KK reduction of Horndeski and the speed of gravity.

based on papers with M.Valencia-Villegas and A.Shtennikova

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I) Horndeski theory and generalizations

II) Kaluza-Klein reduction

III) Profit

IV) Final remarks

Horndeski theory

$$
\begin{aligned}
S &= \int \mathrm{d}^4 x \sqrt{-g}\left(\mathcal{L}_2 + \mathcal{L}_3 + \mathcal{L}_4 + \mathcal{L}_5\right), \\
\mathcal{L}_2 &= F(\pi, X), \\
\mathcal{L}_3 &= K(\pi, X)\Box\pi, \\
\mathcal{L}_4 &= -G_4(\pi, X)R + 2G_{4X}(\pi, X)\left[(\Box\pi)^2 - \pi_{;\mu\nu}\pi^{;\mu\nu}\right], \\
\mathcal{L}_5 &= G_5(\pi, X)G^{\mu\nu}\pi_{;\mu\nu} + \frac{1}{3}G_{5X}\left[(\Box\pi)^3 - 3\Box\pi\pi_{;\mu\nu}\pi^{;\mu\nu} + 2\pi_{;\mu\nu}\pi^{;\mu\rho}\pi_{;\rho}{}^{\nu}\right],
\end{aligned}
$$

where π is the Galileon field, $X = g^{\mu\nu}\pi_{,\mu}\pi_{,\nu}$, $\pi_{,\mu} = \partial_\mu\pi$, $\pi_{;\mu\nu} = \nabla_\nu\nabla_\mu\pi$, $\Box\pi = g^{\mu\nu}\nabla_\nu\nabla_\mu\pi$, $G_{4X} = \partial G_4/\partial X$

Motivation

1. Avoid the quantum gravity. Being able to construct everywhere-regular weak-gravity solutions.

2. Have sufficiently much freedom to modify gravity and scalar dynamics in different ways

3. Theory of a very general form under several assumptions

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   general covariance, locality and 1 additional degree of freedom.

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(1) requires NEC violation.

History

Galileons → Covariant Galileons → Generalized Galileons

(Nicolis et al 0811.2197) **(Deffayet et al 0901.1314)** **(Deffayet et al 1103.3260)**

||

Horndeski

(Horndeski 1974)

$$\begin{aligned}\mathcal{L} = c_1\phi + c_2 X - c_3 X\Box\phi + c_4 X\left[(\Box\phi)^2 - \partial_\mu\partial_\nu\phi\partial^\mu\partial^\nu\phi\right] \\ - \frac{c_5}{3}X\left[(\Box\phi)^3 - 3\Box\phi\partial_\mu\partial_\nu\phi\partial^\mu\partial^\nu\phi + 2\partial_\mu\partial_\nu\phi\partial^\nu\partial^\lambda\phi\partial_\lambda\partial^\mu\phi\right]\end{aligned}$$

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Horndeski

(Horndeski 1974)

$$\mathcal{L} = c_1\phi + c_2 X - c_3 X\Box\phi + \frac{c_4}{2}X^2 R + c_4 X\left[(\Box\phi)^2 - \phi^{\mu\nu}\phi_{\mu\nu}\right] + c_5 X^2 G^{\mu\nu}\phi_{\mu\nu} - \frac{c_5}{3}X\left[(\Box\phi)^3 - 3\Box\phi\phi^{\mu\nu}\phi_{\mu\nu} + 2\phi_{\mu\nu}\phi^{\nu\lambda}\phi^{\mu}_{\lambda}\right]$$

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Horndeski

(Horndeski 1974)

$$
\begin{aligned}
\mathcal{L} = {} & G_2(\phi, X) - G_3(\phi, X)\Box\phi + G_4(\phi, X)R + G_{4X}\left[(\Box\phi)^2 - \phi^{\mu\nu}\phi_{\mu\nu}\right] \\
& + G_5(\phi, X)G^{\mu\nu}\phi_{\mu\nu} - \frac{G_{5X}}{6}\left[(\Box\phi)^3 - 3\Box\phi\phi^{\mu\nu}\phi_{\mu\nu} + 2\phi_{\mu\nu}\phi^{\nu\lambda}\phi^{\mu}_{\lambda}\right]
\end{aligned}
$$

Horndeski theory

$$
\begin{aligned}
S &= \int \mathrm{d}^4 x \sqrt{-g}\left(\mathcal{L}_2 + \mathcal{L}_3 + \mathcal{L}_4 + \mathcal{L}_5\right), \\
\mathcal{L}_2 &= F(\pi, X), \\
\mathcal{L}_3 &= K(\pi, X)\Box\pi, \\
\mathcal{L}_4 &= -G_4(\pi, X)R + 2G_{4X}(\pi, X)\left[(\Box\pi)^2 - \pi_{;\mu\nu}\pi^{;\mu\nu}\right], \\
\mathcal{L}_5 &= G_5(\pi, X)G^{\mu\nu}\pi_{;\mu\nu} + \frac{1}{3}G_{5X}\left[(\Box\pi)^3 - 3\Box\pi\pi_{;\mu\nu}\pi^{;\mu\nu} + 2\pi_{;\mu\nu}\pi^{;\mu\rho}\pi_{;\rho}{}^{\nu}\right],
\end{aligned}
$$

where π is the Galileon field, $X = g^{\mu\nu}\pi_{,\mu}\pi_{,\nu}$, $\pi_{,\mu} = \partial_\mu\pi$, $\pi_{;\mu\nu} = \nabla_\nu\nabla_\mu\pi$, $\Box\pi = g^{\mu\nu}\nabla_\nu\nabla_\mu\pi$, $G_{4X} = \partial G_4/\partial X$.

General lagrangian with 2 tensor and 1 scalar DOF

General relativity, 1-field inflations, non-minimal coupling

K-essence/k-inflation

kinetic gravity braiding/G-inflation

f(R)-gravity, Gauss-Bonnet term, f(G-B)

No Ostrogradski ghost

second order equations of motion in Horndeski, despite second derivatives is the Lagrangian

Can break NEC without linear instabilities

$$\pi = \pi_0 + \chi, \qquad g_{\mu\nu} = \tilde{g}_{\mu\nu} + h_{\mu\nu}$$

$$L_\zeta^{(2)} = \frac{1}{2} U \dot{\zeta}^2 - \frac{1}{2} V (\partial_i \zeta)^2 - \frac{1}{2} W \zeta^2$$

$$U\omega^2 = V\mathbf{p}^2 + W \ ,$$

- stability requirement: $U > 0 \ , \quad V > 0 \ , \quad W \geq 0 \ .$

beyond Horndeski

$$S = \int \mathrm{d}^4 x \sqrt{-g}\left(\mathcal{L}_2 + \mathcal{L}_3 + \mathcal{L}_4 + \mathcal{L}_5 + \mathcal{L}_{\mathcal{BH}}\right),$$

$$\mathcal{L}_2 = F(\pi, X),$$

$$\mathcal{L}_3 = K(\pi, X)\Box\pi,$$

$$\mathcal{L}_4 = -G_4(\pi, X)R + 2G_{4X}(\pi, X)\left[(\Box\pi)^2 - \pi_{;\mu\nu}\pi^{;\mu\nu}\right],$$

$$\mathcal{L}_5 = G_5(\pi, X)G^{\mu\nu}\pi_{;\mu\nu} + \frac{1}{3}G_{5X}\left[(\Box\pi)^3 - 3\Box\pi\pi_{;\mu\nu}\pi^{;\mu\nu} + 2\pi_{;\mu\nu}\pi^{;\mu\rho}\pi_{;\rho}{}^{\nu}\right],$$

$$\mathcal{L}_{\mathcal{BH}} = F_4(\pi, X)\epsilon^{\mu\nu\rho}{}_{\sigma}\epsilon^{\mu'\nu'\rho'\sigma}\pi_{,\mu}\pi_{,\mu'}\pi_{;\nu\nu'}\pi_{;\rho\rho'} +$$

$$+F_5(\pi, X)\epsilon^{\mu\nu\rho\sigma}\epsilon^{\mu'\nu'\rho'\sigma'}\pi_{,\mu}\pi_{,\mu'}\pi_{;\nu\nu'}\pi_{;\rho\rho'}\pi_{;\sigma\sigma'}$$

beyond Horndeski

$$
\begin{aligned}
S &= \int \mathrm{d}^4 x \sqrt{-g}\left(\mathcal{L}_2 + \mathcal{L}_3 + \mathcal{L}_4 + \mathcal{L}_5 + \mathcal{L}_{\mathcal{BH}}\right), \\
\mathcal{L}_2 &= F(\pi, X), \\
\mathcal{L}_3 &= K(\pi, X)\Box\pi, \\
\mathcal{L}_4 &= -G_4(\pi, X)R + 2G_{4X}(\pi, X)\left[(\Box\pi)^2 - \pi_{;\mu\nu}\pi^{;\mu\nu}\right], \\
\mathcal{L}_5 &= G_5(\pi, X)G^{\mu\nu}\pi_{;\mu\nu} + \frac{1}{3}G_{5X}\left[(\Box\pi)^3 - 3\Box\pi\pi_{;\mu\nu}\pi^{;\mu\nu} + 2\pi_{;\mu\nu}\pi^{;\mu\rho}\pi_{;\rho}{}^{\nu}\right], \\
\mathcal{L}_{\mathcal{BH}} &= F_4(\pi, X)\epsilon^{\mu\nu\rho}{}_{\sigma}\epsilon^{\mu'\nu'\rho'\sigma}\pi_{,\mu}\pi_{,\mu'}\pi_{;\nu\nu'}\pi_{;\rho\rho'} + \\
&\qquad + F_5(\pi, X)\epsilon^{\mu\nu\rho\sigma}\epsilon^{\mu'\nu'\rho'\sigma'}\pi_{,\mu}\pi_{,\mu'}\pi_{;\nu\nu'}\pi_{;\rho\rho'}\pi_{;\sigma\sigma'}
\end{aligned}
$$

$$
F_4 \; G_{5X} X = -3F_5 \; \left[G_4 - 2XG_{4X} + \frac{1}{2}G_{5\pi}X\right],
$$

beyond Horndeski

$$
\begin{aligned}
S &= \int \mathrm{d}^4 x \sqrt{-g}\left(\mathcal{L}_2 + \mathcal{L}_3 + \mathcal{L}_4 + \mathcal{L}_5 + \mathcal{L}_{\mathcal{BH}}\right), \\
\mathcal{L}_2 &= F(\pi, X), \\
\mathcal{L}_3 &= K(\pi, X)\Box\pi, \\
\mathcal{L}_4 &= -G_4(\pi, X)R + 2G_{4X}(\pi, X)\left[(\Box\pi)^2 - \pi_{;\mu\nu}\pi^{;\mu\nu}\right], \\
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\mathcal{L}_{\mathcal{BH}} &= F_4(\pi, X)\epsilon^{\mu\nu\rho}{}_{\sigma}\epsilon^{\mu'\nu'\rho'\sigma}\pi_{,\mu}\pi_{,\mu'}\pi_{;\nu\nu'}\pi_{;\rho\rho'} + \\
&\quad + F_5(\pi, X)\epsilon^{\mu\nu\rho\sigma}\epsilon^{\mu'\nu'\rho'\sigma'}\pi_{,\mu}\pi_{,\mu'}\pi_{;\nu\nu'}\pi_{;\rho\rho'}\pi_{;\sigma\sigma'}
\end{aligned}
$$

$$
F_4\ G_{5X}X = -3F_5\ \left[G_4 - 2XG_{4X} + \frac{1}{2}G_{5\pi}X\right],
$$

H → BH $\qquad g_{\mu\nu} \to g_{\mu\nu} + \Gamma(\pi, X)\partial_\mu\pi\partial_\nu\pi.$

DHOST theory

$$
\begin{aligned}
S &= \int \mathrm{d}^4 x \sqrt{-g}\left(\mathcal{L}_2 + \mathcal{L}_3 + \mathcal{L}_4 + \mathcal{L}_5\right), \\
\mathcal{L}_2 &= F(\pi, X), \\
\mathcal{L}_3 &= K(\pi, X)\Box\pi, \\
\mathcal{L}_4 &= F_2(\pi, X)R + \sum_{i=1}^{5} A_i(\pi, X)\, L_i^{(2)}, \\
\mathcal{L}_5 &= F_3(\pi, X)G^{\mu\nu}\pi_{;\mu\nu} + \sum_{j=1}^{10} B_j(\pi, X)\, L_j^{(3)},
\end{aligned}
$$

DHOST theory

$$\mathcal{L}_{\mathsf{Quad}} = \sum_{i=1}^{5} A_i(\pi, X)\, L_i^{(2)}\,,$$

DHOST theory

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$$L_1^{(2)} = (\pi_{\mu\nu})^2\,, \quad L_2^{(2)} = (\Box\pi)^2\,, \quad L_3^{(2)} = \Box\pi\,(\pi_{\mu\nu}\pi^{\mu}\pi^{\nu})\,,$$

$$L_4^{(2)} = (\pi_{\mu\rho}\pi^{\mu})^2\,,\ L_5^{(2)} = (\pi_{\mu\nu}\pi^{\mu}\pi^{\nu})^2\,,$$

DHOST theory

$$\mathcal{L}_{\text{Quad}} = \sum_{i=1}^{5} A_i(\pi, X)\, L_i^{(2)} \,,$$

$$L_1^{(2)} = (\pi_{\mu\nu})^2 \,, \quad L_2^{(2)} = (\Box\pi)^2 \,, \quad L_3^{(2)} = \Box\pi\,(\pi_{\mu\nu}\pi^\mu\pi^\nu) \,,$$

$$L_4^{(2)} = (\pi_{\mu\rho}\pi^\mu)^2 \,,\; L_5^{(2)} = (\pi_{\mu\nu}\pi^\mu\pi^\nu)^2 \,,$$

$$A_2 = -A_1 \,,$$

$$A_4 = \frac{1}{8(F_2 - XA_1)^2}\Big[-16XA_1^3 + 4(3F_2 + 16XF_{2X})A_1^2 \\ -(16X^2F_{2X} - 12XF_2)A_3A_1 - X^2F_2A_3^2 \\ -16F_{2X}(3F_2 + 4XF_{2X})A_1 + 8F_2(XF_{2X} - F_2)A_3 + 48F_2F_{2X}^2\Big] \,,$$

$$A_5 = \frac{(4F_{2X} - 2A_1 + XA_3)\left(-2A_1^2 - 3XA_1A_3 + 4F_{2X}A_1 + 4F_2A_3\right)}{8(F_2 - XA_1)^2} \,.$$

DHOST theory

$$\mathcal{L}_{\mathsf{Cubic}} = \sum_{j=1}^{10} B_j(\pi, X)\, L_j^{(3)}$$

DHOST theory

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$$L_1^{(3)} = (\Box\pi)^3\,, \quad L_2^{(3)} = \Box\pi(\pi_{\mu\nu})^2\,, \quad L_3^{(3)} = (\pi_{\mu\nu})^3\,,$$

$$L_4^{(3)} = (\Box\pi)^2(\pi_{\mu\nu}\pi^{\mu}\pi^{\nu})\,, \qquad L_5^{(3)} = \Box\pi\,(\pi_{\mu\nu}\pi^{\mu})^2\,,$$

$$L_6^{(3)} = (\pi_{\rho\sigma})^2\,(\pi_{\mu\nu}\pi^{\mu}\pi^{\nu})\,, \qquad L_7^{(3)} = \pi^{\mu\nu}\pi_{\nu\rho}\pi^{\rho\sigma}\pi_{\mu}\pi_{\sigma}\,,$$

$$L_8^{(3)} = (\pi^{\mu\nu}\pi_{\mu})^2(\pi^{\rho\sigma}\pi_{\rho}\pi_{\sigma})\,, \qquad L_9^{(3)} = \Box\pi(\pi^{\rho\sigma}\pi_{\rho}\pi_{\sigma})^2\,,$$

$$L_{10}^{(3)} = (\pi^{\rho\sigma}\pi_{\rho}\pi_{\sigma})^3$$

DHOST theory

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$$L_6^{(3)} = (\pi_{\rho\sigma})^2\,(\pi_{\mu\nu}\pi^\mu\pi^\nu)\,, \qquad L_7^{(3)} = \pi^{\mu\nu}\pi_{\nu\rho}\pi^{\rho\sigma}\pi_\mu\pi_\sigma\,,$$

$$L_8^{(3)} = (\pi^{\mu\nu}\pi_\mu)^2(\pi^{\rho\sigma}\pi_\rho\pi_\sigma)\,, \qquad L_9^{(3)} = \Box\pi(\pi^{\rho\sigma}\pi_\rho\pi_\sigma)^2\,,$$

$$L_{10}^{(3)} = (\pi^{\rho\sigma}\pi_\rho\pi_\sigma)^3$$

+ Relations between F_3 and B_j

DHOST theory

$$S = \int \mathrm{d}^4 x \sqrt{-g}\, \left(\mathcal{L}_2 + \mathcal{L}_3 + \mathcal{L}_4 + \mathcal{L}_5 \right),$$

$$\mathcal{L}_2 = F(\pi, X),$$

$$\mathcal{L}_3 = K(\pi, X) \Box \pi,$$

$$\mathcal{L}_4 = F_2(\pi, X) R + \sum_{i=1}^{5} A_i(\pi, X)\, L_i^{(2)},$$

$$\mathcal{L}_5 = F_3(\pi, X) G^{\mu\nu} \pi_{;\mu\nu} + \sum_{j=1}^{10} B_j(\pi, X)\, L_j^{(3)},$$

+ Relations

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\end{aligned}
$$

\+ Relations

H → DHOST $\qquad g_{\mu\nu} \to \Omega^2(\pi, X)\, g_{\mu\nu} + \Gamma(\pi, X) \partial_\mu \pi \partial_\nu \pi.$

Horndeski theory in 5d

$$S = \int \mathrm{d}^5 x \sqrt{g}\, \mathcal{L}_\pi ,$$
$$\mathcal{L}_\pi = \mathcal{L}_2 + \mathcal{L}_3 + \mathcal{L}_4 + \mathcal{L}_5 + \mathcal{L}_6$$

$$\mathcal{L}_2 = F(\pi, X),$$
$$\mathcal{L}_3 = K(\pi, X)\Box\pi,$$
$$\mathcal{L}_4 = -G_4(\pi, X)R + 2G_{4X}(\pi, X)\left[(\Box\pi)^2 - \pi_{;MN}\pi^{;MN}\right],$$
$$\mathcal{L}_5 = G_5(\pi, X)G^{MN}\pi_{;MN} + \frac{1}{3}G_{5X}(\pi, X)\left[(\Box\pi)^3 - 3\Box\pi\pi_{;MN}\pi^{;MN} + 2\pi_{;MN}\pi^{;MP}\pi_{;P}{}^{N}\right],$$
$$\mathcal{L}_6 = \frac{3}{4}G_6(\pi, X)\left(R^2 - 4R^{AB}R_{AB} + R^{ABCD}R_{ABCD}\right)$$
$$+\, 3\, G_{6X}(\pi, X)*$$
$$\left(-R\left((\Box\pi)^2 - \pi^{;AB}\pi_{;AB}\right) + 4R^{AB}\left(\Box\pi\, \pi_{;AB} - \pi_{;A}{}^{C}\pi_{;CB}\right) - 2R^{ABCD}\pi_{;AC}\pi_{;BD}\right)$$
$$+\, G_{6XX}(\pi, X)*$$
$$\left((\Box\pi)^4 - 6\,\pi^{;AB}\pi_{;AB}(\Box\pi)^2 + 8\,\Box\pi\,\pi^{;AB}\pi_{;B}{}^{C}\pi_{;CA} + 3\left(\pi^{;AB}\pi_{;AB}\right)^2 - 6\pi^{;AB}\pi_{;B}{}^{C}\pi_{;C}{}^{D}\pi_{;DA}\right)$$

KK reduction

$$\mathbb{R}^5 \qquad \longrightarrow \qquad \mathbb{R}^4 \times \mathbb{S}^1 \qquad\qquad R^{(\mathbb{S}^1)} \to 0$$

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$$g_{mn} = \left(\begin{array}{cc} g_{\mu\,\nu} - \phi^2\, A_\mu\, A_\nu & \phi^2\, A_\mu \\ \phi^2\, A_\nu & -\phi^2 \end{array} \right)$$

KK reduction

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$$\mathrm{GR} \qquad \longrightarrow \qquad \mathrm{GR} \quad + \quad \mathrm{EM} \quad + \quad \mathrm{dilaton}_\phi$$

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$$\text{GR} \qquad \longrightarrow \qquad \text{GR} \quad + \quad \text{EM} \quad + \quad \text{dilaton}_\phi$$

Let us perform **KK** reduction for **H**, **BH** and **DHOST** theories

KK compactification of Horndeski theory and generalizations

$$R^5 \qquad \longrightarrow \qquad R^4 \times S^1$$

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* Generalized Galileons ⟶ Generalized Galileons

2nd derivatives in the action ⟶ 2nd derivatives in the action

KK compactification of Horndeski theory and generalizations

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no higher derivatives in EOMs $\xrightarrow{\text{H}}$ no higher derivatives in EOMs

KK compactification of Horndeski theory and generalizations

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no higher derivatives in EOMs $\underrightarrow{\text{H}}$ no higher derivatives in EOMs

degenerate kinetic matrix $\underrightarrow{\text{BH, DHOST}}$ degenerate kinetic matrix

KK compactification of Horndeski theory and generalizations

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no higher derivatives in EOMs $\xrightarrow{\text{H}}$ no higher derivatives in EOMs

degenerate kinetic matrix $\xrightarrow{\text{BH, DHOST}}$ degenerate kinetic matrix

* Metric + scalar$_\pi$ ⟶ Metric + vector + scalar$_\pi$ + scalar$_\phi$

[U(1) gauge]

KK compactification of Horndeski theory and generalizations

$$R^5 \qquad \longrightarrow \qquad R^4 \times S^1$$

$$\mathbf{H}\ (g_{MN} + \pi) \qquad \longrightarrow \qquad \mathbf{H}\ (g_{\mu\nu} + \pi) \qquad + \qquad \cdots.$$

KK compactification of Horndeski theory and generalizations

$$R^5 \qquad \longrightarrow \qquad R^4 \times S^1$$

$$\mathbf{H}\ (g_{MN} + \pi) \qquad \longrightarrow \qquad \mathbf{H}\ (g_{\mu\nu} + \pi) \qquad + \qquad \cdots\cdot$$

$$\mathbf{BH}\ (g_{MN} + \pi) \qquad \longrightarrow \qquad \mathbf{BH}\ (g_{\mu\nu} + \pi) \qquad + \qquad \cdots\cdot$$

KK compactification of Horndeski theory and generalizations

$$R^5 \qquad \longrightarrow \qquad R^4 \times S^1$$

$$\mathbf{H}\ (g_{MN} + \pi) \quad \longrightarrow \quad \mathbf{H}\ (g_{\mu\nu} + \pi) \quad + \quad \cdots$$

$$\mathbf{BH}\ (g_{MN} + \pi) \quad \longrightarrow \quad \mathbf{BH}\ (g_{\mu\nu} + \pi) \quad + \quad \cdots$$

$$\mathbf{DHOST}\ (g_{MN} + \pi) \ \longrightarrow \ \mathbf{DHOST}\ (g_{\mu\nu} + \pi) \quad + \quad \cdots$$

KK compactification of Horndeski theory and generalizations

$$R^5 \quad \longrightarrow \quad R^4 \times S^1$$

$$\mathbf{H}\ (g_{MN} + \pi) \quad \longrightarrow \quad \mathbf{H}\ (g_{\mu\nu} + \pi) \quad + \quad \cdots$$

$$\mathbf{BH}\ (g_{MN} + \pi) \quad \longrightarrow \quad \mathbf{BH}\ (g_{\mu\nu} + \pi) \quad + \quad \cdots$$

$$\mathbf{DHOST}\ (g_{MN} + \pi) \longrightarrow \mathbf{DHOST}\ (g_{\mu\nu} + \pi) \quad + \quad \cdots$$

$\cdots$ = Modified Maxwell theory + dilaton interactions

$$S = \int \mathrm{d}^5 x \sqrt{g}\, \mathcal{L}_\pi ,$$

$$\mathcal{L}_\pi = \mathcal{L}_2 + \mathcal{L}_3 + \mathcal{L}_4 + \mathcal{L}_5 + \mathcal{L}_6$$

$$\mathcal{L}_2 = F(\pi, X),$$

$$\mathcal{L}_3 = K(\pi, X)\Box\pi,$$

$$\mathcal{L}_4 = -G_4(\pi, X)R + 2G_{4X}(\pi, X)\left[(\Box\pi)^2 - \pi_{;MN}\pi^{;MN}\right],$$

$$\mathcal{L}_5 = G_5(\pi, X)G^{MN}\pi_{;MN} + \frac{1}{3}G_{5X}(\pi, X)\left[(\Box\pi)^3 - 3\Box\pi\pi_{;MN}\pi^{;MN} + 2\pi_{;MN}\pi^{;MP}\pi_{;P}{}^{N}\right],$$

$$\mathcal{L}_6 = \frac{3}{4}G_6(\pi, X)\left(R^2 - 4R^{AB}R_{AB} + R^{ABCD}R_{ABCD}\right)$$

$$+\, 3\, G_{6X}(\pi, X)*$$

$$\left(-R\left((\Box\pi)^2 - \pi^{;AB}\pi_{;AB}\right) + 4R^{AB}\left(\Box\pi\,\pi_{;AB} - \pi_{;A}{}^{C}\pi_{;CB}\right) - 2R^{ABCD}\pi_{;AC}\pi_{;BD}\right)$$

$$+\, G_{6XX}(\pi, X)*$$

$$\left((\Box\pi)^4 - 6\,\pi^{;AB}\pi_{;AB}(\Box\pi)^2 + 8\,\Box\pi\,\pi^{;AB}\pi_{;B}{}^{C}\pi_{;CA} + 3\left(\pi^{;AB}\pi_{;AB}\right)^2 - 6\pi^{;AB}\pi_{;B}{}^{C}\pi_{;C}{}^{D}\pi_{;DA}\right)$$

$$g_{AB} = \begin{pmatrix} g_{\mu\,\nu} - \phi^2\, A_\mu\, A_\nu & \phi^2\, A_\mu \\ \phi^2\, A_\nu & -\phi^2 \end{pmatrix}$$

$$\mathcal{L}^{5d}_{\mathsf{H}_\pi} \to \mathcal{L}^{KK}_{H} = \mathcal{L}_{\mathsf{H}_\pi} + \mathcal{L}_{A} + \mathcal{L}_{\phi}\,,$$

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$$\mathcal{L}_{\phi} = \mathcal{L}_{K\phi} + \mathcal{L}_{4\phi} + \mathcal{L}_{5\phi} + \mathcal{L}_{6\phi} \, ,$$

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$$\mathcal{L}_\phi = \mathcal{L}_{K\phi} + \mathcal{L}_{4\phi} + \mathcal{L}_{5\phi} + \mathcal{L}_{6\phi} \, ,$$

$$\mathcal{L}_A = \mathcal{L}_{4A} + \mathcal{L}_{5A} + \mathcal{L}_{6A} \, ,$$

$$\mathcal{L}_{K\phi} = \frac{1}{\phi} K \phi^{;\alpha} \pi_{;\alpha} \, ,$$

$$\mathcal{L}_{4\phi} = \frac{2}{\phi} G_4 \left(\Box \phi \right) + \frac{4}{\phi} G_{4X} \left(\Box \pi \right) \phi^{;\alpha} \pi_{;\alpha} \, ,$$

$$\begin{aligned} \mathcal{L}_{5\phi} = \frac{1}{\phi} G_5 \left(\left(\Box \phi \right) \left(\Box \pi \right) - \phi^{;\alpha\beta} \pi_{;\alpha\beta} \right) - \frac{1}{2\phi} G_5 R \phi^{;\alpha} \pi_{;\alpha} \\ + \frac{1}{\phi} G_{5X} \phi^{;\alpha} \pi_{;\alpha} \left(\left(\Box \pi \right)^2 - \pi_{;\alpha\beta} \pi^{;\alpha\beta} \right) \, , \end{aligned}$$

$$\begin{aligned} \mathcal{L}_{6\phi} = \frac{6}{\phi} G_6 G^{\alpha\beta} \phi_{;\alpha\beta} + \frac{12}{\phi} G_{6X} G^{\alpha\beta} \pi_{;\alpha\beta} \phi^{;\gamma} \pi_{;\gamma} + \frac{12}{\phi} G_{6X} \phi^{;\alpha\beta} \pi_{;\beta\gamma} \pi^{;\gamma}{}_{\alpha} \\ + \frac{6}{\phi} G_{6X} \left(\left(\Box \phi \right) \left(\Box \pi \right)^2 - \left(\Box \phi \right) \pi_{;\alpha\beta} \pi^{;\alpha\beta} - 2 \left(\Box \pi \right) \phi^{;\alpha\beta} \pi_{;\alpha\beta} \right) \\ + \frac{4}{\phi} G_{6XX} \phi^{;\alpha} \pi_{;\alpha} \left(\left(\Box \pi \right)^3 - \left(\Box \pi \right) \pi_{;\alpha\beta} \pi^{;\alpha\beta} + 2 \pi_{;\alpha\beta} \pi^{;\beta}{}_{\gamma} \pi^{;\gamma\alpha} \right) \, . \end{aligned}$$

$$\mathcal{L}_{4A} = -\frac{\phi^2}{4} G_4 F_{\alpha\beta} F^{\alpha\beta} + \phi^2 G_{4X} F_{\alpha\gamma} F^{\alpha}{}_{\beta} \pi^{;\beta} \pi^{;\gamma}$$

$$\begin{aligned}
\mathcal{L}_{5A} \quad = \quad & G_5 \phi^2 \left(\frac{1}{2} F^{\alpha\beta} F_{\alpha}{}^{\gamma} \pi_{;\beta\gamma} - \frac{1}{8} F^{\alpha\beta} F_{\alpha\beta} \Box\pi + \frac{1}{2} F^{\alpha\beta} \nabla^{\gamma} F_{\alpha\gamma} \pi_{;\beta} \right) \\
+ \quad & G_5 \phi^2 \left(\frac{3}{2\phi} F^{\alpha\beta} F_{\alpha}{}^{\gamma} \phi_{;\beta} \pi_{;\gamma} - \frac{3}{8\phi} F^{\alpha\beta} F_{\alpha\beta} \phi^{;\gamma} \pi_{;\gamma} \right) \\
+ \quad & G_{5X} \phi^2 \left(\frac{1}{2} F^{\alpha\beta} F_{\alpha}{}^{\gamma} \left(\Box\pi \right) \pi_{;\beta} \pi_{;\gamma} - \frac{1}{2} F^{\alpha\beta} F^{\gamma\delta} \pi_{;\alpha\gamma} \pi_{;\beta} \pi_{;\delta} \right)
\end{aligned}$$

$$
\begin{aligned}
\mathcal{L}_{6A} =& G_6\phi^2\left(\frac{3}{8}F_{\alpha\beta}F^{\alpha\beta}R-\frac{9}{4\phi}F_{\alpha\beta}F^{\alpha\beta}\left(\Box\phi\right)+\frac{9\phi^2}{64}F_{\alpha\beta}F_{\gamma\delta}F^{\alpha\beta}F^{\gamma\delta}+3F_{\alpha\beta}F_{\gamma}{}^{\alpha}R^{\gamma\beta}\right.\\
&-\frac{9}{2\phi}F_{\alpha\beta}F_{\gamma}{}^{\alpha}\phi^{;\gamma\beta}-\frac{9\phi^2}{32}F_{\alpha\beta}F_{\gamma}{}^{\alpha}F_{\delta}{}^{\beta}F^{\gamma\delta}-\frac{9}{\phi^2}F_{\alpha\beta}F_{\gamma}{}^{\alpha}\phi^{;\beta}\phi^{;\gamma}-\frac{9}{\phi}F_{\alpha}{}^{\beta}\nabla^{\gamma}F_{\beta\gamma}\phi^{;\alpha}\\
&+\frac{3}{2}\nabla^{\alpha}F_{\alpha\beta}\nabla^{\gamma}F_{\gamma}{}^{\beta}+\frac{9}{4}R^{\alpha\gamma\beta\delta}F_{\alpha\beta}F_{\gamma\delta}-\frac{3}{\phi}F^{\alpha\beta}\nabla^{\gamma}F_{\alpha\beta}\phi_{;\gamma}+\frac{3}{\phi}F_{\alpha}{}^{\beta}\nabla^{\alpha}F_{\beta\gamma}\phi^{;\gamma}\\
&\left.-\frac{9}{2\phi^2}F_{\alpha\beta}F^{\alpha\beta}\phi_{;\gamma}\phi^{;\gamma}-\frac{15}{16}\nabla^{\alpha}F_{\beta\gamma}\nabla_{\alpha}F^{\beta\gamma}+\frac{3}{8}\nabla^{\alpha}F_{\beta\gamma}\nabla^{\beta}F_{\alpha}{}^{\gamma}\right)\\
&+G_{6X}\left(-\frac{3}{4}F_{\alpha\beta}F^{\alpha\beta}(\Box\pi)^2-\frac{9}{2\phi}F_{\alpha\beta}F^{\alpha\beta}\left(\Box\pi\right)\phi^{;\gamma}\pi_{;\gamma}+\frac{3}{2}RF_{\alpha\beta}F_{\gamma}{}^{\alpha}\pi^{;\beta}\pi^{;\gamma}\right.\\
&+\frac{3}{4}F_{\alpha\beta}F^{\alpha\beta}\pi_{;\gamma\delta}\pi^{;\gamma\delta}+\frac{9\phi^2}{8}F_{\alpha\beta}F_{\gamma\delta}F_{f}{}^{\alpha}F^{\gamma\delta}\pi^{;\beta}\pi^{;f}-6F_{\alpha\beta}F_{\gamma}{}^{\alpha}\left(\Box\pi\right)\pi^{;\gamma\beta}\\
&-\frac{9}{\phi}F_{\alpha\beta}F_{\gamma}{}^{\alpha}\pi^{;\gamma\beta}\phi^{;\delta}\pi_{;\delta}-\frac{18}{\phi}F_{\alpha\beta}F_{\gamma}{}^{\alpha}\left(\Box\pi\right)\phi^{;\gamma}\pi^{;\beta}-6F_{\alpha}{}^{\beta}\left(\Box\pi\right)\nabla^{\gamma}F_{\beta\gamma}\pi^{;\alpha}\\
&+6F_{\alpha\beta}F_{\gamma}{}^{\alpha}\nabla^{\gamma}\pi_{;\delta}\pi^{;\beta\delta}+3F_{\alpha\beta}F_{\gamma\delta}R^{\alpha\gamma}\pi^{;\beta}\pi^{;\delta}-\frac{9\phi^2}{4}F_{\alpha\beta}F_{\gamma}{}^{\alpha}F_{\delta}{}^{\beta}F_{f}{}^{\gamma}\pi^{;\delta}\pi^{;f}\\
&-\frac{18}{\phi}F_{\alpha\beta}F_{\gamma\delta}\pi^{;\alpha\gamma}\phi^{;\beta}\pi^{;\delta}+6F_{\alpha\beta}\nabla^{\gamma}F_{\gamma\delta}\pi^{;\alpha\delta}\pi^{;\beta}-\frac{9}{2}F_{\alpha\beta}F_{\gamma\delta}\pi^{;\alpha\gamma}\pi^{;\beta\delta}\\
&\left.+6F_{\alpha}{}^{\beta}\nabla^{\gamma}F_{\beta\delta}\pi_{;\gamma}{}^{\delta}\pi^{;\alpha}+\frac{18}{\phi}F_{\alpha\beta}F_{\gamma}{}^{\alpha}\pi^{;\gamma\delta}\phi_{;\delta}\pi^{;\beta}\right)\\
&-3\phi^2G_{6XX}\left(F_{\alpha\beta}F_{\gamma}{}^{\alpha}\pi^{;\beta}\pi^{;\gamma}(\Box\pi)^2+2F_{\alpha\beta}F_{\gamma\delta}\left(\Box\pi\right)\pi^{;\alpha\gamma}\pi^{;\beta}\pi^{;\delta}\right.\\
&\left.-F_{\alpha\beta}F_{\gamma}{}^{\alpha}\pi_{;\delta\kappa}\pi^{;\delta\kappa}\pi^{;\beta}\pi^{;\gamma}-2F_{\alpha\beta}F_{\gamma\delta}\pi_{;\alpha}{}^{\kappa}\pi_{;\kappa}{}^{\gamma}\pi^{;\beta}\pi^{;\delta}\right)
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\begin{aligned}
S &= \int \mathrm{d}^4 x \sqrt{-g}\,(\mathcal{L}_2 + \mathcal{L}_3 + \mathcal{L}_4 + \mathcal{L}_5)\,,\\
\mathcal{L}_2 &= F(\pi, X),\\
\mathcal{L}_3 &= K(\pi, X)\Box\pi,\\
\mathcal{L}_4 &= f_2(\pi, X)R + \sum_{i=1}^{5} a_i(\pi, X)\, L_i^{(2)},\\
\mathcal{L}_5 &= f_3(\pi, X)G^{\mu\nu}\pi_{;\mu\nu} + \sum_{j=1}^{10} b_j(\pi, X)\, L_j^{(3)},
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g_{AB} = \begin{pmatrix} g_{\mu\,\nu} - \phi^2 A_\mu A_\nu & \phi^2 A_\mu \\ \phi^2 A_\nu & -\phi^2 \end{pmatrix}
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$$g_{AB} = \begin{pmatrix} g_{\mu\,\nu} - A_\mu\, A_\nu & A_\mu \\ A_\nu & -1 \end{pmatrix}$$

we put $\phi = \text{const}$

$$\mathcal{L}^{5d}_{\mathsf{DHOST}_\pi} \to \mathcal{L}^{KK}_{\mathsf{DHOST}_\pi} = \mathcal{L}_{\mathsf{DHOST}_\pi} + \mathcal{L}_A \,,$$

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$$\mathcal{L}_{\mathrm{DHOST}_\pi} \sim \mathcal{L}^{5d}_{\mathrm{DHOST}_\pi} \,,$$

$$\begin{aligned}
\mathcal{L}_A &= -\frac{f_2(\pi,X)}{4}\, F_{\mu\nu}F^{\mu\nu} + \frac{a_1(\pi,X)}{2}(F_{\mu\nu}\pi^\mu)^2 \\
&+ \frac{f_3(\pi,X)}{8}\left(4F_{\mu\nu}\nabla_\rho F^{\nu\rho}\pi^\mu + F_{\mu\nu}F^{\mu\nu}\Box\pi - 4F_\mu{}^\nu F^{\mu\rho}\pi_{\nu\rho}\right) \\
&+ \frac{b_2(\pi,X)+b_6(\pi,X)}{2}(F_{\mu\nu}\pi^\mu)^2 + \frac{b_3(\pi,X)}{4}F_{\mu\nu}F_{\rho\sigma}\pi^{\mu\rho}\pi^\nu\pi^\sigma
\end{aligned}$$

- Now, one can forget about 5 dimension and KK procedure.

  It can be considered a trick to obtain the desired Lagrangian $\mathcal{L}_A$
- Alternatively one can find the desired combinations among all general types of terms

  Might be more general, but much harder.

Profit

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$$\mathcal{L}_{4A} = -\frac{\phi^2}{4} G_4 F_{\alpha\beta} F^{\alpha\beta} + \phi^2 G_{4X} F_{\alpha\gamma} F^{\alpha}{}_{\beta} \pi^{;\beta} \pi^{;\gamma}$$

$$\begin{aligned}
\mathcal{L}_{5A} \quad = \quad & G_5 \phi^2 \left(\frac{1}{2} F^{\alpha\beta} F_{\alpha}{}^{\gamma} \pi_{;\beta\gamma} - \frac{1}{8} F^{\alpha\beta} F_{\alpha\beta} \Box\pi + \frac{1}{2} F^{\alpha\beta} \nabla^{\gamma} F_{\alpha\gamma} \pi_{;\beta} \right) \\
+ \quad & G_5 \phi^2 \left(\frac{3}{2\phi} F^{\alpha\beta} F_{\alpha}{}^{\gamma} \phi_{;\beta} \pi_{;\gamma} - \frac{3}{8\phi} F^{\alpha\beta} F_{\alpha\beta} \phi^{;\gamma} \pi_{;\gamma} \right) \\
+ \quad & G_{5X} \phi^2 \left(\frac{1}{2} F^{\alpha\beta} F_{\alpha}{}^{\gamma} \left(\Box\pi \right) \pi_{;\beta} \pi_{;\gamma} - \frac{1}{2} F^{\alpha\beta} F^{\gamma\delta} \pi_{;\alpha\gamma} \pi_{;\beta} \pi_{;\delta} \right)
\end{aligned}$$

$$
\begin{aligned}
\mathcal{L}_{6A} =& G_6\phi^2 \left(\frac{3}{8} F_{\alpha\beta} F^{\alpha\beta} R - \frac{9}{4\phi} F_{\alpha\beta} F^{\alpha\beta} \left(\Box\phi\right) + \frac{9\phi^2}{64} F_{\alpha\beta} F_{\gamma\delta} F^{\alpha\beta} F^{\gamma\delta} + 3 F_{\alpha\beta} F_{\gamma}{}^{\alpha} R^{\gamma\beta} \right. \\
& - \frac{9}{2\phi} F_{\alpha\beta} F_{\gamma}{}^{\alpha} \phi^{;\gamma\beta} - \frac{9\phi^2}{32} F_{\alpha\beta} F_{\gamma}{}^{\alpha} F_{\delta}{}^{\beta} F^{\gamma\delta} - \frac{9}{\phi^2} F_{\alpha\beta} F_{\gamma}{}^{\alpha} \phi^{;\beta} \phi^{;\gamma} - \frac{9}{\phi} F_{\alpha}{}^{\beta} \nabla^{\gamma} F_{\beta\gamma} \phi^{;\alpha} \\
& + \frac{3}{2} \nabla^{\alpha} F_{\alpha\beta} \nabla^{\gamma} F_{\gamma}{}^{\beta} + \frac{9}{4} R^{\alpha\gamma\beta\delta} F_{\alpha\beta} F_{\gamma\delta} - \frac{3}{\phi} F^{\alpha\beta} \nabla^{\gamma} F_{\alpha\beta} \phi_{;\gamma} + \frac{3}{\phi} F_{\alpha}{}^{\beta} \nabla^{\alpha} F_{\beta\gamma} \phi^{;\gamma} \\
& \left. - \frac{9}{2\phi^2} F_{\alpha\beta} F^{\alpha\beta} \phi_{;\gamma} \phi^{;\gamma} - \frac{15}{16} \nabla^{\alpha} F_{\beta\gamma} \nabla_{\alpha} F^{\beta\gamma} + \frac{3}{8} \nabla^{\alpha} F_{\beta\gamma} \nabla^{\beta} F_{\alpha}{}^{\gamma} \right) \\
& + G_{6X} \left(- \frac{3}{4} F_{\alpha\beta} F^{\alpha\beta} (\Box\pi)^2 - \frac{9}{2\phi} F_{\alpha\beta} F^{\alpha\beta} \left(\Box\pi\right) \phi^{;\gamma} \pi_{;\gamma} + \frac{3}{2} R F_{\alpha\beta} F_{\gamma}{}^{\alpha} \pi^{;\beta} \pi^{;\gamma} \right. \\
& + \frac{3}{4} F_{\alpha\beta} F^{\alpha\beta} \pi_{;\gamma\delta} \pi^{;\gamma\delta} + \frac{9\phi^2}{8} F_{\alpha\beta} F_{\gamma\delta} F_{f}{}^{\alpha} F^{\gamma\delta} \pi^{;\beta} \pi^{;f} - 6 F_{\alpha\beta} F_{\gamma}{}^{\alpha} \left(\Box\pi\right) \pi^{;\gamma\beta} \\
& - \frac{9}{\phi} F_{\alpha\beta} F_{\gamma}{}^{\alpha} \pi^{;\gamma\beta} \phi^{;\delta} \pi_{;\delta} - \frac{18}{\phi} F_{\alpha\beta} F_{\gamma}{}^{\alpha} \left(\Box\pi\right) \phi^{;\gamma} \pi^{;\beta} - 6 F_{\alpha}{}^{\beta} \left(\Box\pi\right) \nabla^{\gamma} F_{\beta\gamma} \pi^{;\alpha} \\
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& - \frac{18}{\phi} F_{\alpha\beta} F_{\gamma\delta} \pi^{;\alpha\gamma} \phi^{;\beta} \pi^{;\delta} + 6 F_{\alpha\beta} \nabla^{\gamma} F_{\gamma\delta} \pi^{;\alpha\delta} \pi^{;\beta} - \frac{9}{2} F_{\alpha\beta} F_{\gamma\delta} \pi^{;\alpha\gamma} \pi^{;\beta\delta} \\
& \left. + 6 F_{\alpha}{}^{\beta} \nabla^{\gamma} F_{\beta\delta} \pi_{;\gamma}{}^{\delta} \pi^{;\alpha} + \frac{18}{\phi} F_{\alpha\beta} F_{\gamma}{}^{\alpha} \pi^{;\gamma\delta} \phi_{;\delta} \pi^{;\beta} \right) \\
& - 3\phi^2 G_{6XX} \left(F_{\alpha\beta} F_{\gamma}{}^{\alpha} \pi^{;\beta} \pi^{;\gamma} (\Box\pi)^2 + 2 F_{\alpha\beta} F_{\gamma\delta} \left(\Box\pi\right) \pi^{;\alpha\gamma} \pi^{;\beta} \pi^{;\delta} \right. \\
& \left. - F_{\alpha\beta} F_{\gamma}{}^{\alpha} \pi_{;\delta\kappa} \pi^{;\delta\kappa} \pi^{;\beta} \pi^{;\gamma} - 2 F_{\alpha\beta} F_{\gamma\delta} \pi_{;\alpha}{}^{\kappa} \pi_{;\kappa}{}^{\gamma} \pi^{;\beta} \pi^{;\delta} \right)
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Profit 2) Phenomenologically favored by **GW170817**

Modifications of Maxwell theory that are obtained from **KK** are selftuned in a way, so gravitons and photons propagate at the same speed for wide class of Generalized Galileon theories.

$$c_g^2 = c^2$$

This is not very surprising, since both modes comes from 5-dimensional metric.

Modern Universe cosmology

- For Horndeski theory (and beyond Horndeski and DHOST theories)

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- There is an additional phenomenological restriction, if we study modern Universe (models of dark energy and dark matter)

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- For trivial Maxwell electrodynamics ($c = 1$) it means $c_{\mathcal{T}} = 1$ too.

- For **KK** modified Maxwell $c^2 = c_{\mathcal{T}}^2 \neq 1$

- scalar-tensor theories have two dynamical sectors

$$S = \int \mathrm{d}t\mathrm{d}^3x a^3 \Big[\frac{\mathcal{G}_\mathcal{T}}{8}\left(\dot{h}^T_{ik}\right)^2 - \frac{\mathcal{F}_\mathcal{T}}{8a^2}\left(\partial_i h^T_{kl}\right)^2 + \mathcal{G}_\mathcal{S}\dot{\zeta}^2 - \mathcal{F}_\mathcal{S}\frac{(\partial_i\zeta)^2}{a^2}\Big]$$

- We do not care about scalar sector now

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- Instead we consider additional U(1) vector field

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The speeds of sound for tensor and vector modes are, respectively,

$$c_\mathcal{T}^2 = \frac{\mathcal{F}_\mathcal{T}}{\mathcal{G}_\mathcal{T}}, \qquad c^2 = c_\mathcal{V}^2 = \frac{\mathcal{F}_\mathcal{V}}{\mathcal{G}_\mathcal{V}}$$

- Horndeski theory:

$$\begin{aligned}
\mathcal{G}_{\mathcal{T}} &= 2G_4 - 4G_{4X}X + G_{5\pi}X - 2HG_{5X}X\dot{\pi}, \\
\mathcal{F}_{\mathcal{T}} &= 2G_4 - G_{5\pi}X - 2G_{5X}X\ddot{\pi}.
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\end{aligned}$$

- DHOST theory:

$$\begin{aligned}
\mathcal{G}_{\tau} &= 2f_2 + 2\ddot{\pi}Xf_{3,X} - Xf_{3,\pi} - 2Xa_1 \\
&+ 2X(3\dot{\pi}H + \ddot{\pi})b_2 + 6\dot{\pi}XHb_3 + 2\ddot{\pi}X^2b_6\,, \\
\mathcal{F}_{\tau} &= 2f_2 - 2\ddot{\pi}Xf_{3,X} + Xf_{3,\pi}\,,
\end{aligned}$$

conventional Maxwell $c = 1$

- Horndeski:
- $G_4 = G_4(\pi)$
- $G_5 = const$
- Beyond Horndeski
- $F_4 = \frac{2G_{4X}}{X}$
- $G_5 = const$
- DHOST
- $a_1 = 0$
- $f_3 = 0\,, \quad b_i = 0$

Modified Maxwell

- Horndeski:

$$G_4 = G_4(\pi, X)$$
$$G_5 = G_5(\pi)$$

- Beyond Horndeski

$$G_4 = G_4(\pi, X)$$
$$F_4 = F_4(\pi, X)$$
$$G_5 = G_5(\pi)$$

- DHOST

$$f_2 = f_2(\pi, X)$$
$$a_1 = a_1(\pi, X)$$
$$a_3 = a_3(\pi, X)$$

Final Remarks

- Some subclasses of luminal Horndeski with modified Maxwell were known by disformal trick

  BH with $F_4 = \frac{2G_{4X}}{X}$ $\xrightarrow{\text{disformal transformation}}$ H + modified EM

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! **Dark Energy can be made with beyond Horndeski theory**

- Same speeds relation holds above spherically symmetrical dynamical background

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2 tensor modes	2 vector modes	2 scalars
$\swarrow\searrow$	$\swarrow\searrow$	$\Downarrow$
odd even	odd even	2 even

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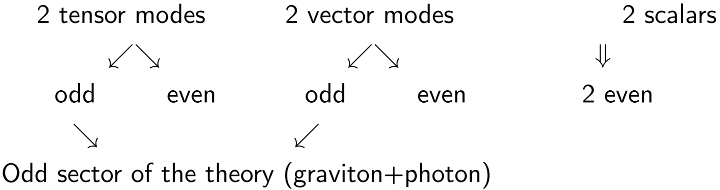

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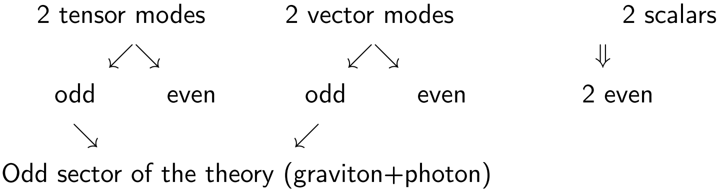

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- No decay for photons
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$= ...$only second derivatives