Towards AdS harmonic superspace

Nikita Zaigraev

Bogoliubov Laboratory of Theoretical Physics, JINR
Moscow Institute of Physics and Technology

February 10, 2025

Problems of the Modern Mathematical Physics 2025, Dubna

Work in progress with E.A. Ivanov

$\mathcal{N}=2$ supersymmetry, harmonic superspace and $\mathcal{N}=2$ higher spins

- Harmonic superspace [Galperin, Ivanov, Kalitsyn, Ogievetsky, Sokatchev 1984] is the universal method to deal with off-shell $\mathcal{N}=2$ supersymmetry theories:

$$\mathbb{HR}^{4+2|8} = \mathbb{R}^{4|8} \times S^2 = \{x^m, \theta^{\alpha i}, \bar{\theta}^{\dot{\alpha} i}, u_i^{\pm}\}.$$

- $\mathcal{N}=2$ higher-spin theories (see talk by E.Ivanov) have natural geometric formulation in terms of unconstrained analytical prepotentials [Buchbinder, Ivanov, N.Z. 2021]:

$$\left\{ h^{++\alpha(s-1)\dot{\alpha}(s-1)}, h^{++\alpha(s-1)\dot{\alpha}(s-2)+}, h^{++\alpha(s-2)\dot{\alpha}(s-1)+}, h^{++\alpha(s-2)\dot{\alpha}(s-2)} \right\}.$$

- These prepotentials are the higher-spin generalization of linearized (over flat $\mathcal{N}=2$ harmonic superspace) $\mathcal{N}=2$ "minimal" Einstein supergravity (obtained using $\mathcal{N}=2$ vector and non-linear $\mathcal{N}=2$ tensor compensators).
- These higher-spin prepotentials naturally couple to the hypermultiplet:

$$\mathcal{D}^{++} \quad \to \quad \mathcal{D}^{++} + \kappa_s h^{++\alpha(s-2)\dot{\alpha}(s-2)M} \partial_M \partial^{s-2}_{\alpha(s-2)\dot{\alpha}(s-2)} (J)^{P(s)}.$$

! Gravitational interaction of massless higher-spin fields exists in the first nontrivial order on AdS background [Fradkin, Vasiliev 1987].

? How to construct $\mathcal{N}=2$ AdS harmonic superspace, $\mathcal{N}=2$ AdS supergravity and $\mathcal{N}=2$ *AdS* higher-spins?

- AdS (super)space is a special solution of the equation of motion of (super)gravity with a cosmological term:

$$\mathcal{N}=0: \qquad e_a{}^m(x) = \frac{1}{1+\mu\bar{\mu}x^2}\delta_a^m,$$

$$\mathcal{N}=1: \qquad \begin{cases} \mathcal{H}^m(x,\theta,\bar{\theta}) = e_a{}^m\left[\theta\sigma^a\bar{\theta} + \frac{1}{4}\theta^2\bar{\theta}^2\epsilon^{abcd}\omega_{bcd}\right], \\ \varphi(x,\theta) = e^{-1/3}(x)[1+\theta^2\bar{\mu}]. \end{cases}$$

$$\mathcal{N}=2: \qquad \textbf{???}$$

- Flat and AdS (super)space are only ones that posses maximall (super)symmetry:

$$\mathfrak{so}(3,2) \simeq \mathfrak{sp}(4), \qquad \mathfrak{osp}(\mathcal{N}|4).$$

- Superfield formulation of $\mathfrak{osp}(1|4)$ supersymmetry [Ivanov, Sorin 1979].
- $\mathcal{N}=1$ higher-superspin superfields on the AdS superspace [Kuzenko, Sibiryakov 1994].
- $AdS_4^{\mathcal{N}}$ superspace is superconformally flat [Bandos, Ivanov, Lukierski, Sorokin 2002].

- $\mathcal{N}=2$ AdS superalgebra $\mathfrak{osp}(2|4)$ can be considered as the subalgebra of $\mathcal{N}=2$ superconformal $\mathfrak{su}(2,2|2)$ superalgebra:

$$\{Q^i_\alpha, \bar{Q}_{\dot\alpha k}\} = 2\delta^i_k \sigma^m_{\alpha\dot\alpha} P_m \ , \quad \{S_{\alpha k}, \bar{S}^i_{\dot\alpha}\} = 2\delta^i_k \sigma^m_{\alpha\dot\alpha} K_m \ ,$$
$$\{Q^i_\alpha, S^{\beta k}\} = \epsilon^{ik}(\sigma^{mn})^\beta_\alpha M_{mn} + 2i\epsilon^{ik}\delta^\beta_\alpha (D + iR) - 4i\delta^\beta_\alpha T^{(ik)} \ ,$$
$$[K_m, Q^i_\alpha] = (\sigma_m)_{\alpha\dot\alpha}\bar{S}^{\dot\alpha i} \ , \quad [P_m, S_{\alpha i}] = (\sigma_m)_{\alpha\dot\alpha}\bar{Q}^{\dot\alpha}_i \ ,$$
$$[M_{mn}, P_s] = i(\eta_{ns}P_m - \eta_{ms}P_n) \ , \ [M_{mn}, Q^i_\alpha] = -\frac{1}{2}(\sigma_{mn})^\beta_\alpha Q^i_\beta \ ,$$
$$[T^{ij}, T^{lk}] = -i(\epsilon^{il}T^{jk} + \epsilon^{jk}T^{il}) \ .$$

- $\mathcal{N}=2$ AdS supersymmetry generators

$$\Psi^i_\alpha := Q^i_\alpha + c^{ik}S_{k\alpha}, \qquad \bar\Psi_{\dot\alpha i} := \overline{\Psi^i_\alpha} = \bar{Q}_{\dot\alpha i} + c_{ik}\bar{S}^k_{\dot\alpha},$$
$$c^{ik} = c^{ki}, \qquad \overline{c^{ik}} = c_{ik} = \epsilon_{il}\epsilon_{kj}c^{lj}$$

satisfy

$$\{\Psi^i_\alpha, \Psi^j_\beta\} = -4c^{(ij)}M_{(\alpha\beta)} + 4i\epsilon_{\alpha\beta}\epsilon^{ij}\underbrace{c^{kl}T_{kl}}_{U(1)\in SU(2)_{conf}} \ .$$

! So, $c^{(ij)}$ break $SU(2)_{conf}$ to $U(1)$. $c^{kl}T_{kl}$ is an analogue of the central charge $Z = \partial_5$ in the flat $\mathcal{N}=2$ supersymmetry.

- *AdS* supersymmetry generators give *AdS* translations:

$$\{\Psi^i_\alpha, \bar{\Psi}_{\dot{\alpha}j}\} = 4\delta^i_j P_{\alpha\dot{\alpha}} + 4\underbrace{c^{ik}c_{kj}}_{\delta^i_j|c|^2} K_{\alpha\dot{\alpha}} = 4\delta^i_j \underbrace{\left(P_{\alpha\dot{\alpha}} + |c|^2 K_{\alpha\dot{\alpha}}\right)}_{R_{\alpha\dot{\alpha}}}.$$

- The commutator of *AdS* translations gives:

$$[R_m, R_n] = -4i|c|^2 L_{[mn]}.$$

- AdS translations have the non-trivial commutator with AdS supersymmetry:

$$[R_m, \Psi^i_\alpha] = (\sigma_m)_{\alpha\dot{\alpha}}\, c^{ik}\bar{\Psi}^{\dot{\alpha}}_k.$$

! This relation implies, that field masses in AdS supermultiplets are different in contrast to Poincaré supersymmetry.

- If fix $c^{(ij)} = 0$, we restore $\mathcal{N} = 2$ Poincaré superalgebra (without central charge).

? How to construct superfield background with $\mathcal{N} = 2$ AdS supersymmetry in $\mathcal{N} = 2$ harmonic superspace?

Toy example: Weyl-invariant gravity formulation

- In the conformal formulation of Einstein gravity [Deser 1970], in addition to the metric g_{mn} a scalar field ϕ is added. We consider the action for a scalar field conformally interacting with gravity

$$S[\phi, g_{mn}] = -3 \int d^4x\sqrt{-g}\,\left(g^{mn}\partial_m\phi\partial_n\phi + \frac{1}{6}R\phi^2 - \frac{\xi^2}{3}\phi^4\right).$$

- The action is invariant under diffeomorphisms

$$x^m \to x'^m = x'^m(x),$$
$$g_{mn}(x) \to g'_{mn}(x') = g_{kl}(x)\frac{\partial x^k}{\partial x'^m}\frac{\partial x^l}{\partial x'^n},$$
$$\phi(x) \to \phi'(x') = \phi(x)$$

and Weyl transformations:

$$g_{mn}(x) \to g'_{mn}(x) = e^{2\omega(x)}g_{mn}(x),$$
$$\phi(x) \to \phi'(x) = e^{-\omega(x)}\phi(x).$$

- Using Weyl transformations, one can fix gauge $\phi = \frac{1}{\kappa}$. After that we restore ordinary Einstein-Hilbert action with cosmological term:

$$S_{EH} = \int d^4x\sqrt{-g}\left(-\frac{1}{2\kappa^2}R + \frac{\xi^2}{\kappa^4}\right).$$

Toy example: Weyl-invariant gravity formulation

- Thus, the gravitational background in the conformal formalism is fixed by specifying a metric and conformal compensator:
$$g_{mn}(x), \qquad \phi(x) = \kappa^{-1}.$$
- However, using the Weyl transformations the gravitational background can be equivalently described by the configuration
$$e^{2\omega(x)} g_{mn}(x), \qquad \phi(x) = \kappa^{-1} e^{-\omega(x)}.$$
This method is well-suited for conformally flat spaces.
- In this approach, the $(A)dS$ space is described by
$$g_{mn}(x) = \eta_{mn}, \qquad \phi(x) = \kappa^{-1} a(x), \qquad \text{where} \qquad a(x) := \frac{1}{1 + m^2 x^2}.$$
- $\phi(x)$ is the solution of e.o.m. of $\lambda\phi^4$ theory.
- This background is invariant with respect to coordinate transformations:
$$x^m \to x'^m \simeq x^m + a^m + m^2 \left(2(ax)x^m - a^m x^2\right) + \omega^{[m}{}_{n]} x^n$$
accompanied by Weyl transformations with parameter
$$\omega(x) \simeq 2(kx).$$
- These transformations form $\mathfrak{sp}(4) \simeq \mathfrak{so}(3,2)$ algebra.

$\mathcal{N}=2$ Einstein supergravity: principal version

- $\mathcal{N}=2$ Einstein supergravity in harmonic superspace [Galperin, Ivanov, Ogievetsky, Sokatchev 1987]:

$$S_{\mathcal{N}=2} = \frac{1}{2}\int d\zeta^{(-4)}\, q^{+a}\mathfrak{D}^{++}q^{+}_{a} - \frac{1}{4\kappa^2}\int d^4x d^8\theta du\, E\,\mathcal{H}^{+5}\mathcal{H}^{--5}.$$

- Harmonic derivative:

$$\mathfrak{D}^{++} = \partial^{++} + \theta^{+\hat{\alpha}}\partial^{+}_{\hat{\alpha}} + \underbrace{\mathcal{H}^{++\alpha\dot{\alpha}}\partial_{\alpha\dot{\alpha}} + \mathcal{H}^{++\hat{\alpha}+}\partial^{-}_{\hat{\alpha}} + \mathcal{H}^{(+4)}\partial^{--}}_{\mathcal{N}=2\text{ Weyl multiplet }(24_B+24_F\ d.o.f.)} + \mathcal{H}^{++5}\partial_5,$$

- $\mathcal{N}=2$ Maxwell compensator $(8_B+8_F$ d.o.f.) is described by the analytic prepotential $\mathcal{H}^{++5}$.
- Hypermultiplet compensator $(\infty_{\mathbf{B}}+\infty_{\mathbf{F}}$ d.o.f.) is described by the doublet

$$q^{+}_{a} = (q^{+}, -\tilde{q}^{+}), \qquad q^{+a} := \widetilde{q^{+}_{a}} = \epsilon^{ab}q^{+}_{b} = (\tilde{q}^{+}, q^{+}),$$

which depends on x^5 coordinate as:

$$q^{+}(\zeta, x^5) = \exp\left(-\frac{i\xi}{2\kappa}x^5\right) q^{+}(\zeta).$$

- ! Nontrivial $\xi \neq 0$ case corresponds to the cosmological constant.
- Harmonic zero curvature equation:

$$[\mathfrak{D}^{++} - \mathcal{H}^{(+4)}\mathfrak{D}^{--}, \mathfrak{D}^{--}] = \mathcal{D}^0 \qquad \Rightarrow \qquad \mathcal{H}^{--\alpha\dot{\alpha}}, \mathcal{H}^{--\hat{\alpha}\pm}, \mathcal{H}^{--5}.$$

$\mathcal{N}=2$ Einstein supergravity: gauge transformations

- $\mathcal{N}=2$ Einstein supergravity action

$$S_{\mathcal{N}=2} = \frac{1}{2}\int d\zeta^{(-4)}\, q^{+a}\mathfrak{D}^{++}q^{+}_{a} - \frac{1}{4\kappa^2}\int d^4x d^8\theta du\, E\, \mathcal{H}^{+5}\mathcal{H}^{--5}.$$

is invariant under the analyticity-preserved superdiffeomorphisms:

$$\begin{aligned}
x^{\alpha\dot\alpha} &\to x'^{\alpha\dot\alpha} = x^{\alpha\dot\alpha} + \lambda^{\alpha\dot\alpha}(\zeta),\\
\theta^{+\hat\alpha} &\to \theta'^{+\hat\alpha} = \theta^{+\hat\alpha} + \lambda^{+\hat\alpha}(\zeta),\\
\theta^{-\hat\alpha} &\to \theta'^{-\hat\alpha} = \theta^{-\hat\alpha} + \lambda^{-\hat\alpha}(\zeta,\theta^-),\\
u^{+i} &\to u'^{+i} = u^{+i} + \lambda^{++}(\zeta)u^{-}_{i},\\
u^{-i} &\to u'^{-i} = u^{-i},\\
x^5 &\to x'^5 = x^5 + \lambda^5(\zeta)
\end{aligned}$$

and prepotential transformations:

$$\mathfrak{D}^{++}(\zeta) \;\to\; \mathfrak{D}'^{++}(\zeta') = \mathfrak{D}^{++}(\zeta) - \lambda^{++}\mathcal{D}^0,$$

$$q^+_a(\zeta,x^5) \to q'^+_a(\zeta',x'^5) = \left(\text{Ber}\left|\frac{\partial\zeta}{\partial\zeta'}\right|\right)^{\frac{1}{2}} q^+_a(\zeta,x^5).$$

- Full $\mathcal{N}=2$ superspace measure E transforms as

$$E(\zeta,\theta^-) \to E'(\zeta',\theta'^-) = \text{Ber}\left|\frac{\partial(\zeta,\theta^-)}{\partial(\zeta',\theta'^-)}\right| E(\zeta,\theta^-),$$

and can be constructed only in terms of $\mathcal{N}=2$ Weyl supermultiplet prepotentials.

- Flat $\mathcal{N} = 2, D = 4$ harmonic superspace corresponds to background values of Weyl prepotentials:

$$\mathcal{H}^{++\alpha\dot{\alpha}}_{flat} = -4i\theta^{+\alpha}\bar{\theta}^{+\dot{\alpha}}, \qquad \mathcal{H}^{++\hat{\alpha}+}_{flat} = 0, \qquad \mathcal{H}^{(+4)}_{flat} = 0$$

and covariant harmonic derivative is reduced to

$$\mathfrak{D}^{++} \to \mathcal{D}^{++}_{flat} = \partial^{++} - 4i\theta^{+\alpha}\bar{\theta}^{+\dot{\alpha}}\partial_{\alpha\dot{\alpha}} + \theta^{+\hat{\alpha}}\partial^{+}_{\hat{\alpha}}.$$

- Flat background of Weyl supergravity is invariant under the rigid $\mathcal{N} = 2$ superconformal symmetry $\mathfrak{su}(2,2|2)$, which is given by:

$$\begin{cases} \lambda^{\alpha\dot{\alpha}}_{sc} = & a^{\alpha\dot{\alpha}} + l^{(\alpha}_{\ \beta)}x^{\beta\dot{\alpha}} + \bar{l}^{(\dot{\alpha}}_{\ \dot{\beta})}x^{\alpha\dot{\beta}} - 4i\left(\epsilon^{\alpha i}\bar{\theta}^{+\dot{\alpha}} + \theta^{+\alpha}\bar{\epsilon}^{\dot{\alpha}i}\right)u^-_i + x^{\dot{\alpha}\rho}k_{\rho\dot{\rho}}x^{\dot{\rho}\alpha} \\ & + a x^{\alpha\dot{\alpha}} - 4i\theta^{+\alpha}\bar{\theta}^{+\dot{\alpha}}\lambda^{(ij)}u^-_i u^-_j - 4i\left(x^{\alpha\dot{\rho}}\bar{\eta}^i_{\dot{\rho}}\bar{\theta}^{+\dot{\alpha}} + \theta^{+\alpha}\eta^i_{\rho}x^{\rho\dot{\alpha}}\right)u^-_i, \\ \lambda^{+\alpha}_{sc} = & \epsilon^{\alpha i}u^+_i + l^{(\alpha}_{\ \beta)}\theta^{+\beta} + \frac{1}{2}\theta^{+\alpha}(a+ib) + x^{\alpha\dot{\beta}}k_{\beta\dot{\beta}}\theta^{+\beta} + x^{\alpha\dot{\alpha}}\bar{\eta}^i_{\dot{\alpha}}u^+_i \\ & + \theta^{+\alpha}\left(\lambda^{(ij)}u^+_i u^-_j + 4i\theta^{+\rho}\eta^i_{\rho}u^-_i\right), \\ \bar{\lambda}^{+\dot{\alpha}}_{sc} = & \bar{\epsilon}^{\dot{\alpha}i}u^+_i + \bar{l}^{(\dot{\alpha}}_{\ \dot{\beta})}\bar{\theta}^{+\dot{\beta}} + \frac{1}{2}\bar{\theta}^{+\dot{\alpha}}(a-ib) + x^{\dot{\alpha}\beta}k_{\beta\dot{\beta}}\bar{\theta}^{+\dot{\beta}} + x^{\alpha\dot{\alpha}}\eta^i_{\alpha}u^+_i \\ & + \bar{\theta}^{+\dot{\alpha}}\left(\lambda^{(ij)}u^+_i u^-_j - 4i\bar{\theta}^{+\dot{\rho}}\bar{\eta}^i_{\dot{\rho}}u^-_i\right), \\ \lambda^{++}_{sc} = & \lambda^{ij}u^+_i u^+_j + 4i\theta^{+\alpha}\bar{\theta}^{+\dot{\alpha}}k_{\alpha\dot{\alpha}} + 4i\left(\theta^{+\alpha}\eta^i_{\alpha} + \bar{\eta}^i_{\dot{\alpha}}\bar{\theta}^{+\dot{\alpha}}\right)u^+_i. \end{cases}$$

- The analytical prepotential of the $\mathcal{N}=2$ vector multiplet is transformed as

$$\delta_\lambda \mathcal{H}^{++5} := \mathcal{H}'^{++5}(\zeta) - \mathcal{H}^{++5}(\zeta) = \mathfrak{D}^{++}\lambda^5 - \hat{\Lambda}\mathcal{H}^{++5}.$$

- Using Abelian gauge freedom (parameter λ^5), one can impose the Wess-Zumino-type gauge:

$$\mathcal{H}^{++5}_{WZ} = i(\theta^+)^2\phi - i(\bar\theta^+)^2\bar\phi - 4i\theta^{+\alpha}\bar\theta^{+\dot\alpha}A_{\alpha\dot\alpha} + (\bar\theta^+)^2\theta^{+\alpha}\kappa^i_\alpha u^-_i + (\theta^+)^2\bar\theta^{+\dot\alpha}\bar\kappa^i_{\dot\alpha}u^-_i + (\theta^+)^4 D^{(ij)}u^-_i u^-_j .$$

- Using Weyl supergravity gauge freedom,

$$\delta\phi(x) = [a(x) + ib(x)]\,\phi(x)$$

  one can fix value of the scalar $\phi = 1$ and break the dilatation and $U(1)_R$ gauge freedom.
- So, we fix the non-trivial background value for superfield H^{++5}:

$$\boxed{\mathcal{H}^{++5}_{flat} = i\left[(\theta^+)^2 - (\bar\theta^+)^2\right]}.$$

  This leads to the additional constraints on the rigid symmetries of the background.

$\mathcal{N}=2$ flat harmonic superspace: Vector compensator

- The condition of background invariance is

$$\delta_\lambda \mathcal{H}^{++5}_{flat} = \mathcal{D}^{++}\lambda^5 - 2i\lambda^{+\hat{\alpha}}\theta^+_{\hat{\alpha}} = 0,$$

- We substitute $\lambda^{+\hat{\alpha}}_{sc}$ and obtain the harmonic equation:

$$\mathcal{D}^{++}\lambda^5 = 2i\theta^{+\alpha}\epsilon^i_\alpha u^+_i + i(\theta^+)^2(a+ib) + i(\theta^+)^2 x^{\alpha\dot{\alpha}}k_{\alpha\dot{\alpha}} \\ - 2ix^{\alpha\dot{\alpha}}\theta^+_\alpha\bar{\eta}^i_{\dot{\alpha}}u^+_i + (\theta^+)^2\lambda^{(ij)}u^+_i u^-_j + \text{tilde conj.}$$

- This equation has the analytic solution $\lambda^5(\zeta)$ only if $\boxed{a = b = k_{\alpha\dot{\alpha}} = \bar{\eta}^i_{\dot{\alpha}} = 0}$:

$$\lambda^5 = 2i\epsilon^{\hat{\alpha}i}\theta^+_{\hat{\alpha}}u^-_i + \frac{1}{2}\left[(\theta^+)^2 + (\bar{\theta}^+)^2\right]\lambda^{(ij)}u^-_i u^-_j .$$

- So, the background value of $\mathcal{H}^{++5}_{flat}$ breaks rigid background $\mathcal{N}=2$ superconformal symmetry to $\mathcal{N}=2$ supersymmetry and $SU(2)_{conf}$:

$$\mathfrak{su}(2,2|2) \quad \to \quad \mathfrak{siso}(1,3|2) \oplus \mathfrak{su}(2).$$

$\mathcal{N}=2$ flat harmonic superspace: hypermultiplet compensator

- Weyl transformation of hypermultiplet in infinitesimal form is

$$\delta_\lambda q^+_a = -\frac{1}{2}\Omega q^+_a - \hat{\Lambda} q^+_a - \lambda^5 \partial_5 q^+_a$$

- If we fix background value

$$\left(q^+_a\right)_{flat} = f^b_a u^+_b \kappa^{-1},$$

invariance condition gives:

$$\lambda^{ij} u^+_i u^+_j f^b_a u^-_b = \lambda^{ij} u^+_i u^-_j f^b_a u^+_b \quad \Rightarrow \quad \frac{2}{3} f^b_a \lambda^i_b u^+_i = \frac{1}{3} f^b_a \lambda^i_b u^+_i \quad \Rightarrow \quad \boxed{\lambda^{(ij)} = 0}.$$

- In such a way we break $SU(2)$ background invariance.

We conclude that the flat harmonic superspace is parametrized by

$$\boxed{\begin{cases} \mathcal{H}^{++\alpha\dot{\alpha}}_{flat} = -4i\theta^{+\alpha}\bar{\theta}^{+\dot{\alpha}}, \qquad \mathcal{H}^{++\hat{\alpha}+}_{flat} = 0, \qquad \mathcal{H}^{(+4)}_{flat} = 0, \\ \mathcal{H}^{++5}_{flat} = i\left[(\theta^+)^2 - (\bar{\theta}^+)^2\right], \\ \left(q^+_a\right)_{flat} = f^b_a u^+_b \kappa^{-1}. \end{cases}}$$

- How to generalize these results to $\mathcal{N}=2$ AdS superspace?

Linearized version of $\mathcal{N}=2$ principal supergravity

- We consider the linearization on the flat background (for simplicity $f_a^b = \delta_a^b$):

$$Q_a^+ = u_a^+ + q_a^+, \qquad \mathcal{H}^{++M} = \mathcal{H}^{++M}_{flat} + h^{++M}.$$

$$\delta_\lambda Q_a^+ = -\frac{1}{2}\Omega Q_a^+ - \hat{\Lambda} Q_a^+, \qquad \Rightarrow \qquad \delta_\lambda q_a^+ = \frac{1}{2}\Omega u_a^+ - \lambda^{++} u_a^-$$

- Covariant superfields:

$$\begin{aligned}
G^{++\alpha\dot\alpha} &= h^{++\alpha\dot\alpha} + 4ih^{++\alpha+}\bar\theta^{-\dot\alpha} + 4i\theta^{-\alpha}h^{++\dot\alpha+} - 4ih^{(+4)}\theta^{-\alpha}\bar\theta^{-\dot\alpha}, \\
G^{++5} &= h^{++5} - 2ih^{++\hat\alpha+}\bar\theta^{-}_{\hat\alpha} + ih^{(+4)}\theta^{-\hat\alpha}\theta^{-}_{\hat\alpha}, \\
G^{++\hat\alpha+} &= -h^{++\hat\alpha+} + h^{(+4)}\theta^{-\hat\alpha}, \\
G^{(+4)} &= h^{(+4)}.
\end{aligned}$$

- Linearized action:

$$\begin{aligned}
S_{flat} = &\int d^4x d^8\theta du \left\{ G^{++\alpha\dot\alpha} G^{++}_{\alpha\dot\alpha} + 4G^{++5}G^{--5} \right\} \\
&+ \int d\zeta^{(-4)} \left[q^{+a}\mathcal{D}^{++}q_a^+ + q^{+a}h^{(+4)}u_a^- + u^{+a}\hat{\mathcal{H}}^{++}q_a^+ + \frac{1}{2}(q^{+a}u_a^+)^2 \right].
\end{aligned}$$

$\mathcal{N}=2$ superalgebra of flat covariant derivatives

- Harmonic derivatives:

$$\mathcal{D}^{++} = \partial^{++} - 4i\theta^{+\alpha}\bar{\theta}^{+\dot{\alpha}}\partial_{\alpha\dot{\alpha}} + \theta^{+\hat{\alpha}}\partial^{+}_{\hat{\alpha}} + i\left[(\theta^{+})^2 - (\bar{\theta}^{+})^2\right]\partial_5,$$
$$\mathcal{D}^{--} = \partial^{--} - 4i\theta^{-\alpha}\bar{\theta}^{-\dot{\alpha}}\partial_{\alpha\dot{\alpha}} + \theta^{-\hat{\alpha}}\partial^{-}_{\hat{\alpha}} + i\left[(\theta^{-})^2 - (\bar{\theta}^{-})^2\right]\partial_5,$$
$$\mathcal{D}^{0} = \partial^{0} + \theta^{+\hat{\alpha}}\partial^{-}_{\hat{\alpha}} - \theta^{-\hat{\alpha}}\partial^{+}_{\hat{\alpha}}.$$

- Spinor derivatives:

$$\mathcal{D}^{+}_{\hat{\alpha}} = \partial^{+}_{\hat{\alpha}},$$
$$\mathcal{D}^{-}_{\alpha} = -\partial^{-}_{\alpha} + 4i\bar{\theta}^{-\dot{\alpha}}\partial_{\alpha\dot{\alpha}} - 2i\theta^{-}_{\alpha}\partial_5,$$
$$\bar{\mathcal{D}}^{-}_{\dot{\alpha}} = -\partial^{-}_{\dot{\alpha}} - 4i\theta^{-\alpha}\partial_{\alpha\dot{\alpha}} - 2i\bar{\theta}^{-}_{\dot{\alpha}}\partial_5.$$

- Vector derivative:

$$\mathcal{D}_{\alpha\dot{\alpha}} = \partial_{\alpha\dot{\alpha}}$$

- Covariant derivative satisfy relations:

$$[\mathcal{D}^{++}, \mathcal{D}^{--}] = \mathcal{D}^{0}, \qquad [\mathcal{D}^{0}, \mathcal{D}^{\pm\pm}] = \pm 2\mathcal{D}^{\pm\pm},$$
$$\{\mathcal{D}^{+}_{\alpha}, \mathcal{D}^{-}_{\beta}\} = 2i\epsilon_{\alpha\beta}\partial_5, \qquad \{\bar{\mathcal{D}}^{+}_{\dot{\alpha}}, \bar{\mathcal{D}}^{-}_{\dot{\beta}}\} = 2i\epsilon_{\dot{\alpha}\dot{\beta}}\partial_5,$$
$$\{\mathcal{D}^{+}_{\alpha}, \bar{\mathcal{D}}^{-}_{\dot{\alpha}}\} = -4i\mathcal{D}_{\alpha\dot{\alpha}} = -\{\bar{\mathcal{D}}^{+}_{\dot{\alpha}}, \mathcal{D}^{-}_{\alpha}\}.$$

Superconformally flat harmonic superspace

- In Weyl-invariant gravity formulation:

$$g_{mn}(x) \to g'_{mn}(x) = e^{2\omega(x)} g_{mn}(x),$$
$$\phi(x) \to \phi'(x) = e^{-\omega(x)} \phi(x)$$

conformally-flat spaces are described by

$$g_{mn}(x), \qquad \phi(x),$$

which posses a gauge fixing:

$$\eta_{mn}, \qquad \phi(x) - \text{unspecified}.$$

- So we define superconformally flat harmonic superspace as a superspace, specified by values of Weyl prepotentials and compensators:

$$\left\{ \mathcal{H}^{++\alpha\dot{\alpha}}, \mathcal{H}^{++\hat{\alpha}+}, \mathcal{H}^{(+4)} \right\}, \qquad \mathcal{H}^{++5}, \quad q^+_a$$

which allow gauge with

$$\left\{ \mathcal{H}^{++\alpha\dot{\alpha}}, \mathcal{H}^{++\hat{\alpha}+}, \mathcal{H}^{(+4)} \right\} = \left\{ \mathcal{H}^{++\alpha\dot{\alpha}}_{flat}, \mathcal{H}^{++\hat{\alpha}+}_{flat}, \mathcal{H}^{(+4)}_{flat} \right\}.$$

How to specify AdS harmonic superspace?

1. We fix the flat gauge for Weyl supermultiplet prepotentials:

$$\left\{\mathcal{H}^{++\alpha\dot{\alpha}}, \mathcal{H}^{++\hat{\alpha}+}, \mathcal{H}^{(+4)}\right\} = \left\{\mathcal{H}^{++\alpha\dot{\alpha}}_{flat}, \mathcal{H}^{++\hat{\alpha}+}_{flat}, \mathcal{H}^{(+4)}_{flat}\right\}.$$

   This background is invariant under the superconformal $\mathfrak{su}(2,2|4)$ transformations.

2. We looking for the solutions of the superfield equation of motion for $\mathcal{N}=2$ vector and hypermultiplet compensators:

$$\begin{cases} \mathcal{D}^{++} q^{+}_{AdS} = \frac{i\xi}{2\kappa} \mathcal{H}^{++5}_{AdS} q^{+}_{AdS}, \\ \left(\mathcal{D}^{+}\right)^{4} \mathcal{H}^{--5}_{AdS} = -\frac{i\xi}{\kappa} \tilde{q}^{+}_{AdS} q^{+}_{AdS}. \end{cases}$$

   Generally, these solutions fully break $\mathfrak{su}(2,2|4)$.

3. We require that solutions q^{+}_{AdS} and $\mathcal{H}^{++5}_{AdS}$ are invariant under $\mathfrak{osp}(2|4)$ transformations.

After that, one can consider linearization

$$\mathcal{H}^{++M} = \mathcal{H}^{++M}_{AdS} + h^{++M}, \qquad Q^{+}_{a} = (q^{+}_{a})_{AdS} + q^{+}_{a},$$

define linearized gauge transformation laws, construct invariant action for linearized $\mathcal{N}=2$ AdS supergravity, generalize to higher spins, e.t.c.

❗ It is also possible to search for solutions with presented nontrivial Weyl multiplet superfields. These solutions are gauge equivalent to solutions presented here. ❗

Vector compensator

- Among the components of the vector multiplet, only scalar fields can take nontrivial vacuum values, otherwise the background Lorentz invariance will be violated.
- So, we obtain ansatz for the compensator superfield $\mathcal{H}^{++5}$:

$$\mathcal{H}^{++5}_{AdS} = i\left[(\theta^+)^2 - (\bar{\theta}^+)^2\right] a(x) + A(\theta^+)^4 c_{ij} u^-_i u^-_j a^n(x),$$

  where A and n are some unknown coefficients. This leads to the additional constraints on the rigid symmetries of the background.
- The condition of background invariance is

$$\delta_\lambda H^{++5}_{AdS} = \mathcal{D}^{++}\lambda^5 - \hat{\Lambda}_{AdS} H^{++5}_{AdS} = 0.$$

- We will denote:

$$\delta_{AdS} := \hat{\Lambda}_{AdS}.$$

- As in the purely bosonic case, we consider the combination of translations and special conformal transformations:

$$\begin{cases} \lambda^{\alpha\dot{\alpha}} = a^{\alpha\dot{\alpha}} + x^{\alpha\dot{\rho}} k_{\rho\dot{\rho}} x^{\rho\dot{\alpha}} = a^{\alpha\dot{\alpha}} + 2x^{\alpha\dot{\alpha}}(kx) - x^2 k^{\alpha\dot{\alpha}}, \\ \lambda^{+\alpha} = x^{\alpha\dot{\rho}} k_{\rho\dot{\rho}} \theta^{+\rho} = (kx)\theta^{+\alpha} + x^{(\alpha\dot{\rho}} k_{\rho)\dot{\rho}} \theta^{+\rho}, \\ \bar{\lambda}^{+\dot{\alpha}} = x^{\dot{\alpha}\rho} k_{\rho\dot{\rho}} \bar{\theta}^{+\dot{\rho}} = (kx)\bar{\theta}^{+\dot{\alpha}} + x^{(\dot{\alpha}\rho} k_{\rho\dot{\rho})} \bar{\theta}^{+\dot{\rho}}, \\ \lambda^{++} = 4i\theta^{+\rho}\bar{\theta}^{+\dot{\rho}} k_{\rho\dot{\rho}}, \end{cases}$$

  where we identify $k_{\alpha\dot{\alpha}} = m^2 a_{\alpha\dot{\alpha}}$.
- $a(x)$ has simple transformation law:

$$\delta_{AdS}\, a^n(x) = -2n(kx)a^n = -2m^2 n(ax)a^n(x).$$

- We also note that the transformation of Grassmanian coordinates consists of two contributions: weight and rotation leading to simple transformation laws:

$$\delta_{AdS}\left(\theta^+\right)^2 = 2(kx)\left(\theta^+\right)^2, \qquad \delta_{AdS}\left(\bar{\theta}^+\right)^2 = 2(kx)\left(\bar{\theta}^+\right)^2.$$

- Using these equation we obtain for AdS translation of $\mathcal{H}^{++5}_{AdS}$:

$$\delta_{AdS}\mathcal{H}^{++5}_{AdS} = A(\theta^+)^4 c^{ij} u^-_i u^-_j \, [4 - 2n]\, a^n.$$

  So we fix $n = 2$ and parameter $\lambda^5_{AdS} = 0$ in AdS translation sector.

AdS supersymmetry

Parameters in the supersymmetry sector are:

$$\begin{cases} \lambda^{\alpha\dot{\alpha}} = -4i\epsilon^{-\alpha}\bar{\theta}^{+\dot{\alpha}} - 4i\theta^{+\alpha}\eta^{-}_{\rho}x^{\rho\dot{\alpha}}, \\ \lambda^{+\alpha} = \epsilon^{+\alpha} - 2i(\theta^{+})^2\eta^{-\alpha} = \epsilon^{+\alpha} + 4i(\theta^{+}\eta^{-})\theta^{+\alpha}, \\ \bar{\lambda}^{+\dot{\alpha}} = x^{\alpha\dot{\alpha}}\eta^{+}_{\alpha}, \\ \lambda^{++} = 4i\theta^{+\alpha}\eta^{+}_{\alpha} = 4i(\theta^{+}\eta^{+}). \end{cases}$$

Using relations

$$\delta_{AdS}(\theta^{+})^2 = 2(\epsilon^{+}\theta^{+}), \qquad \delta_{AdS}(\bar{\theta}^{+})^2 = -2x^{\alpha\dot{\alpha}}\eta^{+}_{\alpha}\bar{\theta}^{+}_{\dot{\alpha}},$$

$$\delta_{AdS}(\theta^{+})^4 = 2(\bar{\theta}^{+})^2(\epsilon^{+}\theta^{+}) - 2(\theta^{+})^2x^{\alpha\dot{\alpha}}\eta^{+}_{\alpha}\bar{\theta}^{+}_{\dot{\alpha}},$$

$$\delta_{AdS}a(x) = \lambda^{\alpha\dot{\alpha}}\partial_{\alpha\dot{\alpha}}a(x) = 4im^2\left(\epsilon^{-\alpha}\bar{\theta}^{+\dot{\alpha}}x_{\alpha\dot{\alpha}} + (\theta^{+}\eta^{-})x^2\right)a^2(x),$$

we obtain for *AdS* supersymmetry variation:

$$\begin{aligned} \delta_{AdS}\mathcal{H}^{++5}_{AdS} = {} & 2i\left(\epsilon^{+\alpha}\theta^{+}_{\alpha} + x^{\alpha\dot{\alpha}}\eta^{+}_{\alpha}\bar{\theta}^{+}_{\dot{\alpha}}\right)a(x) \\ & - 4m^2(\theta^{+})^2\epsilon^{-\alpha}\bar{\theta}^{+\dot{\alpha}}x_{\alpha\dot{\alpha}}a^2(x) + 4m^2(\bar{\theta}^{+})^2(\theta^{+}\eta^{-})x^2a^2(x) \\ & - \frac{4}{3}A(\theta^{+})^2c^{i}_{k}\eta^{\alpha k}\bar{\theta}^{+\dot{\alpha}}x_{\alpha\dot{\alpha}}u^{-}_{i}\,a^2(x) + \frac{4}{3}A(\bar{\theta}^{+})^2(c^{i}_{k}\epsilon^{k}\theta^{+})u^{-}_{i}\,a^2(x) \\ & - 2A(\theta^{+})^2\eta^{\alpha k}\bar{\theta}^{+\dot{\alpha}}x_{\alpha\dot{\alpha}}c^{ij}u^{+}_{(k}u^{-}_{i}u^{-}_{j)}a^2(x) + 2A(\bar{\theta}^{+})^2(\epsilon^{k}\theta^{+})c^{ij}u^{+}_{(k}u^{-}_{i}u^{-}_{j)}a^2(x). \end{aligned}$$

AdS supersymmetry

- We need represent this variation as the harmonic derivative of analytic parameter λ^5_{AdS}.
- One can resolve this equation as

$$\lambda^5_{AdS} = \left(\epsilon^{-\alpha}\theta^+_\alpha + x^{\alpha\dot\alpha}\eta^-_\alpha\bar\theta^+_{\dot\alpha}\right) a(x) \\ - 4m^2\left[(\theta^+)^2 - (\bar\theta^+)^2\right]\left(\eta^{\alpha k}\bar\theta^{+\dot\alpha}x_{\alpha\dot\alpha} + \epsilon^{\alpha k}\theta^+_\alpha\right) c^{ij} u^-_{(k} u^-_i u^-_{j)} a^2(x)$$

if identify parameters:

$$A = 6m^2, \qquad \boxed{\eta^i_\alpha = m^2 c^i_k \epsilon^k_\alpha}, \qquad \boxed{\epsilon^i_\alpha = -c^i_k \eta^k_\alpha}, \qquad \boxed{c^{ik}c_{jk} = |c|^2\delta^i_j = \frac{1}{m^2}\delta^i_j}.$$

In such a way, we reproduce identification of supersymmetry and S-supersymmetry parameters in $\mathfrak{su}(2,2|2)$ superalgebra.

- Final result for $\mathcal{N} = 2$ vector compensator:

$$\boxed{\mathcal{H}^{++5}_{AdS} = i\left[(\theta^+)^2 - (\bar\theta^+)^2\right] a(x) + 6m^2(\theta^+)^4 a^2(x)\, c^{(ij)} u^-_i u^-_j\,.}$$

- Similarly, the AdS vacuum value of the hypermultiplet can also be determined.

Outlook

- *AdS covariant derivatives in harmonic superspace*
- *Dynamical actions for linearized $\mathcal{N} = 2$ AdS supergravity*
- $\mathcal{N} = 2$ *harmonic theories on supergravity background*
- *Dynamical actions for $\mathcal{N} = 2$ higher-spin AdS supermultiplets*
- *Structure of conserved $\mathcal{N} = 2$ AdS supercurrents*
- $\mathcal{N} = 2$ *Fradkin-Vasiliev mechanism*

Thank you for your attention!