# *On holographic correspondence: complexity, warped CFT and bulk paths*

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# Motivation

- Understanding quantum dynamics of strongly interacting objects

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- In Holography - dynamics on the bdy and corresponding bulk processes/reconstruction
  - Integrable or chaotic behavior?
    - in flat & curved spaces (highly nontrivial)
  - Minimal knowledge to (almost) completely describe a system
    $\rightarrow$ paterns and codes , optimization

- Complexity of (quantum) integrable systems & strongly interacting compact objects - an indicator for chaotic behavior
- example: for Krylov spaces

$$b_n \sim n^{\delta}, \quad \delta \geq 1 \text{ - chaotic}, \quad 0 < \delta < 1 \text{ - integrable}$$

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  - the naive notion of C(t): a correlator for operator $A(t)$ (autocorrelation function) $C(t) = \min_B C_B(t)$

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- Consider a basis $B = \{|B_i\rangle\, i = 0, 1, \dots\}$ and def *cost finction*

$$C_B(t) = \sum_n c_n |\langle \psi_n | B_n \rangle|^2, \quad c_n \textit{ positive increasing}, \ |B_0\rangle = |\psi(t_0)\rangle$$

$|\langle \psi_n | B_n \rangle|^2 \equiv p_B(n, t)$: probabilities of being in each basis vector.

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- The allowed transformations $U(\sigma)$ - as path ordered exponentials

$$V = i\frac{dU}{d\tau}U^\dagger = T_\alpha V^\alpha \quad \Longrightarrow \quad U(\sigma) = \mathcal{P}e^{-i\int_{s_i}^{\sigma} V(s)ds}$$

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- bi-invariant metric

$$ds^2_{bi-inv} = \mathrm{Tr}(V^\dagger V)d\tau^2 \tag{1}$$

- The length of a path from $s_i$ to $s_f$ going through $|\Psi(\sigma)\rangle$

$$\ell(|\Psi(\sigma)\rangle) = \int_{s_i}^{s_f} ds(\sigma).$$

- Def. complexity $\mathcal{C}$: the minimal length/geodesics between states driven by generators $G(s)$

$$\mathcal{C}(|\Psi(s_i)\rangle, |\Psi(s_f)\rangle) = \min_{V(s)} \ell(|\Psi(\sigma)\rangle).$$

# Geometric Complexity

• Nielsen's complexity: evolution operator $\Longleftrightarrow$ path' length w/ b.c. & minimized length + penalty factors $\mu_\alpha \quad \Longrightarrow$

$$C_N(t) = \min_V \int_0^t d\tau \left( \sum_\alpha \mathrm{Tr}(T_\alpha V)^2 + \mu_\alpha \, \mathrm{Tr}(T_\alpha V)^2 \right)^{1/2} ,$$

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• Objective: geodesics connecting the identity to a target unitary $U_{target} = \exp\{-i\mathcal{H}t\}$ at a chosen moment t, with $\mathcal{H}$ being the physical Hamiltonian.

• accounting for penalties in the metric

$$\sqrt{\sum_\alpha [\mathrm{Tr}(T_\alpha V)^2 + \mu_\alpha \, \mathrm{Tr}(T_\alpha V)^2]} = \sqrt{y_n Q_{nm} y_m}$$

$$\Longrightarrow \quad Q_{nm} = \sum_\alpha \mu_\alpha \langle n|T_\alpha|n\rangle \langle m|T_\alpha^\dagger|m\rangle, \quad (2)$$

# Volume - Momentum Complexity relations

- Momentum Constraint (MC) - defined for any Cauchy surface $\Sigma$, initial data: $h_{ab}$ and $K_{ab}$ $\implies$

$$\nabla^a K_{ab} - \nabla_b K = -8\pi G P_b, \qquad P_b = -N_\Sigma^\mu T_{\mu\nu} e_b^\nu, \tag{3}$$

$P_b$ - pulled-back momentum flux through $\Sigma$.

- If $\Sigma$ is extremal $\implies K = 0$
- Integrate the expression over $\Sigma$

$$\int_\Sigma P_C = -\frac{1}{8\pi G}\int_{\partial\Sigma} dS^a K_{ab} C^b + \frac{1}{8\pi G}\int_\Sigma K^{ab}\nabla_a C_b, \tag{4}$$

$P_C \equiv P_a C^a$ - the momentum along the C-field.

- The complexity current

$$J_a = -\frac{1}{d-1} K_{ab} C^b. \tag{5}$$

$\Longrightarrow$ complexity rate for VC: a bulk integral over extremal $\Sigma$:

$$\dot{C}[\Sigma] = \frac{d-1}{8\pi G} \int_{\Sigma} \nabla_a J^a. \tag{6}$$

• The idea: construct reps of the unitary acting on the states

$$U(\sigma) = \overleftarrow{\mathcal{P}} \exp\left[ -\int_0^{\sigma} ds \mathcal{H}(s) \right], \qquad \text{where} \quad \mathcal{H}(s) \equiv \sum_I Y^I \mathcal{O}_I, \tag{7}$$

• Example: construction of unitary with (circuit) parameter $\sigma$ and complex parameters $(\alpha_\mu, \beta_\mu, \lambda_{\mu\nu}, \gamma_D)$ depending on it (but with the requirement to be unitary)

$$\boxed{U(\sigma) \equiv e^{i\alpha(\sigma)\cdot P}\, e^{i\gamma_D(\sigma)D} \left( \prod_{\mu<\nu} e^{i\lambda\mu\nu L_{\mu\nu}} \right) e^{i\beta(\sigma)\cdot K}.} \tag{8}$$

Nielsen's approach: the optimal circuit with the minimized cost

$$\mathcal{D}(U(\sigma)) \equiv \int_0^1 ds\, F(U(s), Y^I(s)), \tag{9}$$

$F$ - cost function: 1) Smoothness, 2) Positivity, 3) Triangle inequality and 4) Positive homogeneity.

$$C(|\psi_T\rangle) \equiv \mathrm{Min}\, \mathcal{D}. \tag{10}$$

**The first law of volume complexity**

Combining the first- and second-order variations one finds *the first law of complexity* as

$$\delta\mathcal{C} = \underbrace{p_a \delta x^a}_{\delta C'} + \frac{1}{2}\underbrace{\delta p_a \delta x^a}_{\delta C''}, \qquad p_a = \frac{\partial F}{\partial \dot{x}^a}.$$

# Spread states and Operator growth

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• Unitary evolution mixes the initial state $|\psi\rangle$ with other quantum states as time evolves

$$|\psi(t)\rangle = e^{-i\mathcal{H}t}|\psi(0)\rangle = \sum_{n=0}^{\infty} \frac{(-i\mathcal{H}t)^n}{n!}|\psi\rangle = \sum_{n=0}^{\infty} \frac{(-it)^n}{n!}|\psi_n\rangle. \tag{11}$$

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• The Gram-Schmidt procedure applied to generate an ordered, orthonormal basis $K = \{|K_0\rangle, |K_1\rangle, \dots\}$.

- consider a basis $B = \{|B_i\rangle\, i = 0, 1, \dots\}$ and def *cost finction*

$$C_B(t) = \sum_n c_n |\langle\psi_n|B_n\rangle|^2, \qquad c_n \textit{ positive increasing}, \;\; |B_0\rangle = |\psi(t_0)\rangle$$

- def *Complexity*

$$\boxed{C(t) = \min_B C_B(t)}$$

# Spread states and Operator growth

- Operator growth

$$\mathcal{O}(t) = e^{i\mathcal{H}t}\,\mathcal{O}(0)\,e^{-i\mathcal{H}t} = \sum_{n=0}^{\infty} \frac{(it)^n}{n!}\tilde{\mathcal{O}}_n, \tag{12}$$

where

$$\tilde{\mathcal{O}}_0 = \mathcal{O}, \quad \tilde{\mathcal{O}}_1 = [\mathcal{H}, \mathcal{O}], \quad \tilde{\mathcal{O}}_2 = [\mathcal{H}, [\mathcal{H}, \mathcal{O}]] \ldots \tag{13}$$

As time progresses, a simple operator $\mathcal{O}(t)$ "grows" in the space of operators of the theory becoming more "complex".
- the idea: use $\tilde{\mathcal{O}}_n$ to construct states of the basis $\{|\mathcal{O}_n(0))\}$

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- Notion of Liouvillian (superoperator)

$$\mathcal{L} := [\mathcal{H}, *] \quad \Longrightarrow \quad \tilde{\mathcal{O}}_n = \mathcal{L}^n \mathcal{O}(0) \quad \Longrightarrow \quad \mathcal{O}(t) = e^{i\mathcal{L}t}\mathcal{O}(0). \tag{14}$$

- Subtlety: $|\mathcal{O}_n(0)) = \mathcal{O}_n|0\rangle$ may not be orthogonal (and $\{|\mathcal{O}_n(0))\}$ may not define a basis)

# Constructing Krylov spaces

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- The algorithm of orthogonalization (Arnoldi iteration)

1. set $b_0 \equiv 0$ and $|\mathcal{O}_{-1}) \equiv 0$
2. Define $|\mathcal{O})_0 = \frac{1}{\sqrt{(\mathcal{O}|\mathcal{O})}}\mathcal{O})$
3. For $n = 1$:
   - $|A_1) = \mathcal{L}|\mathcal{O}_0)$
   - $b_1 = ||A_1||$
   - If $b_1 \neq 0$ define$|\mathcal{O}_1) = \frac{1}{b_1}|A_1)$
4. For $n > 1$:
   - $|A_n) = \mathcal{L}|\mathcal{O}_{n-1}) - b_{n-1}|\mathcal{O}_{n-2})$
   - $b_n = ||A_n|| \equiv \sqrt{(A_n|A_n)}$
   - If $b_n = 0$ stop the procedure; if not, define $|\mathcal{O}_n) = \frac{1}{b_n}|A_n)$ and go to step 4.

# About Krylov basis

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- Construct

$$S_B = \{C_B^{(0)}, C_B^{(1)}, C_B^{(2)}, \dots\}, \qquad C_B^{(n)} = \frac{d^n}{dt^n} C_B^{(}t)|_{t=0}.$$

- If there is a k such that $C_{B_1}^{(m)} = C_{B_2}^{(m)}$ for $m < k$ and $C_{B_1}^{(k)} < C_{B_2}^{(k)}$ for $m = k$ then $C_{B_1}(t) < C_{B_2}(t)$ in a domain $0 \leq t \leq \tau$ for some $\tau < T$ .

Define $S_{B_1}/S_{B_2} = \{C_{B_1}^{(0)}/C_{B_2}^{(0)}, C_{B_1}^{(1)}/C_{B_2}^{(1)}, C_{B_1}^{(2)}/C_{B_2}^{(2)}, \dots\}$
If $S_{B_i} \in$ k-jet, $i = 1, 2 \quad \Longrightarrow \quad S_{B_1}/S_{B_2} = \{1, \dots\}$.
If for (k+1)- jet $S_{B_1}/S_{B_2} = \{1, \dots, \alpha < 1\}$ then $S_{B_1}(t) < S_{B_2}(t)$

$\Longrightarrow$ Krylov basis $K = \{|K_i\rangle\}$ minimizes the cost function $C_B$!

- The Krylov space: spanned by $\{P_n(\mathcal{L})|\hat{\mathcal{O}})\}$

$$|\hat{\mathcal{O}}_n) := |P_n(\mathcal{L})\hat{\mathcal{O}}), \qquad n = 0, 1, \ldots$$

- If $(\hat{\mathcal{O}}_m|\mathcal{L}|\hat{\mathcal{O}}_n)$ is a Hermitian matrix

$$\mathcal{L}_{nm} \equiv \begin{pmatrix} 0 & b_1 & 0 & 0 & \cdots \\ b_1 & 0 & b_2 & 0 & \cdots \\ 0 & b_2 & 0 & b_3 & \cdots \\ 0 & 0 & b_3 & 0 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}. \tag{15}$$

$\Longrightarrow$ a three-term recurrence relation

$$\mathcal{L}P_n(\mathcal{L}) = b_{n+1}P_{n+1}(\mathcal{L}) + b_n P_{n-1}(\mathcal{L}) \tag{16}$$

$\Longrightarrow$ by Favard's theorem $\exists$ measure wrw $P_n(\mathcal{L})$ are orthogonal.

# Moments and Hankel determinant

• A key quantity containing equivalent information is the moment matrix $\mathfrak{M}$ defined by

$$\mathfrak{M}_0 = \begin{pmatrix} \int x^0 d\omega & \int x d\omega & \cdots & \int x^n d\omega \\ \int x d\omega & \int x^2 d\omega & \cdots & \int x^{n+1} d\omega \\ . & . & \cdots & \\ \int x^n d\omega & \int x^{n+1} d\omega & \cdots & \int x^{2n} d\omega \end{pmatrix} = \begin{pmatrix} \mu_0 & \mu_1 & \cdots & \mu_n \\ \mu_1 & \mu_2 & \cdots & \mu_{n+1} \\ . & . & \cdots & \\ \mu_n & \mu_{n+1} & \cdots & \mu_{2n} \end{pmatrix}$$

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- Hankel determinant $D_n$

$$D_n = \det_{1 \le i,j \le N} (\mu_{i+j}) = \begin{vmatrix} \mu_0 & \mu_1 & \cdots & \mu_n \\ \mu_1 & \mu_2 & \cdots & \mu_{n+1} \\ \cdot & \cdot & \cdots & \\ \mu_n & \mu_{n+1} & \cdots & \mu_{2n} \end{vmatrix} \tag{17}$$

# Orthogonal polynomials

- Moments, Hankel and orthogonal polynomial $D_n(x)$

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- Using $D_n$ and $D_(x) \Longrightarrow$ define an orthogonal polynomial

$$P_n(x) = \frac{D_n(x)}{\sqrt{D_{n-1} D_n}} \tag{19}$$

- Using recurent relations one finds the relations to Lanczos coefficients

$$\boxed{b_n^2 = \frac{D_{n-1} D_{n+1}}{D_n^2}, \qquad a_n = \ln \frac{D_n}{D_{n-1}}.} \tag{20}$$

# Krylov complexity

- Decomposition of $\mathcal{O}(t)$ in terms of the Krylov elements:

$$|\mathcal{O}(t)) = \sum_{n=0}^{K-1} \phi_n(t)|\mathcal{O}_n). \tag{21}$$

- The Liouvillian in Krylov basis

$$\mathcal{L} = \sum_{n=0}^{K-1} b_{n+1} \left[ |\mathcal{O}_n)(\mathcal{O}_{n+1}| + |\mathcal{O}_{n+1})(\mathcal{O}_n| \right] \tag{22}$$

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- *Krylov Complexity and K-entropy (Shannon)*

$$\boxed{\mathcal{K}(t) = \sum n|\phi_n(t)|^2, \qquad S(t) = \sum |\phi_n(t)|^2 \log |\phi_n(t)|^2} \tag{23}$$

# Warped geometry and warped CFT

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  $+$ unitarity, locality & a bounded below spectrum of the dilatation operator - translations and dilatations are enhanced to an infinite-dimensional symmetries.
- For every value of $\mu\ell \neq 3$: $\exists$ other solutions - $SL(2,R) \times U(1)$ $WAdS_3$ geometries. It is achieved by multiplying the fiber metric with a constant warp factor.
  $\Longrightarrow$ breaks $SL(2,R)_L \times SL(2,R)_R$ to $SL(2,R) \times U(1)$.

- $AdS_3$ deformation - a fibration over $AdS_2$ w/ squashing parameter

$$ds^2 = \frac{\ell^2}{4}[-\cosh^2\sigma d\tau^2 + d\sigma^2 + (du + \sinh\sigma d\tau)^2] \quad \Longrightarrow$$
$$ds^2 = \frac{\ell^2}{\nu^2+3}\left[-\cosh^2\sigma d\tau^2 + d\sigma^2 + \frac{4\nu^2}{\nu^2+3}(du + \sinh\sigma d\tau)^2\right], \tag{24}$$

  $\{u, \tau, \sigma\} \in [-\infty, \infty]$, $\nu^2 \geq 1$ - spacelike stretched $AdS_3$; $\nu^2 \leq 1$ - spacelike squashed $AdS_3$.
- Detournay, Hartman and Hofman [1210.0539]: transl. inv. only + chiral scaling symmetry $\Longrightarrow$ one Vir and a U(1) current algebra.
- Holographically: a WCFT can be described as a $SL(2,R) \times U(1)$ Chern-Simons theory in 3d [Castro, Hofman, Iqbal] .

*Comments*: Recently: the Kerr BH background a hidden $SL(2,R) \times U(1)$ ("Love") symmetry in the near zone approximation.

# Warped Conformal Symmetry

- The BH solutions, asymptotic to warped $AdS_3$

$$ds^2 = dt^2 + \frac{l^2}{3+\nu^2}\frac{dr^2}{(r-r_-)(r-r_+)} - 2(\nu r + \frac{1}{2}\sqrt{r_+ r_-(3+\nu^2)})dtd\phi$$
$$+ \frac{r}{4}[3(\nu^2-1)r + (3+\nu^2)(r_+ + r_-) + 4\nu\sqrt{r_+ r_-(3+\nu^2)}]d\phi^2$$

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$$+ \frac{r}{4}[3(\nu^2-1)r + (3+\nu^2)(r_+ + r_-) + 4\nu\sqrt{r_+ r_-(3+\nu^2)}]d\phi^2$$

- The asymptotic algebra

$$\begin{aligned}
[L_m, L_n] &= (m-n)L_{m+n} + \frac{c_V}{12}m^3\delta_{n+m,0} \\
[L_m, J_n] &= -nJ_{m+n}, \\
[J_m, J_n] &= \frac{c_J}{12}m\delta_{m+n,0} = \frac{k}{2}m\delta_{m+n,0},
\end{aligned} \tag{25}$$

$$c_V = \frac{5\nu^2+3}{\nu(\nu^2+3)}\frac{l}{G}, \qquad c_J = \frac{\nu^2+3}{\nu}\frac{l}{G} = k/6. \tag{26}$$

# Symmetries and operators

- Transformations of local operators under global scaling symmetry $x \to \lambda x$ and translational symmetry $x \to x + a$, $y \to y + b$,

$$\Phi_i(\lambda x + a, y + b) = \lambda^{-h_i} \Phi_i(x, y)\,, \tag{27}$$

- Infinitesumally

$$[L_n, \mathcal{O}(x, y)] = [x^{n+1}\partial_x + (n+1)x^n h]\mathcal{O}(x, y)\,, \tag{28}$$

$$[J_n, \mathcal{O}(x, y)] = i x^n \partial_y \mathcal{O}(x, y) \tag{29}$$

$$= -\, x^n Q \mathcal{O}(x, y)\,, \tag{30}$$

- The standard basis

$$|\mathcal{O}^{\{\vec{N},\vec{M}\}}\rangle = L_{-1}^{N_1} L_{-2}^{N_2} \ldots J_{-1}^{M_1} J_{-2}^{M_2} \ldots |\Delta, Q\rangle$$

# A new basis of operators

- $U(1)$ Sugawara

$$T^{\mathrm{sug}}(z) = \sum_n \frac{L_n^{\mathrm{sug}}}{z^{n+2}}, \qquad L_n^{\mathrm{sug}} = \frac{1}{2k}\left(\sum_{m\leq -1} J_m J_{n-m} + \sum_{m\geq 0} J_{n-m} J_m\right), \tag{31}$$

$\Longrightarrow$

$$\begin{aligned} [L_n^{\mathrm{sug}}, L_m^{\mathrm{sug}}] &= (n-m)L_{n+m}^{\mathrm{sug}} + \frac{1}{12}n(n^2-1)\delta_{n+m,0}, \\ [L_n^{\mathrm{sug}}, J_m] &= -mJ_{n+m} \end{aligned} \tag{32}$$

&

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# A new basis

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- Define spectral flow invariant Virasoro generators

$$\mathcal{L}_n \equiv L_n - L_n^{\text{sug}} = L_n - \frac{1}{k}\Big( \sum_{m\leq -1} J_m J_{n-m} + \sum_{m\geq 0} J_{n-m} J_m \Big) .. \tag{34}$$

<u>The key point</u>: $\mathcal{L}_n$ and $J_n$ generators provide a basis that factors the algebra into separate Virasoro and $U(1)$ sectors:

$$\begin{aligned} [\mathcal{L}_n, \mathcal{L}_m] &= (n-m)\mathcal{L}_{n+m} + \frac{c-1}{12} n(n^2-1)\delta_{n+m,0}, \\ [\mathcal{L}_n, J_m] &= 0. \end{aligned} \tag{35}$$

$\Longrightarrow$ states $|\phi\rangle$ that are primary with respect to the $L_n$'s and $J_n$'s, with weight $h$ and charge $q_\phi$, are primary under $\mathcal{L}_n$ as well, with weight

$$h^{(0)} = h - \frac{Q_\phi^2}{2k}. \tag{36}$$

# Primaries

- The primary state $|\Delta, Q\rangle$ under $\mathcal{L}_n$ and $\mathcal{J}_n$,

$$\mathcal{L}_0|\Delta, Q\rangle = \Delta^{inv}|\Delta, Q\rangle, \quad \mathcal{J}_0|\Delta, Q\rangle = -Q|\Delta, Q\rangle\,, \tag{37}$$

$$\mathcal{L}_n|\Delta, Q\rangle = 0, \quad \mathcal{J}_n|\Delta, Q\rangle = 0, \quad \forall n > 0\,, \tag{38}$$

- The conformal weight

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- *Remark*: The advantage of using $\{\mathcal{L}_n, \mathcal{J}_m\}$ basis:

- orthogonality of the corresponding descendant states
- factorization of the norm of mixed states including both, $Vir$ & $U(1)$ descendants

# Descendants

- A descendant operators for $|\Delta, Q\rangle$

$$|\mathcal{O}^{\{\vec{N},\vec{M}\}}\rangle = \mathcal{L}_{-1}^{N_1}\mathcal{L}_{-2}^{N_2}...\mathcal{J}_{-1}^{M_1}\mathcal{J}_{-2}^{M_2}\ldots|\Delta, Q\rangle\,, \quad \vec{N} = N_1, \cdots \& \vec{M} = M_1, \cdots.$$

- The spectral invariant conformal weight and charge

$$\mathcal{L}_0|\mathcal{O}^{\{\vec{N},\vec{M}\}}\rangle = \left(\Delta^{inv} + \sum_{n>0} nN_n\right)|\mathcal{O}^{\{\vec{N},\vec{M}\}}\rangle, \tag{40}$$

$$\mathcal{J}_0|\mathcal{O}^{\{\vec{N},\vec{M}\}}\rangle = -Q|\mathcal{O}^{\{\vec{N},\vec{M}\}}\rangle\,. \tag{41}$$

- The conformal weight

$$h = \Delta + \sum_n nN_n - \frac{Q^2}{k}\,. \tag{42}$$

# $SL(2,R)$ subsector of Virasoro

- The action of $SL(2,R)$ on a Fock state

$$L_0|h,n\rangle = (h+n)|h,n\rangle, \quad L_{-1}|h,n\rangle = \sqrt{(n+1)2h+n}|h,n+1\rangle \tag{43}$$

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- Perelomov construction

$$e^{zL_{-1}}|h\rangle = \sum_{n=0}^{\infty} \frac{z^n}{n!} L_{-1}^n |h\rangle = \sum_{n=0}^{\infty} \frac{z^n}{n!} \sqrt{\frac{n!\Gamma(2h+n)}{\Gamma(2h)}} |h,n\rangle. \tag{45}$$

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- The explicit form of a state

$$\boxed{|z,h\rangle = (1-|z|^2)^h \sum_{n=0}^{\infty} z^n \sqrt{\frac{\Gamma(2h+n)}{n!\Gamma(2h)}}\, |h,n\rangle.} \tag{46}$$

# $SL(2,R)$ subsector of Virasoro

- The state generated by Liouvillian $\mathcal{L} = L_{-1} + L_1$

$$|\mathcal{O}(t)) = e^{i\alpha(L_{-1}+L_1)t}|h\rangle = |z = i\tanh(\alpha t); h = \eta/2\rangle \tag{47}$$

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- Identification between the Krylov basis and the basis vectors

$$|\mathcal{O}(t)) = |h\rangle, \qquad |\mathcal{O}_n) = |h, n\rangle.$$

- The Lanczos coeffcients (from (44)):

$$b_n = \alpha\sqrt{n(2h+n-1)}. \tag{48}$$

$\Longrightarrow$ the wavefunctions are just coefficients of the coherent state.

- Krylov Complexity for $SL(2,R)$

$$K_{\mathcal{O}} = \langle\mathcal{O}(t)|\mathcal{O}(t)\rangle = 2h\sinh^2(\alpha t). \tag{49}$$

# Other subsectors of Virasoro

- Virasoro algebra

$$[L_n, L_m] = (n-m)L_{n+m} + \frac{c}{12}n(n^2-1)\delta_{m+n,0}, \tag{50}$$

- construct $SL(2,\mathbb{R})$ from $L_0$ and $L_k = L^\dagger_{-k}$ using

$$[L_k, L_{-k}] = 2kL_0 + \frac{c}{12}k(k^2-1), \qquad [L_0, L_{\pm k}] = \mp k L_{\pm k}. \tag{51}$$

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$$\tilde{L}_\pm = \frac{1}{k}L_{\pm k}, \qquad \tilde{L}_0 = \frac{1}{k}\left(L_0 + \frac{c}{12}k(k^2-1)\right). \tag{52}$$

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$$\begin{aligned} \Longrightarrow \quad D_k(\xi) &= e^{\xi L_{-k} - \bar{\xi} L_k} \\ &= e^{i\phi\frac{\tanh(kr)}{k}L_{-k}} e^{-\frac{2}{k}\log(\cosh(kr))\left(L_0 + \frac{c}{12}k(k^2-1)\right)} e^{-i\phi\frac{\tanh(kr)}{k}L_k}. \end{aligned} \tag{53}$$

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- Autocorrelation function for $SL$ case

$$C(t) = (1|\psi_{\mathcal{O}}(t)) = \frac{1}{\cosh^{2h}(\alpha t)}$$

- In oscillator basis $\alpha_n = \frac{i}{\sqrt{2}}\frac{\partial}{\partial u_n},\ \alpha_{-n} = -i\sqrt{2}nu_n,\ n > 0$

$$\langle f|L_n|u\rangle = \langle u|L_{-n}|f\rangle = l_{-n}f(u) = l_{-n}f(u). \tag{54}$$

- A generic descendant state at level $N = \sum_j jm_j$ is a sum of monomials $u_1^{m_1}u_2^{m_2}u_3^{m_3}\ldots$.

- Operators ($c = 1 + 24\mu^2,\ h = \mu^2 + \lambda^2$)

$$l_0 = h + \sum_{n=1}^{\infty} n\, u_n \frac{\partial}{\partial u_n},$$

$$l_k = \sum_{n=1}^{\infty} n\, u_n \frac{\partial}{\partial u_{n+k}} - \frac{1}{4}\sum_{n=1}^{k-1} \frac{\partial^2}{\partial u_n \partial u_{k-n}} + (\mu + i\lambda)\frac{\partial}{\partial u_k}, \qquad k > 0 \tag{55}$$

$$l_{-k} = \sum_{n=1}^{\infty} (n+k)\, u_{n+k}\frac{\partial}{\partial u_n} - \sum_{n=1}^{k-1} n(k-n)u_n u_{k-n} + 2k(\mu - i\lambda)u_k, \quad k > 0$$

# The action on descendants

- A generic descendant in oscillator basis is

$$\Phi_{\{m\}}(u) \equiv \frac{u_1^{m_1} u_2^{m_2} \dots}{N_{\{m\}}}, \qquad N_{\{m\}} = \sqrt{\prod_{j=1}^{\infty} \frac{m_j!}{(2j)^{m_j}}}. \tag{56}$$

$$\langle \Phi_{\{m\}}, \Phi_{\{m\}} \rangle = 1, \qquad (\Phi_{\{m\}}, l_0 \Phi_{\{m\}}) = h + \sum_j j m_j = h + N. \tag{57}$$

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- The action of $\mathcal{L}$ on an arbitrary descendant

$$\langle u | \mathcal{L} | \Phi_{\{m\}} \rangle = \xi (l_{-1} + l_1) \Phi_{\{m\}} = \sum_{\sum j r_j = N+1} b_{\{m\} \to \{r_j\}} \Phi_{\{r_j\}}(u) + \sum_{\sum j s_j = N-1} b_{\{m\} \to \{s_j\}} \Phi_{\{s_j\}}(u) \tag{58}$$

# Lanczos coefficients

- Elements of the Lanczos matrix

$$b_{\{m\}\to\{r_j\}} = \left(\Phi_{\{m\}}(u), \xi l_{-1}\Phi_{\{r_j\}}(u)\right)$$

$$\implies l_{-1}\Phi_{\{m_k\}} = \sum_{n=1}^{N} \sqrt{n(n+1)m_n(m_{n+1}+1)}\Phi_{\ldots,m_n-1,m_{n+1}+1,\ldots}(u) + (\mu - i\lambda)\sqrt{2(m_1+1)}\Phi_{m_1+1,m_2,\ldots}(u). \quad (59)$$

$\implies$ two types Lanczos coefficients (Caputa & Datta 2021')

Type 1: $b^{(1)}_{\{m_k\}\to\{\ldots,m_n-1,m_{n+1}+1,\ldots\}} = \alpha\sqrt{n(n+1)m_n(m_{n+1}+1)}$ (60)

Type 2: $b^{(2)}_{\{m+k\}\to\{m_1+1,m_2,\ldots\}} = \alpha(\mu - i\lambda)\sqrt{2(m_1+1)}.$ (61)

- Dimensions

$$\dim_{Lanczos}\left[b_{\{m\}\to\{r_j\}}\right] = p(N)\times p(N+1) \overset{N\to\infty}{\sim} \frac{e^{2\pi\sqrt{2N/3}}}{N^2}$$

$$\dim_{\text{links}} \sim \int_0^\infty dn p(n) \overset{N\to\infty}{\sim} \frac{e^{\pi\sqrt{2N/3}}}{\sqrt{2N}}$$ : suppression by $\sim e^{-\pi\sqrt{2N/3}}$

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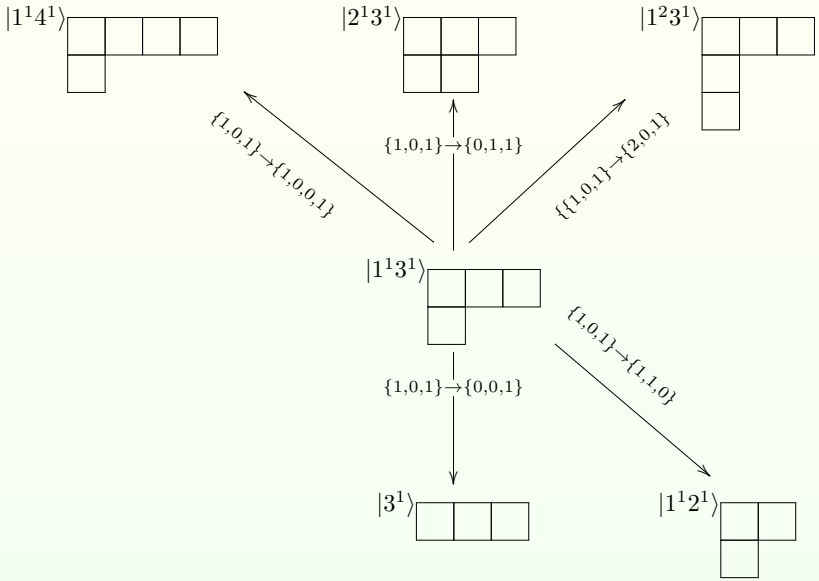
- An example: descendants resulting from the action of $L_{\pm 1}$ on $|1^1 3^1\rangle$



# Lanczos coefficients for typical descendants

⋆ Lanczos coefficients for typical high-level descendants of a heavy primary

- states with $(c, h)$ dependence, $n \ll N$

$$b_{\{m_i\}\to\{\ldots,m_n-1,m_n+1,\ldots\}} \quad \Longrightarrow \quad b_n \sim \sqrt{N}$$

- states without $(c, h)$ dependence, $n \ll N$

$$b_{\{m_i\}\to\{m_1+1,m_2,\ldots\}} \quad \Longrightarrow \quad b_n \sim \sqrt[4]{n}$$

# Expansion over normalized descendants

- $$\Psi_{\mathcal{O}}(t) := \langle u|e^{i\alpha t(l_1+l_{-1})}\mathcal{O}(0)|0\rangle$$
$$= e^{\alpha_0 h}\left[1+\sum_{N=1}^{\infty}\sum_{\sum im_i=N}\varphi_{\{m_i\}}(t)\Phi_{\{m_i\}}(u)\right], \qquad (62)$$
- 'wavefunctions', $\varphi_{\{m_i\}}(t)$, of the primary operator are given by
$$\varphi_{\{m_i\}}(t) = \frac{z^N}{\cosh^{2h}(\alpha t)}\frac{[2(\mu-i\lambda)]^{\sum m_j}}{\sqrt{\prod_i T_{i,m_i}}}\ , \qquad \sum_j jm_j = N\ . \qquad (63)$$
with $z = i\tanh(\alpha t),\ \ \alpha_0 = -2h\log\cosh(\alpha t),\ \ T_{j,m} = (2j)^m m_j!$
- The probabilities
$$p_{\{m_j\}}(t) = |\varphi_{\{m_j\}}(t)|^2 = \frac{\tanh^{2N}(\alpha t)}{\cosh^{4h}(\alpha t)}\frac{[4h]^{\sum m_j}}{\prod_i (2i)^{m_i} m_i!}\ .$$

# $Vir$ contribution

- Krylov complexity (see also Caputa,Datta 21')

$$K_{\mathcal{O}}(t) = \sum_{n=0}^{\infty} n \sum_{\sum i m_i = n} |\phi_{m_i}|^2(t) = 2h \sinh^2(\alpha t) \tag{64}$$

$\Longrightarrow$ exponential growth of $K_{\mathcal{O}}(t)$ at late times

$$K_{\mathcal{O}}(t \to \infty) \sim \frac{h}{2} e^{2\alpha t}.$$

- Normalized variance ($\langle \hat{N}^2 ) \rangle = \sum_n n^2 |\varphi_n(t)|^2$)

$$\delta^2_{\mathcal{O}}(t) = \frac{\langle \hat{N}^2 \rangle - K(t)^2}{K(t)^2} \quad \Longrightarrow \quad \delta_{\mathcal{O}}(t \to \infty) \sim \frac{1}{\sqrt{2h}}.$$

# $U(1)$ contribution

- Rescaling of $J_n$:

$$J_n \longrightarrow \mathcal{J}_n = \sqrt{\frac{2}{k}} J_n$$

$\Longrightarrow$ the algebra

$$[\mathcal{J}_n, \mathcal{J}_m] = n\delta_{n+m}.$$

- States $\quad |k_n\rangle = \frac{e^{\beta J_{-1}}}{\sqrt{\langle 0|J_1^n J_{-1}^n|0\rangle}}|0\rangle$

- Autocorrelation function

$$C^U(t) \sim \frac{1}{2\cosh^{2Q}(\beta\frac{k}{2}t)}$$

# Virasoro-Kac-Moody Character

- Properties
  - Modular invariance
  - Vir-KM primaries with positive semidefinite norms
  - the spectrum of primary states satisfy $h \geq Q^2/k$ with $c > 1$,
- *Holographic WCFTs are characterized by a Virasoro-Kac-Moody algebra with a positive central charge but a negative $U(1)$ level.* There are *negative* norm descendants, however they can be resummed into a Virasoro-Kac-Moody character whose contribution to the bootstrap equations is positive $\Longrightarrow$ contains states with complex $U(1)$ charge.
- *Unitary WCFTs (differs from holographic case) have a positive $U(1)$ level and positive norm descendant states.*

# Virasoro-Kac-Moody Character

- Virasoro-Kac-Moody character - product of $U(1)$ and Vir contributions

# Virasoro-Kac-Moody Character

- Virasoro-Kac-Moody character - product of $U(1)$ and Vir contributions
  - the contribution of the $U(1)$ descendants

$$\prod_{n=1}^{\infty} \frac{1}{1+q^n} = q^{1/24} \frac{\eta(\tau)}{\eta(2\tau)}.$$

  - the contribution of the Vir descendants

$$(1-\delta^{(0)}q) \prod_{n=1}^{\infty} \frac{1}{1-q^n} = q^{1/24} \frac{1}{\eta(\tau)} (1-\delta^{(0)}q).$$

  - the full Virasoro-Kac-Moody character

$$\chi_{h,n}(\tau,\kappa) = q^{h+2/24-c/24} \frac{1}{\eta(2\tau)} r^n (1-\delta^{(0)}q); \qquad Q \in \mathbb{R}$$

- This character is independent of the basis used for the Vir descendants!

• Consider Vir-KM primary w/ *real* charge $Q$

$$\Longrightarrow \qquad \left| \prod_{n_i, m_i} P_{-n_i}^{m_i} |h, Q\rangle \right|^2 = \begin{cases} 1 & \textit{if} \quad \sum_i m_i = \textit{even} \\ -1 & \textit{if} \quad \sum_i m_i = \textit{odd} \end{cases}$$

- The vacuum w/ real (or vanishing) charge: contains states w/ positive *and* negative norms! $\rightarrow$ the character is given above

• Descendant states with imaginary charge due to the antihermicity of the $P_n$ - make only positive contributions to the corresponding Vir-KM character!

All of the $U(1)$ descendant states carry positive norm

$\Longrightarrow$ the unitary Vir-KM character:

$$\chi_{h,n}(\tau, \zeta) = \frac{1}{\eta^2(\tau)} q^{h+2/24-c/24} r^n (1 - \delta^{(0)} q); \qquad Q \in i\mathbb{R} \qquad (65)$$

**Conclusion:** the characters of unitary Vir-KM representations with positive central charge, positive level, and $h \geq h_0 = 0$ are independent of the charge of the state.

# The warped system

- Autocorrelation functions

$$C^W(t) \sim \frac{1}{\cosh^{2h}(\alpha t)} \frac{1}{\cosh^{2Q}(\beta \frac{k}{2} t)}$$

- Krylov complexity

$$K_W(t) \simeq 2hQ \cosh^2(\alpha t) \cosh^2(\beta \frac{k}{2} t)$$

  - Operator growth

$$K_W(t) \sim e^{(2\alpha + \beta k)t}$$

- Normalized variance

$$\delta_W(t \to \infty) \sim \frac{1}{\sqrt{2hQ}}.$$

- Information metric

$$ds^2 = \frac{Q}{1 - |z_2|^2} dz_1 d\bar{z}_1 + \frac{Q|z_1|^2 + 2h(1 - |z_2|^2)}{(1 - |z_2|^2)^3} dz_2 d\bar{z}_2$$

# The rate of Complexity and Momentum

- The conjecture ($K$ - Krylov complexity)

$$\frac{dK}{dt} = / \propto P$$

- For the dual part: consider vacuum solution $r_+ = r_- = 0$

$\Longrightarrow$ ADM form

$$ds^2 = -N^2(r)dt^2 + \ell^2 R^2(r)\left(d\phi + N_\phi(r)dt\right)^2 + \frac{\ell^4 dr^2}{4R^2(r)N^2(r)} \ ,$$

$$R^2(r) = \frac{3(\nu^2-1)}{4}r^2, \quad N^2(r) = \frac{4\ell^2(\nu u^2+3)}{3(\nu^2-1)}, \quad N^\phi = \frac{8\nu}{3(\nu^2-1)r}.$$

- for small enough times

$$\Pi_r = \frac{\ell^4}{4R^2(r)N^2(r)}\dot{r} \simeq (a+b)t.$$

Comparing with

$$\dot{K}_W \simeq 4hQ(\alpha + \frac{\beta k}{2})t$$

one can easily identify the corresponding coefficients $a$ and $b$ .

# Conclusions

- ⋆ Considerations of the operator growth in 2d WCFT's show:
  - Lanczos coefficients essentially depend on the details of descendant states
  - a subset of them does saturate the upper bound of linear growth (as conjectured)
  - K-complexity: universal but is not sensitive enough to distinguish WCFT from $SL(2,R) \times U(1)$ case
  - K-complexity defined for subclasses of vertices (analogous to Caputa,Datta'21)
  - We confirm the conjecture $\dot{K}_W \simeq P_r$ for the case of warped spaces.
- ⋆ Future directions:
  - Lanczos coefficients for $W_3$; do they still obey the maximal bound?
  - relations to dipole deformations? embedding in higher dimensional cases?
  - study complexity of multi-gluonic compound states in QCD?
  - ...

# END

THANK YOU!