be described exactly in all loops with the help of the NSVZ β-functions

M. A. Shifman, Int. J. Mod. Phys. A **11** (1996), 5761.

$$\frac{\beta_1}{\alpha_1^2} = -\frac{3}{5}\cdot\frac{1}{2\pi}\Big[-11 + \mbox{tr}\Big(\frac{1}{6}\gamma_Q + \frac{4}{3}\gamma_U + \frac{1}{3}\gamma_D + \frac{1}{2}\gamma_L + \gamma_E\Big) + \frac{1}{2}\gamma_{H_u} + \frac{1}{2}\gamma_{H_d}\Big];$$
$$\frac{\beta_2}{\alpha_2^2} = -\frac{1}{2\pi(1-\alpha_2/\pi)}\Big[-1 + \mbox{tr}\Big(\frac{3}{2}\gamma_Q + \frac{1}{2}\gamma_L\Big) + \frac{1}{2}\gamma_{H_u} + \frac{1}{2}\gamma_{H_d}\Big];$$
$$\frac{\beta_3}{\alpha_3^2} = -\frac{1}{2\pi(1-3\alpha_3/2\pi)}\Big[3 + \mbox{tr}\Big(\gamma_Q + \frac{1}{2}\gamma_U + \frac{1}{2}\gamma_D\Big)\Big].$$

They relate three gauge β-functions of the theory to the anomalous dimensions of the chiral matter superfields. The renormalization group functions (RGFs) are defined by the equations

$$\beta_i(\alpha,Y) = \frac{d\alpha_i}{d\ln\mu}\Big|_{\alpha_0,Y_0=\mbox{const}};\qquad \gamma_i(\alpha,Y) = \frac{d\ln Z_i}{d\ln\mu}\Big|_{\alpha_0,Y_0=\mbox{const}},$$

where the subscript 0 denotes the bare values.

The exact equations describing the renormalization of the MSSM Yukawa couplings

RGFs describing the renormalization of the Yukawa couplings and of the parameter $\boldsymbol{\mu}$ can also be related to the anomalous dimensions of the matter superfields due to the nonrenormalization of the superpotential

M. T. Grisaru, W. Siegel, M. Rocek, Nucl. Phys. B **159** (1979), 429.

$$\frac{dY_U}{d\ln\mu} = \frac{1}{2}\Big(\gamma_{H_u} Y_U + (\gamma_Q)^T Y_U + Y_U \gamma_U\Big);$$
$$\frac{dY_D}{d\ln\mu} = \frac{1}{2}\Big(\gamma_{H_d} Y_D + (\gamma_Q)^T Y_D + Y_D \gamma_D\Big);$$
$$\frac{dY_E}{d\ln\mu} = \frac{1}{2}\Big(\gamma_{H_d} Y_E + (\gamma_L)^T Y_E + Y_E \gamma_E\Big);$$
$$\frac{d\boldsymbol{\mu}}{d\ln\mu} = \frac{1}{2}\Big(\gamma_{H_u} + \gamma_{H_d}\Big)\boldsymbol{\mu}.$$

It is important that these equations are valid in the HD+MSL scheme because in this scheme all renormalization constants contain only powers of $\ln\Lambda/\mu$, where Λ is the dimensionful regularization parameter.

The renormalization group equations for the Yukawa couplings can be multiplied by the corresponding inverse matrices. After that, it is possible to calculate traces of the resulting equations using the formula

$$\mbox{tr}\Big[M^{-1}\frac{dM}{d\ln\mu}\Big] = \frac{d}{d\ln\mu}\mbox{tr}\ln M = \frac{d}{d\ln\mu}\ln\det M,$$

Then (taking into account that the indices numerating generators range from 1 to 3) we see that the equations describing how the determinants of the Yukawa matrices depend on the renormalization point μ are written as

$$\gamma_{\det Y_U} \equiv \frac{d\ln\det Y_U}{d\ln\mu} = \mbox{tr}\Big[(Y_U)^{-1}\frac{dY_U}{d\ln\mu}\Big] = \frac{1}{2}\Big(3\gamma_{H_u} + \mbox{tr}\big(\gamma_Q+\gamma_U\big)\Big);$$

$$\gamma_{\det Y_D} \equiv \frac{d\ln\det Y_D}{d\ln\mu} = \mbox{tr}\Big[(Y_D)^{-1}\frac{dY_D}{d\ln\mu}\Big] = \frac{1}{2}\Big(3\gamma_{H_d} + \mbox{tr}\big(\gamma_Q+\gamma_D\big)\Big);$$

$$\gamma_{\det Y_E} \equiv \frac{d\ln\det Y_E}{d\ln\mu} = \mbox{tr}\Big[(Y_E)^{-1}\frac{dY_E}{d\ln\mu}\Big] = \frac{1}{2}\Big(3\gamma_{H_d} + \mbox{tr}\big(\gamma_L+\gamma_E\big)\Big).$$

They can be solved together with the NSVZ equations and the equation describing the renormalization of the parameter $\boldsymbol{\mu}$.

Collecting the above equations we obtain the system of differential equations describing the renormalization of the MSSM parameters exactly in all orders

$$\frac{d}{d\ln\mu}\Big(\frac{5}{3}\cdot\frac{2\pi}{\alpha_1}\Big) = -11 + \mbox{tr}\Big(\frac{1}{6}\gamma_Q + \frac{4}{3}\gamma_U + \frac{1}{3}\gamma_D + \frac{1}{2}\gamma_L + \gamma_E\Big) + \frac{1}{2}\gamma_{H_u} + \frac{1}{2}\gamma_{H_d};$$

$$\frac{d}{d\ln\mu}\Big(\frac{2\pi}{\alpha_2} + 2\ln\alpha_2\Big) = -1 + \mbox{tr}\Big(\frac{3}{2}\gamma_Q + \frac{1}{2}\gamma_L\Big) + \frac{1}{2}\gamma_{H_u} + \frac{1}{2}\gamma_{H_d};$$

$$\frac{d}{d\ln\mu}\Big(\frac{2\pi}{\alpha_3} + 3\ln\alpha_3\Big) = 3 + \mbox{tr}\Big(\gamma_Q + \frac{1}{2}\gamma_U + \frac{1}{2}\gamma_D\Big);$$

$$\frac{d\ln\det Y_U}{d\ln\mu} = \frac{1}{2}\Big(3\gamma_{H_u} + \mbox{tr}\big(\gamma_Q + \gamma_U\big)\Big);$$

$$\frac{d\ln\det Y_D}{d\ln\mu} = \frac{1}{2}\Big(3\gamma_{H_d} + \mbox{tr}\big(\gamma_Q + \gamma_D\big)\Big);$$

$$\frac{d\ln\det Y_E}{d\ln\mu} = \frac{1}{2}\Big(3\gamma_{H_d} + \mbox{tr}\big(\gamma_L + \gamma_E\big)\Big);$$

$$\frac{d\ln\boldsymbol{\mu}}{d\ln\mu} = \frac{1}{2}\Big(\gamma_{H_u} + \gamma_{H_d}\Big).$$

The anomalous dimensions of the chiral matter superfields and $\boldsymbol{\mu}$ can be eliminated, thereby obtaining a differential equation which contains only derivatives of the gauge and Yukawa couplings.

Eliminating the anomalous dimensions of the matter superfields

First, we eliminate $\text{tr}(\gamma_L)$, $\text{tr}(\gamma_E)$, and $\gamma_{H_u} + \gamma_{H_d}$. The resulting equations contain γ_Q, γ_D, and γ_U only in the combination $\text{tr}(2\gamma_Q + \gamma_U + \gamma_D)$,

$$\frac{d}{d\ln\mu}\Big(\frac{2\pi}{\alpha_3} + 3\ln\alpha_3\Big) = 3 + \frac{1}{2}\text{tr}\Big(2\gamma_Q + \gamma_U + \gamma_D\Big);$$

$$\frac{d}{d\ln\mu}\Big(\frac{2\pi}{\alpha_2} + 2\ln\alpha_2 + \frac{5}{3}\cdot\frac{2\pi}{\alpha_1} - 2\ln\boldsymbol{\mu} - 2\ln\det Y_E + 2\ln\det Y_D\Big) = -12 + \frac{4}{3}\text{tr}\Big(2\gamma_Q + \gamma_U + \gamma_D\Big);$$

$$\frac{d}{d\ln\mu}\Big(\ln\det Y_D + \ln\det Y_U - 3\ln\boldsymbol{\mu}\Big) = \frac{1}{2}\text{tr}\Big(2\gamma_Q + \gamma_U + \gamma_D\Big).$$

This allows either eliminating the one-loop constants or eliminating the parameter $\boldsymbol{\mu}$. The resulting equations take the form

$$\frac{d}{d\ln\mu}\Big(\frac{2\pi}{\alpha_3} + 3\ln\alpha_3 + \frac{\pi}{2\alpha_2} + \frac{1}{2}\ln\alpha_2 + \frac{5\pi}{6\alpha_1} - \frac{1}{2}\ln\det Y_E - \frac{7}{6}\ln\det Y_D - \frac{5}{3}\ln\det Y_U + \frac{9}{2}\ln\boldsymbol{\mu}\Big) = 0,$$

$$\frac{d}{d\ln\mu}\Big(\frac{2\pi}{\alpha_3} + 3\ln\alpha_3 - \frac{\pi}{\alpha_2} - \ln\alpha_2 - \frac{5\pi}{3\alpha_1} + \ln\det Y_E - \frac{2}{3}\ln\det Y_D + \frac{1}{3}\ln\det Y_U - 9\ln\mu\Big) = 0,$$

respectively.

Integrating the first equation we obtain the expression RGI_1, which does not explicitly depend on the scale μ, but contains the parameter $\boldsymbol{\mu}$. Integrating the second equation gives the expression RGI_2 independent of $\boldsymbol{\mu}$, but containing the scale μ,

$$\mathrm{RGI}_1 = \frac{\boldsymbol{\mu}^{9/2}\,(\alpha_3)^3\,(\alpha_2)^{1/2}}{\big(\det Y_E\big)^{1/2}\,\big(\det Y_U\big)^{5/3}\,\big(\det Y_D\big)^{7/6}}\,\exp\Big(\frac{2\pi}{\alpha_3}+\frac{\pi}{2\alpha_2}+\frac{5\pi}{6\alpha_1}\Big);$$

$$\mathrm{RGI}_2 = \frac{(\alpha_3)^3\det Y_E\,\big(\det Y_U\big)^{1/3}}{\mu^9\,\alpha_2\,\big(\det Y_D\big)^{2/3}}\,\exp\Big(\frac{2\pi}{\alpha_3}-\frac{\pi}{\alpha_2}-\frac{5\pi}{3\alpha_1}\Big).$$

Instead of the renormalization group invariants (RGI_1, RGI_2) it is possible to use the equivalent set (RGI_3, RGI_4), where the expressions

$$\mathrm{RGI}_3 \equiv \Big(\frac{\mathrm{RGI}_1}{\mathrm{RGI}_2}\Big)^{2/3} = \frac{\boldsymbol{\mu}^3\,\mu^6\,\alpha_2}{\big(\det Y_E\big)\,\big(\det Y_U\big)^{4/3}\,\big(\det Y_D\big)^{1/3}}\,\exp\Big(\frac{\pi}{\alpha_2}+\frac{5\pi}{3\alpha_1}\Big);$$

$$\mathrm{RGI}_4 \equiv \big(\mathrm{RGI}_1\big)^{2/3}\big(\mathrm{RGI}_2\big)^{1/3} = \frac{\boldsymbol{\mu}^3\,(\alpha_3)^3}{\mu^3\det Y_U\,\det Y_D}\,\exp\Big(\frac{2\pi}{\alpha_3}\Big)$$

also have a rather simple form.

Differentiating $\ln(\text{RGI}_3)$ and $\ln(\text{RGI}_4)$ with respect to $\ln\mu$ we obtain the equations

$$0 = \Big(\frac{1}{\alpha_2} - \frac{\pi}{\alpha_2^2}\Big)\beta_2 - \frac{5\pi}{3{\alpha_1}^2}\beta_1 + 6 + 3\gamma_{\mu} - \gamma_{\det Y_E} - \frac{4}{3}\gamma_{\det Y_U} - \frac{1}{3}\gamma_{\det Y_D};$$
$$0 = \Big(\frac{3}{\alpha_3} - \frac{2\pi}{\alpha_3^2}\Big)\beta_3 - 3 + 3\gamma_{\mu} - \gamma_{\det Y_U} - \gamma_{\det Y_D}.$$

The scheme dependence of these equations becomes essential starting from the order $O(\alpha^2, \alpha Y^2, Y^4)$ corresponding to the three-loop approximation for the β-functions and to the two-loop approximation for the anomalous dimensions.

In the HD+MSL scheme they should be satisfied in all orders independently of the regularization parameters

$$A \equiv \int\limits_0^\infty dx\, \ln x\, \frac{d}{dx}\frac{1}{R(x)}; \qquad a_{\varphi,3} \equiv \frac{M_{\varphi,3}}{\Lambda}; \qquad a_{\varphi,2} \equiv \frac{M_{\varphi,2}}{\Lambda};$$
$$B \equiv \int\limits_0^\infty dx\, \ln x\, \frac{d}{dx}\frac{1}{F^2(x)}; \qquad a_3 \equiv \frac{M_3}{\Lambda}; \qquad a_2 \equiv \frac{M_2}{\Lambda}; \qquad a_1 \equiv \frac{M_1}{\Lambda},$$

where $R(x)$ and $F(x)$ are the higher derivative regulator functions, and M_i are the Pauli–Villars masses.

The three-loop verification, the HD+MSL scheme

The three-loop β-functions for the MSSM in the HD+MSL scheme have been calculated in

O. Haneychuk, V. Shirokova, K.S., JHEP **09** (2022), 189.

$$\frac{\beta_1(\alpha,Y)}{\alpha_1^2} = -\frac{1}{2\pi}\cdot\frac{3}{5}\Big\{-11-\frac{199\alpha_1}{60\pi}-\frac{9\alpha_2}{4\pi}-\frac{22\alpha_3}{3\pi}+\frac{1}{8\pi^2}\mbox{tr}\Big(\frac{13}{3}Y_U Y_U^+ +\frac{7}{3}Y_D Y_D^+ +3Y_E Y_E^+\Big)$$
$$+\frac{1}{2\pi^2}\Big[\frac{5131\alpha_1^2}{3600}+\frac{27\alpha_2^2}{16}+\frac{88\alpha_3^2}{9}+\frac{23\alpha_1\alpha_2}{40}+\frac{137\alpha_1\alpha_3}{45}+\alpha_2\alpha_3+\frac{2189\alpha_1^2}{100}\Big(\ln a_1+1+\frac{A}{2}\Big)$$
$$+\frac{9\alpha_2^2}{4}\Big(7\ln a_2-6\ln a_{\varphi,2}+1+\frac{A}{2}\Big)-22\alpha_3^2\Big(3\ln a_{\varphi,3}-2\ln a_3+1+\frac{A}{2}\Big)\Big]+\frac{1}{8\pi^3}\mbox{tr}\Big(Y_U Y_U^+\Big)$$
$$\times\Big[2\alpha_2+2\alpha_3+(B-A)\Big(\frac{169\alpha_1}{180}+\frac{13\alpha_2}{4}+\frac{52\alpha_3}{9}\Big)\Big]+\frac{1}{8\pi^3}\mbox{tr}\Big(Y_D Y_D^+\Big)\Big[\frac{\alpha_2}{2}+2\alpha_3+(B-A)$$
$$\times\Big(\frac{49\alpha_1}{180}+\frac{7\alpha_2}{4}+\frac{28\alpha_3}{9}\Big)\Big]+\frac{1}{8\pi^3}\mbox{tr}\Big(Y_E Y_E^+\Big)\Big[\frac{3\alpha_2}{2}+(B-A)\Big(\frac{27\alpha_1}{20}+\frac{9\alpha_2}{4}\Big)\Big]-\frac{1}{(8\pi^2)^2}\Big[\frac{15}{4}$$
$$\times\mbox{tr}\Big((Y_U Y_U^+)^2\Big)+\frac{11}{4}\mbox{tr}\Big((Y_D Y_D^+)^2\Big)+\frac{9}{4}\mbox{tr}\Big((Y_E Y_E^+)^2\Big)+\frac{19}{6}\mbox{tr}\Big(Y_D Y_D^+ Y_U Y_U^+\Big)+\frac{17}{4}\Big(\mbox{tr}(Y_U Y_U^+)\Big)^2$$
$$+\frac{5}{4}\Big(\mbox{tr}(Y_D Y_D^+)\Big)^2+\frac{5}{4}\Big(\mbox{tr}(Y_E Y_E^+)\Big)^2+\frac{25}{6}\mbox{tr}\Big(Y_E Y_E^+\Big)\,\mbox{tr}\Big(Y_D Y_D^+\Big)\Big]\Big\}+O(\alpha^3,\alpha^2Y^2,\alpha Y^4,Y^6);$$

$$\frac{\beta_2(\alpha,Y)}{\alpha_2^2} = \ldots + O(\alpha^3,\alpha^2Y^2,\alpha Y^4,Y^6); \qquad \frac{\beta_3(\alpha,Y)}{\alpha_3^2} = \ldots + O(\alpha^3,\alpha^2Y^2,\alpha Y^4,Y^6).$$

The two-loop anomalous dimensions in the HD+MSL scheme can be calculated starting from the anomalous dimensions of the chiral matter superfields

$$\gamma_{\det Y_U}(\alpha,Y) = \frac{1}{2}\Big(3\gamma_{H_u}(\alpha,Y) + \text{tr}\,\gamma_Q(\alpha,Y) + \text{tr}\,\gamma_U(\alpha,Y)\Big)$$

$$= -\frac{13\alpha_1}{20\pi} - \frac{9\alpha_2}{4\pi} - \frac{4\alpha_3}{\pi} + \frac{1}{16\pi^2}\,\text{tr}\Big(12\,Y_U Y_U^+ + Y_D Y_D^+\Big) + \frac{1}{2\pi^2}\Big[\frac{169\alpha_1^2}{1200} + \frac{27\alpha_2^2}{16} + \frac{16\alpha_3^2}{3}$$

$$+\frac{3\alpha_1\alpha_2}{8} + \frac{17\alpha_1\alpha_3}{15} + 3\alpha_2\alpha_3 - \frac{27\alpha_2^2}{2}\Big(\ln a_{\varphi,2} + 1 + \frac{A}{2}\Big) - 36\alpha_3^2\Big(\ln a_{\varphi,3} + 1 + \frac{A}{2}\Big)$$

$$+\frac{429\alpha_1^2}{100}\Big(\ln a_1 + 1 + \frac{A}{2}\Big) + \frac{63\alpha_2^2}{4}\Big(\ln a_2 + 1 + \frac{A}{2}\Big) + 24\alpha_3^2\Big(\ln a_3 + 1 + \frac{A}{2}\Big)\Big] + \frac{1}{16\pi^3}$$

$$\times\,\text{tr}(Y_U Y_U^+)\Big[\frac{7\alpha_1}{10} + \frac{3\alpha_2}{2} + 12\alpha_3 + (B-A)\Big(\frac{13\alpha_1}{5} + 9\alpha_2 + 16\alpha_3\Big)\Big] + \frac{1}{16\pi^3}\text{tr}(Y_D Y_D^+)$$

$$\times\Big[\frac{\alpha_1}{10} + (B-A)\Big(\frac{7\alpha_1}{60} + \frac{3\alpha_2}{4} + \frac{4\alpha_3}{3}\Big)\Big] - \frac{1}{(16\pi^2)^2}\Big[31\,\text{tr}\Big((Y_U Y_U^+)^2\Big) + 2\,\text{tr}\Big((Y_D Y_D^+)^2\Big)$$

$$+11\,\text{tr}\Big(Y_D Y_D^+ Y_U Y_U^+\Big) + 9\Big(\text{tr}(Y_U Y_U^+)\Big)^2 + 3\Big(\text{tr}(Y_D Y_D^+)\Big)^2 + \text{tr}(Y_E Y_E^+)\,\text{tr}(Y_D Y_D^+)\Big]$$

$$+O\Big(\alpha^3, \alpha^2 Y^2, \alpha Y^4, Y^6\Big);$$

$$\gamma_{\det Y_D}(\alpha,Y) = \frac{1}{2}\Big(3\gamma_{H_d}(\alpha,Y) + \text{tr}\,\gamma_Q(\alpha,Y) + \text{tr}\,\gamma_D(\alpha,Y)\Big) = \ldots + O(\alpha^3, \alpha^2 Y^2, \alpha Y^4, Y^6);$$

$$\gamma_{\det Y_E}(\alpha,Y) = \frac{1}{2}\Big(3\gamma_{H_d}(\alpha,Y) + \text{tr}\,\gamma_L(\alpha,Y) + \text{tr}\,\gamma_E(\alpha,Y)\Big) = \ldots + O(\alpha^3, \alpha^2 Y^2, \alpha Y^4, Y^6);$$

$$\gamma_{\mu}(\alpha,Y) = \frac{1}{2}\Big(\gamma_{H_u}(\alpha,Y) + \gamma_{H_d}(\alpha,Y)\Big) = \ldots + O(\alpha^3, \alpha^2 Y^2, \alpha Y^4, Y^6).$$

The three-loop verification, the HD+MSL scheme

Substituting the above expressions for RGFs in the HD+MSL scheme we see that in the considered approximation the derivatives of the expressions RGI_3 and RGI_4 vanish independently of the values of the regularization parameters,

$$\Big[\Big(\frac{1}{\alpha_2}-\frac{\pi}{\alpha_2^2}\Big)\beta_2-\frac{5\pi}{3\alpha_1^2}\beta_1+6+3\gamma_{\mu}-\gamma_{\det Y_E}-\frac{4}{3}\gamma_{\det Y_U}-\frac{1}{3}\gamma_{\det Y_D}\Big]_{\text{HD+MSL}}=O\Big(\alpha^3,\alpha^2Y^2,\alpha Y^4,Y^6\Big);$$

$$\Big[\Big(\frac{3}{\alpha_3}-\frac{2\pi}{\alpha_3^2}\Big)\beta_3-3+3\gamma_{\mu}-\gamma_{\det Y_U}-\gamma_{\det Y_D}\Big]_{\text{HD+MSL}}=O\Big(\alpha^3,\alpha^2Y^2,\alpha Y^4,Y^6\Big).$$

Therefore, in the considered approximation the expressions RGI_1 and RGI_2 also do not depend on the renormalization point μ in the HD+MSL scheme.

Certainly, this is quite expected because the HD+MSL scheme is NSVZ in all orders.

However, the most popular renormalization prescription in the supersymmetric case is the $\overline{\text{DR}}$ scheme. The matter is that dimensional regularization

G. 't Hooft and M. J. G. Veltman, Nucl. Phys. B **44** (1972), 189;
C. G. Bollini and J. J. Giambiagi, Nuovo Cim. B **12** (1972), 20;
J. F. Ashmore, Lett. Nuovo Cim. **4** (1972), 289;
G. M. Cicuta and E. Montaldi, Lett. Nuovo Cim. **4** (1972), 329.

explicitly breaks supersymmetry, because the numbers of boson and fermion degrees of freedom differently depend on the space-time dimension.
That is why in the supersymmetric case it is more convenient to use its modification called dimensional reduction

W. Siegel, Phys. Lett. B **84** (1979), 193.

In this case the γ-matrices are taken in the integer dimension (usually, $D=4$), while the loop integrals are calculated in the dimension $D=4-\varepsilon$.

The $\overline{\text{DR}}$ scheme is obtained if the dimensional reduction is supplemented by modified minimal subtraction.

The three-loop verification, the $\overline{\text{DR}}$ scheme

The three-loop β-functions for the MSSM in the $\overline{\text{DR}}$ scheme have been calculated in

I. Jack, D. R. T. Jones, A. F. Kord, Annals Phys. **316** (2005), 213.

$$\frac{\beta_1(\alpha,Y)}{\alpha_1^2} = -\frac{1}{2\pi}\cdot\frac{3}{5}\Big\{-11-\frac{199\alpha_1}{60\pi}-\frac{9\alpha_2}{4\pi}-\frac{22\alpha_3}{3\pi}+\frac{1}{8\pi^2}\mbox{tr}\Big(\frac{13}{3}Y_U Y_U^+ +\frac{7}{3}Y_D Y_D^+ +3Y_E Y_E^+\Big)$$
$$+\frac{1}{2\pi^2}\Big(\frac{32117\alpha_1^2}{1800}+\frac{27\alpha_2^2}{8}-\frac{121\alpha_3^2}{18}+\frac{23\alpha_1\alpha_2}{40}+\frac{137\alpha_1\alpha_3}{45}+\alpha_2\alpha_3\Big)+\frac{1}{8\pi^3}\mbox{tr}\Big(Y_U Y_U^+\Big)\Big(\frac{169\alpha_1}{360}$$
$$+\frac{29\alpha_2}{8}+\frac{44\alpha_3}{9}\Big)+\frac{1}{8\pi^3}\mbox{tr}\Big(Y_D Y_D^+\Big)\Big(\frac{49\alpha_1}{360}+\frac{11\alpha_2}{8}+\frac{32\alpha_3}{9}\Big)+\frac{1}{8\pi^3}\mbox{tr}\Big(Y_E Y_E^+\Big)\Big(\frac{27\alpha_1}{40}+\frac{21\alpha_2}{8}\Big)$$
$$-\frac{1}{(8\pi^2)^2}\Big[7\,\mbox{tr}\Big((Y_U Y_U^+)^2\Big)+\frac{9}{2}\mbox{tr}\Big((Y_D Y_D^+)^2\Big)+\frac{9}{2}\mbox{tr}\Big((Y_E Y_E^+)^2\Big)+\frac{29}{6}\mbox{tr}\Big(Y_D Y_D^+ Y_U Y_U^+\Big)$$
$$+\frac{15}{2}\Big(\mbox{tr}(Y_U Y_U^+)\Big)^2+3\Big(\mbox{tr}(Y_D Y_D^+)\Big)^2+2\Big(\mbox{tr}(Y_E Y_E^+)\Big)^2+7\,\mbox{tr}\Big(Y_E Y_E^+\Big)\,\mbox{tr}\Big(Y_D Y_D^+\Big)\Big]\Big\}$$
$$+O(\alpha^3,\alpha^2Y^2,\alpha Y^4,Y^6);$$

$$\frac{\beta_2(\alpha,Y)}{\alpha_2^2} = \ldots + O(\alpha^3,\alpha^2Y^2,\alpha Y^4,Y^6);$$
$$\frac{\beta_3(\alpha,Y)}{\alpha_3^2} = \ldots + O(\alpha^3,\alpha^2Y^2,\alpha Y^4,Y^6).$$

The three-loop verification, the $\overline{\text{DR}}$ scheme

Again, the two-loop anomalous dimensions in the $\overline{\text{DR}}$ scheme can be calculated starting from the anomalous dimensions of the chiral matter superfields

$$\gamma_{\det Y_U}(\alpha,Y) = \frac{1}{2}\Big(3\gamma_{H_u}(\alpha,Y) + \text{tr}\,\gamma_Q(\alpha,Y) + \text{tr}\,\gamma_U(\alpha,Y)\Big)$$
$$= -\frac{13\alpha_1}{20\pi} - \frac{9\alpha_2}{4\pi} - \frac{4\alpha_3}{\pi} + \frac{1}{16\pi^2}\,\text{tr}\Big(12\,Y_U Y_U^+ + Y_D Y_D^+\Big) + \frac{1}{2\pi^2}\Big[\frac{2743\alpha_1^2}{1200} + \frac{45\alpha_2^2}{16} - \frac{2\alpha_3^2}{3}$$
$$+\frac{3\alpha_1\alpha_2}{8} + \frac{17\alpha_1\alpha_3}{15} + 3\alpha_2\alpha_3\Big] + \frac{1}{16\pi^3}\,\text{tr}(Y_U Y_U^+)\Big[\frac{7\alpha_1}{10} + \frac{3\alpha_2}{2} + 12\alpha_3\Big] + \frac{1}{16\pi^3}\text{tr}(Y_D Y_D^+)$$
$$\times\frac{\alpha_1}{10} - \frac{1}{(16\pi^2)^2}\Big[31\,\text{tr}\Big((Y_U Y_U^+)^2\Big) + 2\,\text{tr}\Big((Y_D Y_D^+)^2\Big) + 11\,\text{tr}\Big(Y_D Y_D^+ Y_U Y_U^+\Big) + 9\Big(\text{tr}(Y_U Y_U^+)\Big)^2$$
$$+3\Big(\text{tr}(Y_D Y_D^+)\Big)^2 + \text{tr}(Y_E Y_E^+)\,\text{tr}(Y_D Y_D^+)\Big] + O\Big(\alpha^3,\alpha^2Y^2,\alpha Y^4,Y^6\Big);$$

$$\gamma_{\det Y_D}(\alpha,Y) = \frac{1}{2}\Big(3\gamma_{H_d}(\alpha,Y) + \text{tr}\,\gamma_Q(\alpha,Y) + \text{tr}\,\gamma_D(\alpha,Y)\Big) = \ldots + O(\alpha^3,\alpha^2Y^2,\alpha Y^4,Y^6);$$

$$\gamma_{\det Y_E}(\alpha,Y) = \frac{1}{2}\Big(3\gamma_{H_d}(\alpha,Y) + \text{tr}\,\gamma_L(\alpha,Y) + \text{tr}\,\gamma_E(\alpha,Y)\Big) = \ldots + O(\alpha^3,\alpha^2Y^2,\alpha Y^4,Y^6);$$

$$\gamma_{\mu}(\alpha,Y) = \frac{1}{2}\Big(\gamma_{H_u}(\alpha,Y) + \gamma_{H_d}(\alpha,Y)\Big) = \ldots + O(\alpha^3,\alpha^2Y^2,\alpha Y^4,Y^6).$$

However, in the $\overline{\text{DR}}$ scheme the derivatives of $\ln \text{RGI}_3$ and $\ln \text{RGI}_4$ with respect to $\ln\mu$ do not vanish in that orders where the scheme dependence becomes essential.

The three-loop verification, the $\overline{\text{DR}}$ scheme

$$\Big[\Big(\frac{1}{\alpha_2}-\frac{\pi}{\alpha_2^2}\Big)\beta_2-\frac{5\pi}{3\alpha_1^2}\beta_1+6+3\gamma_\mu-\gamma_{\det Y_E}-\frac{4}{3}\gamma_{\det Y_U}-\frac{1}{3}\gamma_{\det Y_D}\Big]_{\overline{\text{DR}}}$$

$$=\frac{1}{2\pi^2}\Big(\frac{1243\alpha_1^2}{400}+\frac{17\alpha_2^2}{16}-5\alpha_3^2\Big)+\frac{1}{16\pi^3}\text{tr}(Y_U Y_U^+)\Big(\frac{143\alpha_1}{180}+\frac{11\alpha_2}{4}+\frac{44\alpha_3}{9}\Big)$$

$$+\frac{1}{16\pi^3}\text{tr}(Y_D Y_D^+)\Big(\frac{14\alpha_1}{45}+2\alpha_2+\frac{32\alpha_3}{9}\Big)+\frac{1}{16\pi^3}\text{tr}(Y_E Y_E^+)\Big(\frac{9\alpha_1}{10}+\frac{3\alpha_2}{2}\Big)$$

$$-\frac{1}{(16\pi^2)^2}\Big[11\,\text{tr}\Big((Y_U Y_U^+)^2\Big)+8\,\text{tr}\Big((Y_D Y_D^+)^2\Big)+6\,\text{tr}\Big((Y_E Y_E^+)^2\Big)+\frac{19}{3}\text{tr}\Big(Y_D Y_D^+ Y_U Y_U^+\Big)$$

$$+11\Big(\text{tr}(Y_U Y_U^+)\Big)^2+8\Big(\text{tr}(Y_D Y_D^+)\Big)^2+2\Big(\text{tr}(Y_E Y_E^+)\Big)^2+\frac{26}{3}\text{tr}\Big(Y_E Y_E^+\Big)\,\text{tr}\Big(Y_D Y_D^+\Big)\Big]$$

$$+O\Big(\alpha^3,\alpha^2Y^2,\alpha Y^4,Y^6\Big)\ne O\Big(\alpha^3,\alpha^2Y^2,\alpha Y^4,Y^6\Big);$$

$$\Big[\Big(\frac{3}{\alpha_3}-\frac{2\pi}{\alpha_3^2}\Big)\beta_3-3+3\gamma_\mu-\gamma_{\det Y_U}-\gamma_{\det Y_D}\Big]_{\overline{\text{DR}}}=\frac{1}{2\pi^2}\Big(\frac{363\alpha_1^2}{400}+\frac{9\alpha_2^2}{16}-\frac{21\alpha_3^2}{8}\Big)$$

$$+\frac{1}{16\pi^3}\text{tr}(Y_U Y_U^+)\Big(\frac{13\alpha_1}{30}+\frac{3\alpha_2}{2}+\frac{8\alpha_3}{3}\Big)+\frac{1}{16\pi^3}\text{tr}(Y_D Y_D^+)\Big(\frac{7\alpha_1}{30}+\frac{3\alpha_2}{2}+\frac{8\alpha_3}{3}\Big)$$

$$-\frac{1}{(16\pi^2)^2}\Big[6\,\text{tr}\Big((Y_U Y_U^+)^2\Big)+6\,\text{tr}\Big((Y_D Y_D^+)^2\Big)+6\Big(\text{tr}(Y_U Y_U^+)\Big)^2+6\Big(\text{tr}(Y_D Y_D^+)\Big)^2$$

$$+2\,\text{tr}\Big(Y_E Y_E^+\Big)\,\text{tr}\Big(Y_D Y_D^+\Big)+4\,\text{tr}\Big(Y_D Y_D^+ Y_U Y_U^+\Big)\Big]+O\Big(\alpha^3,\alpha^2Y^2,\alpha Y^4,Y^6\Big)$$

$$\ne O\Big(\alpha^3,\alpha^2Y^2,\alpha Y^4,Y^6\Big).$$

The parameter μ in the MSSM superpotential should be of the order of the electroweak scale, which is impossible to explain in MSSM. The μ problem can be solved in the Next-to-Minimal Supersymmetric Standard Model (NMSSM)

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which contains an additional chiral matter superfield S. This superfield is a singlet with respect to $SU(3)\times SU(2)\times U(1)$. Then it is possible to replace the μ term

$$\Delta W_{\mbox{MSSM}} = \mu\,(H_{u1}\ \ H_{u2}) \begin{pmatrix} 0 & 1\\ -1 & 0 \end{pmatrix} \begin{pmatrix} H_{d1}\\ H_{d2} \end{pmatrix}$$

by the gauge invariant expression

$$\Delta W_{\mbox{NMSSM}} = \lambda S\,(H_{u1}\ \ H_{u2}) \begin{pmatrix} 0 & 1\\ -1 & 0 \end{pmatrix} \begin{pmatrix} H_{d1}\\ H_{d2} \end{pmatrix} + \frac{\kappa}{3} S^3,$$

in which λ and κ are new dimensionless couplings. In this case the effective value of μ is equal to the vacuum expectation value of (the lowest component of) S multiplied by λ and can have an order of the electroweak scale.

RGIs for the NMSSM

Due to the nonrenormalization of the superpotential, the anomalous dimensions of λ and κ satisfy the all-loop equations

$$\gamma_\kappa \equiv \frac{d\ln\kappa}{d\ln\mu} = \frac{3}{2}\gamma_S; \qquad \gamma_\lambda \equiv \frac{d\ln\lambda}{d\ln\mu} = \frac{1}{2}\Big(\gamma_S + \gamma_{H_u} + \gamma_{H_d}\Big).$$

Therefore, the sum which for the MSSM gives $\gamma_{\boldsymbol{\mu}}$ can be written as

$$\frac{1}{2}\Big(\gamma_{H_u} + \gamma_{H_d}\Big) = \gamma_\lambda - \frac{1}{3}\gamma_\kappa.$$

The NSVZ relations for NMSSM are the same as for MSSM (although the anomalous dimensions are different). The equations describing the running of the Yukawa couplings Y_E, Y_U, and Y_D also remain unchanged. That is why RGIs for NMSSM can be obtained from the ones for MSSM after the replacement

$$\boldsymbol{\mu} \ \to\ \lambda\kappa^{-1/3}.$$

The expression $\mbox{RGI}_2$ does not depend on $\boldsymbol{\mu}$ and, therefore, is also RGI for NMSSM. The $\mbox{RGI}_1$ after this replacement takes the form

$$\widetilde{\mbox{RGI}}_1 = \frac{\lambda^{9/2}\,(\alpha_3)^3\,(\alpha_2)^{1/2}}{\kappa^{3/2}\,\Big(\mbox{det}\,Y_E\Big)^{1/2}\Big(\mbox{det}\,Y_U\Big)^{5/3}\Big(\mbox{det}\,Y_D\Big)^{7/6}}\,\exp\Big(\frac{2\pi}{\alpha_3} + \frac{\pi}{2\alpha_2} + \frac{5\pi}{6\alpha_1}\Big).$$

Conclusion

- In certain $\mathcal{N}=1$ supersymmetric theories with multiple gauge couplings one can construct such combinations of various couplings that do not depend on scale in all orders or, in other words, RGIs.
- In particular, in $\mathcal{N}=1$ SQCD interacting with $\mathcal{N}=1$ SQED RGI can be constructed from the strong and electromagnetic coupling constants (if the matter superfields have the same absolute values of the electromagnetic charges). Therefore, in this theory two gauge couplings do not run independently.
- If the (absolute values of the) electromagnetic charges of matter superfields are different, then it is possible to construct an equation relating the β-function of $\mathcal{N}=1$ SQCD to the Adler D-function.
- For the MSSM and NMMSM one can construct two independent RGIs from the gauge couplings, Yukawa couplings and the μ parameter. They are scale independent in all orders in the HD+MSL scheme, when a theory is regularized by higher covariant derivatives, and divergences are removed by minimal subtractions of logarithms.
- The explicit three-loop calculations for the MSSM confirm the renormalization group invariance of the constructed expressions in the HD+MSL scheme. However, in the $\overline{\text{DR}}$ scheme they start to depend on scale in such an approximation where the scheme dependence becomes essential.

Thank you for the attention!