Deformations, branes and integrability

based on the work with Gleb Zverev

Edvard Musaev

MIPT, Dolgoprudny,

Laboratory of HEP

PMMP'2025

The statement

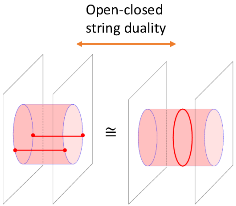
Polyvector unimodular generalized Yang–Baxter deformations of an integrable 2d σ-model are (most probably*) integrable.

*Based on numerical analysis of KAM tori for particular ansätze.

Gauge/gravity duality

A well understood example: weak/strong AdS/CFT correspondence [Maldacena (1997)]

- N Dp-branes can be described equivalently by open or closed strings
- Closed $g_s N \gg 1$: a supergravity background, AdS near the brane
- Open $g_s N \ll 1$: a gauge theory on the brane



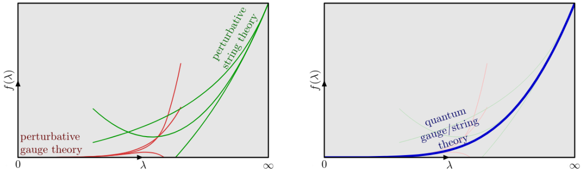


Figure 3: Weak coupling (3, 5, 7 loops) and strong coupling (0, 1, 2 loops) expansions (left) and numerically exact evaluation (right) of some interpolating function $f(\lambda)$.

Fig: Picture taken from N. Beisert et al. [1012.3982]

AdS integrability

- Scaling dimensions of $\mathcal{N}=4$ $d=4$ SYM: $\Delta_{\mathcal{O}}=f(\lambda)$ — functions determined by some (integral) equations.
- $\mathrm{Tr}[\Phi_{I_1}\dots\Phi_{I_L}] \longleftrightarrow$ states of SO(6) spin chain of length L (1-loop anomalous dimensions)

  [Minahan, Zarembo (2003)]
- Type IIB string on $AdS_5\times\mathbb{S}^5$ is integrable (as a classical 2d σ-model)

  [Bena, Polchinski, Roiban (2003)]
- $\mathfrak{psu}(2,2|4)$ & QYBE $\Longrightarrow$ S-matrix for the string on $AdS_5\times\mathbb{S}^5$

  [Staudacher, Beisert ('04,'05), Arutyunov, Frolov, Zamaklar (2007)]
- Classical spinning string solutions in $AdS_5\times\mathbb{S}^5$ correspond to solution of certain integrable systems

  [Gubser, Klebanov, Polyakov (2002), Frolov, Tseytlin (2003), Arutyunov, Frolov, Russo, Tseytlin (2003)]
- Many other models: Gross–Neveu, $AdS_n\times\mathbb{S}^n$, $AdS_4\times\mathbb{CP}^3$ strings
  some reviews: [2408.08414, 1310.4854, 1012.3982, 2301.06486]

Families

- Many of integrable string models belong to integrable families

- Under gauge/gravity duality these generate new (integrable?) field theories:

$\mathrm{AdS}_5 \times \mathbb{S}^5 \Longleftrightarrow \mathcal{N} = 4$ SYM

$\downarrow$ $\qquad\qquad\qquad\qquad$ $\downarrow$

A deformed bg $\Longleftrightarrow$

- adding operators ($\mathrm{Tr}[\Phi^3]$);
- change RG behavior (AdS cut-off);
- non-commutativity;

- Deformation of a string sigma-model $\Longleftrightarrow$ a transformation of the background

$$S = T \int d^2\sigma (G_{\mu\nu} + B_{\mu\nu}) \partial_+ X^\mu \partial_- X^\nu. \tag{1}$$

The plan

- Describe the framework of bi-vector Yang–Baxter deformations (2d σ-model & 10D SUGRA)
- Generalize it to polyvector deformations (include U-dualities)
- Generalization of the classical Yang–Baxter equation
- Show pictures (KAM tori) suggesting a relation between classical integrability and genCYBE

Classical integrability in mechanics

EoM's of an integrable system can be recast in the form of Lax pair equations

$$\frac{d}{dt}L(z) = [L(z), M(z)] = L(z).M(z) - M(z).L(z); \\ L(z), M(z) \in \mathrm{Mat}(N, \mathbb{C}), \quad z \in \mathbb{C}, \tag{2}$$

Integrals of motion:

$$H_m(z) = \mathrm{Tr}\Big[L(z)^m\Big]. \tag{3}$$

Reversely: given matrix $r \in \mathfrak{g} \wedge \mathfrak{g}$ define Poisson bracket

$$\{L_1(u), L_2(v)\} = [r_{12}(u,v), L_1(u)] + [r_{12}(u,v), L_2(v)], \\ 0 = [r_{12}(u-v), r_{13}(u)] + [r_{13}(u), r_{23}(v)] + [r_{12}(u-v), r_{23}(v)]. \tag{4}$$

r-matrix generates integrable systems

[Lax (1968), Sklyanin, Kulish, Semenov-Tyan-Shanski (1980-1983)]

Integrability in 2d field theory

Recasting EoM's in the form of the flatness condition

$$dA + A \wedge A = 0. \tag{5}$$

allows to construct parallel transport operator $U(u;\sigma_1;\sigma_0) = \text{Pexp}\left[\int_{t_0,x_0}^{t_1,x_1} A(u)\right]$ and to define Lax pair:

$$T(u) = \text{Pexp}\left[\oint A(u)\right], \quad M(u) = A_t(u)\Big|_{\sigma^1=0}. \tag{6}$$

Lax equation and conserved currents:

$$\dot{T}(u) = [T(u), M(u)], \quad F_k(u) = \text{Tr}T(u)^k. \tag{7}$$

[Zakharov, Shabat (1971)]

Integrable deformations

- Deformed SU(2) principal chiral model is integrable [Cherednik (1981)]

$$S = -\frac{1}{2}\int d\tau d\sigma \,\mathrm{Tr}\Big[\mathrm{Ad}(\partial_+ g g^{-1}).J.\mathrm{Ad}(\partial_- g g^{-1})\Big],$$
$$J = \mathrm{diag}[J_1, J_2, J_3], \quad \text{deforms Killing form} \tag{8}$$

- Yang-Baxter σ-model for any compact G [Klimcik (2002)]

$$S = -\frac{1}{2}\int d\tau d\sigma \,\mathrm{Tr}\Big[\partial_+ g\, g^{-1}, \frac{(1+\eta^2)^2}{(1+\eta R)}\partial_- g\, g^{-1}\Big], \tag{9}$$

is integrable if classical Yang-Baxter equations is satisfied

$$R(X) := r^{ab} T_a \kappa_{bc} X^c = \mathrm{Tr}_2\Big[r(1\otimes X)\Big], \quad r \in \mathfrak{g}\wedge\mathfrak{g},$$
$$r^{[a_1 b_1} r^{a_2 b_2} f_{b_1 b_2}{}^{a_3]} = 0, \tag{10}$$

Integrable deformations

- Superstring on $\mathrm{AdS}_5 \times \mathbb{S}^5$ is integrable
  [Bena, Polchinski, Roiban (2004)]

$$\text{BOSONIC} \left(\frac{PSU(2,2|4)}{SO(4,1) \times SO(5)} \right) = \frac{SO(4,2)}{SO(4,1)} \frac{SO(6)}{SO(5)} = AdS_5 \times \mathbb{S}^5 \tag{11}$$

algebra

$$\mathfrak{g} = \mathfrak{g}^{(0)} \oplus \mathfrak{g}^{(1)} \oplus \mathfrak{g}^{(2)} \oplus \mathfrak{g}^{(3)},$$
$$A_a = g^{-1}\partial_a g = A_a{}^{(0)} + A_a{}^{(1)} + A_a{}^{(2)} + A_a{}^{(3)} \tag{12}$$

- Its Yang-Baxter deformation is also integrable
  [Vicedo, Delduc, Magro (2013)]

$$S = -\frac{(1+\eta^2)^2}{2(1-\eta^2)} \int d\tau d\sigma\, P_-^{ab}\, \mathrm{STr}\left[A_a . d \circ \frac{1}{1 - \eta R_g \circ d} (A_b) \right] \tag{13}$$

U(1)×U(1) deformation: the gravity side

Class of solutions dual to marginal Leigh-Strassler deformations [Lunin, Maldacena (2005)]

- The initial $\mathcal{N} = 4$ $d = 4$ SYM is dual to $AdS_5 \times \mathbb{S}^5$
- Take Killing vectors from the U(1)×U(1) subgroup of the SO(6) isometry of the 5-sphere

$$\beta^{mn} = \frac{1}{2} r^{ab} k_a{}^m k_b{}^n = \gamma k_{\phi_1}{}^m k_{\phi_2}{}^n \tag{14}$$

$$g_{mn} \xrightarrow{\text{def}} g^{mn} + \beta^{mn} \xrightarrow{\text{new}} G_{mn} + B_{mn} = (g^{-1} + \beta)^{-1}{}_{mn} \tag{15}$$

- This is nothing but a TsT transformation (T-duality — Shift — T-duality)

$$T_{\phi^1} \quad \oplus \quad \phi^2 \to \phi^2 + \gamma \phi^1 \quad \oplus \quad T_{\phi^1} \tag{16}$$

- Deformation is integrable as is any abelian deformation

[Orlando, Reffert, Sekiguchi, Yoshida (2019)]

CFT side

Families of Leigh-Strassler $\mathcal{N}=1$ marginal and relevant deformations of $D=4$ $\mathcal{N}=4$ SYM [Leigh, Strassler (1995)]

$$W = i\kappa \mathrm{Tr}\overbrace{\Big[\underbrace{e^{i\gamma}\Phi_1\Phi_2\Phi_3 - e^{-i\gamma}\Phi_1\Phi_3\Phi_2}_{\beta-\text{deformation}}\Big] + \rho \mathrm{Tr}\Big[\underbrace{\Phi_1^3+\Phi_2^3+\Phi_3^3}_{\rho-\text{deformation}}\Big]}^{\text{marginal}} + \overbrace{\frac{m}{2}\mathrm{Tr}\,\Phi_3^2}^{\text{relevant}} \qquad (17)$$

- γ, ρ — **exactly** marginal, break SUSY to $\mathcal{N}=1$,

  dual to non-commutative deformations of background string geometry

  [Berenshtein, Jejjala, Leigh (2000), Lunin, Maldacena (2005), Kulaxizi (2006)]
- m — triggers RG flow to a strongly coupled IR $\mathcal{N}=1$ SCFT (Leigh–Strassler flow)

  dual to domain wall backgrounds

  [Freedman, Gubser, Pilch, Warner (1999)]

Why non-commutativity?

- TsT along (φ^1, φ^2) turns on B-field

$$B = \gamma\, H\, d\varphi^1 \wedge d\varphi^2$$

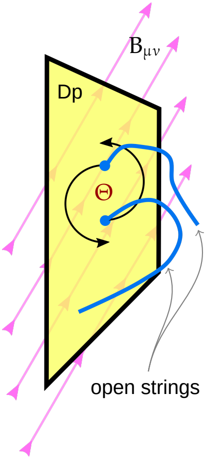
- Open strings' ends on Dp-branes in external B-field do not commute

$$\langle X^\mu(\tau), X^\nu(0)\rangle = -g^{\mu\nu} \log\tau + i\pi\Theta^{\mu\nu}\varepsilon(\tau)$$

open-closed string map:

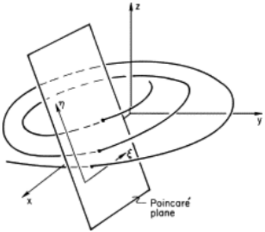
$$g^{-1} + \Theta = (G+B)^{-1}$$

- A similar map exists for (mem)branes and coincides with polyvector deformations of 11D bg's



Real β-deformation of $\mathrm{AdS}_5 \times \mathbb{S}^5$

- Poincare sections: intersection points of phase curves and a given surface in the full phase space;
- Lyapunov exponents show divergence between two trajectories with evolution;
- KAM tori: tori in the phase space of an integrable system wrapped by trajectories.



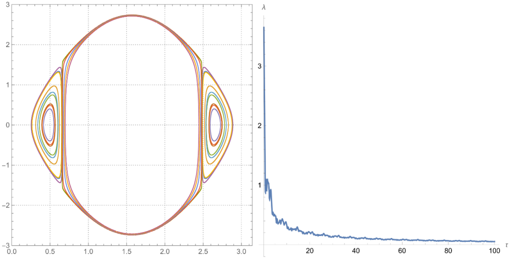
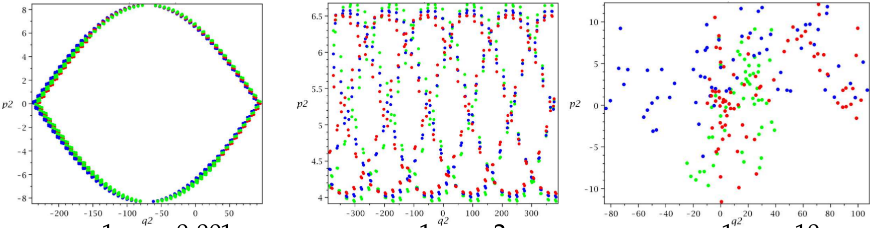


Fig: Poincare sections and Lyapunov exponent

Complex β-deformation of $AdS_5 \times \mathbb{S}^5$

- Supplement TsT by an S-duality transformation $\mathcal{O} = S_\sigma^{-1} T s_\gamma T S_\sigma$.
- $\beta = \gamma + i\sigma$:

$$\tau \to \frac{\tau}{1+\beta\tau} \quad \tau = B_{12} + i\sqrt{G} \tag{18}$$



$\gamma = 1, \sigma = 0.001$ $\gamma = 1, \sigma = 2$ $\gamma = 1, \sigma = 10$

Fig: Picture taken from Giataganas, Pando Zayas, Zoubos [1311.3241]

- Non-Yang-Baxter part of the deformation breaks KAM tori — no integrability

Non-abelian deformations

- Type II supergravity fields: $G_{mn}, B_{mn}, \varphi, C_{(p)}$
- A general bi-vector Yang-Baxter deformation

  [Araujo, Bakhmatov, Colgain, Sakamoto, Sheikh-Jabbari, Yavatanoo (2017)]

$$
\begin{aligned}
&\bullet \quad (G+B)^{-1} = g^{-1} + \beta && \text{no initial flux}\\
&\bullet \quad (G+B)^{-1} = (g+b)^{-1} + \beta && \text{with a flux of } b_{mn}
\end{aligned}
\tag{19}
$$

- Sufficient conditions to have a solution

$$
\begin{aligned}
[k_a, k_b] &= f_{ab}{}^c k_c && \text{(Killing vector algebra)}\\
\beta^{mn} &= k_a{}^m k_b{}^n r^{ab} && \text{(bi-Killing anzats);}\\
r^{b_1[a_1} r^{|b_2|a_2} f_{b_1 b_2}{}^{a_3]} &= 0 && \text{(classical YB equation);}\\
r^{b_1 b_2} f_{b_1 b_2}{}^a k_a{}^m &= I^m = 0 && \text{(unimodularity condition);}
\end{aligned}
\tag{20}
$$

Polyvector deformations

Extend to 11D backgrounds: need a T-duality covariant approach:

- Bi-vector: $(G+B)^{-1} = (g+b)^{-1} + \beta$
- Fields of the 11D SUGRA: G_{mn}, C_{mnk}

T-covariance $\Longrightarrow$ U-covariance (exceptional field theory)

- Allows to construct generalized Yang-Baxter deformations;
- These admit more solutions;
- Relate deformations to certain coordinate transformations
- Hints for integrability of the membrane.

Deformations by U-duality

- Bi-vector deformations sit in the T-duality $O(d,d)$ group

$$\mathcal{O}_\beta = \begin{bmatrix} 1 & 0 \\ \beta & 1 \end{bmatrix} \in O(d,d) \tag{21}$$

- U-duality group of Type II string: is $E_{d(d)}$

[Cremmer, Julia (1979), (1981)]

$$E_{5(5)} = SO(5,5), \quad E_{4(4)} = SL(5), \quad E_{3(3)} = SL(3) \times SL(2). \tag{22}$$

- Using these as generating transformations arrive at polyvector deformations

[Bakhmatov, Colgain, Deger, EtM, Sheikh-Jabbari, (2019)]

$$\Omega^{m_1 m_2 m_3}, \quad \Omega^{m_1 m_2 m_3 m_4 m_5 m_6}, \quad \dots \tag{23}$$

Generalized 3-vector Yang-Baxter equations

3-Killing ansatz

$$\Omega^{mnk} = \rho^{abc} k_a{}^m k_b{}^n k_c{}^k. \tag{24}$$

Sufficient conditions to generate solutions to SUGRA

Linear: unimodularity

$$\rho^{a_1 a_2 a_3} f_{a_2 a_3}{}^{a_4} = 0. \tag{25}$$

Quadratic: the generalized Yang-Baxter equation

$$\rho^{a_1[a_2|a_6|}\rho^{a_3 a_4|a_5|} f_{a_5 a_6}{}^{a_7]} - \rho^{a_2[a_1|a_6|}\rho^{a_3 a_4|a_5|} f_{a_5 a_6}{}^{a_7]} = 0, . \tag{26}$$

sf.

$$\text{CYBE:} \quad r^{b_1[a_1} r^{|b_2|a_2} f_{b_1 b_2}{}^{a_3]} = 0, \quad \text{Uni:} \quad r^{b_1 b_2} f_{b_1 b_2}{}^{a} = 0 \tag{27}$$

[Sakatani, Blair, Malek, Thompson, Colgain, Deger, Sheikh-Jabbari, Bakhmatov, Gubarev, EtM]

Explicit deformation rules

Consider a solution of the form $M_7 \times N_4$ with

$$
\begin{aligned}
ds^2 &= \Delta(y) ds_7{}^2 + ds_4^2, \\
c &= \frac{1}{3!} c_{mnk}(y) dy^m \wedge dy^n \wedge dy^k
\end{aligned}
\tag{28}
$$

Tri-vector transformations in explicit form ($W_m = \varepsilon_{mnkl}\Omega^{nkl}, v^m = \varepsilon^{mnkl} c_{mnk}$)

$$
\begin{aligned}
K^{-1} &= 1 + W_m W^m - 2 W_m v^m + (W_m v^m)^2, \\
G_{\mu\nu} &= K^{-\frac{1}{3}} g_{\mu\nu}, \\
G_{mn} &= K^{\frac{2}{3}} \left(g_{mn} + (1 + v^2) W_m W_n - 2 v_{(m} W_{n)} \right), \\
C^{mnk} &= K^{-1} \Big(c^{mnk} + (1 + v^2) \Omega^{mnk} \Big).
\end{aligned}
\tag{29}
$$

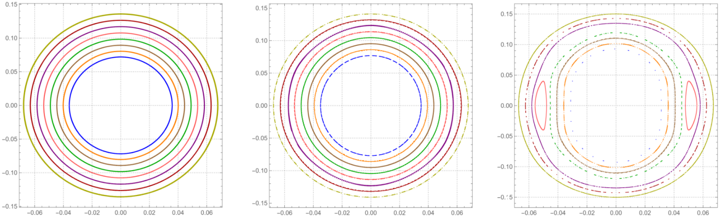
- Defs of $AdS_4 \times \mathbb{S}^7$ along the AdS isometries [Bakhmatov, Gubarev, EtM (2020)]

Lunin-Maldacena $U(1)^3$ deformation

- Deformation of $AdS_4 \times \mathbb{S}^7$ with $\mathbb{S}^7$ reduced to $\mathbb{CP}3$;
- Action on $\tau = C_{123} + i\sqrt{G}$ as

$$\tau \to \frac{\tau}{1+\gamma\tau} \tag{30}$$

- The same as 3-vector deformations with $\Omega = \partial_{\phi_1} \wedge \partial_{\phi_2} \wedge \partial_{\phi_3}$



(a) $\gamma = 0$ (c) $\gamma = 200$ (d) $\gamma = 500$

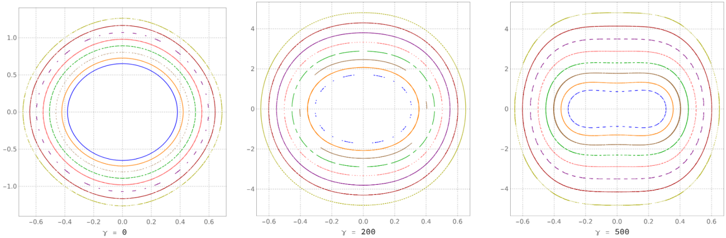
Fig: Poincare sections of IIA string on deformed $AdS_4 \times \mathbb{CP}3$. The expected result: KAM tori are preserved.

Non-abelian PPM deformation

- (2 parameter) deformation of $AdS_4 \times \mathbb{S}^7$ with $\mathbb{S}^7$ reduced to $\mathbb{C}P3$ by

$$\Omega = \rho^{abcd} P_a \wedge P_b \wedge M_{cd}, \quad a, b = 0, 1, 2; \tag{31}$$

- Does not reduce to a bi-vector deformation $\Omega \neq \beta \wedge P$!
- Requires genCYBE to produce a SUGRA solution



(a) $\gamma = 0$ (c) $\gamma = 200$ (d) $\gamma = 500$

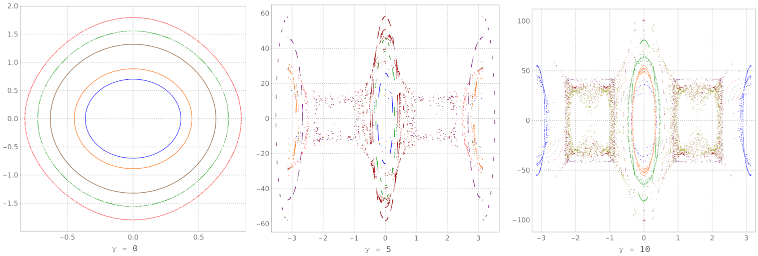
KAM tori are preserved.

Non-abelian non-YB deformation

- (2 parameter) deformation of $\mathrm{AdS}_4 \times \mathbb{S}^7$ with $\mathbb{S}^7$ reduced to $\mathbb{C}\mathrm{P}3$ by

$$\Omega = \mathrm{D} \wedge \mathrm{P} \wedge \mathrm{P} + \mathrm{P} \wedge \mathrm{P} \wedge \mathrm{M} \tag{32}$$

- Does not reduce to a bi-vector deformation $\Omega \neq \beta \wedge \mathrm{P}$
- Does not satisfy genCYBE, satisfies unimodularity, produces a solution to SUGRA eqns



KAM tori get broken.

Intermediate summary

- Bi-vector Yang-Baxter deformation preserve integrability (abelian — proven, non-abelian — many examples);
- Include U-duality: generalize to polyvector deformations.
- These are governed by generalized Yang–Baxter equation on ρ^{abc}.
- Numerical analysis suggests that genCYBE has smth to do with integrability (KAM tori)

Why do we expect integrability here on principle?

3-brackets

The membrane dynamics can be formulated in terms of 3-brackets
[Bagger,Lambert (2007); Gustavsson (2009)]

$$[X, Y, Z] \in \mathfrak{g}, \quad \forall X, Y, Z \in \mathfrak{g}, \tag{33}$$

On a manifold one defines Nambu-Lie structure as a generalisation of the Poisson-Lie structure

$$\begin{aligned} \{f, g, h\} &= \Omega^{mnk}\, \partial_m f\, \partial_n g\, \partial_k h, \\ \{x^m, x^n, \{x^k, x^l, x^p\}\} &+ \text{cyclic} = 0. \end{aligned} \tag{34}$$

It is then natural to require smth of the type

$$\begin{aligned} \frac{d}{dt} L &= [L, M], \\ \{L_1, L_2, L_3\} &= [r_{123}, L_1] + [r_{123}, L_2] + [r_{123}, L_3]. \end{aligned} \tag{35}$$

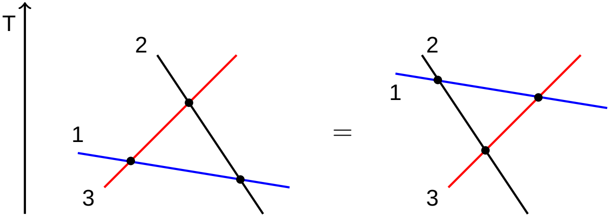
Self-consistency of such defined Nambu-Lie bracket requires r to satisfy generalised YB equation!

Quantum integrability

Quantum R-matrix:

$$R \in \mathrm{End}(V \otimes V) \quad (36)$$

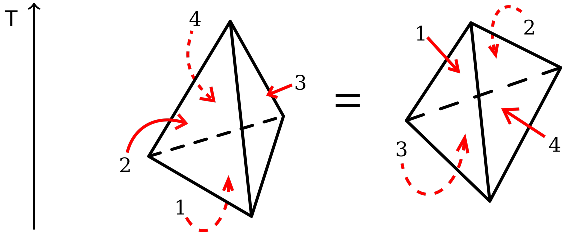
Factorised S-matrix for scattering of particles in 2d



$$\begin{aligned} &R_{23}(v)R_{13}(u)R_{12}(u-v) = R_{12}(u-v)R_{13}(u)R_{23}(v). \\ &R_{12}(u) = \mathrm{id} + \hbar r_{12}(u). \\ &[r_{12}(u-v), r_{13}(u)] + [r_{13}(u), r_{23}(v)] + [r_{12}(u-v), r_{23}(v)] = 0. \end{aligned} \quad (37)$$

Tetrahedron equation

Quantum simplex equation - factorised S-matrix for string scattering



$$R_{234}R_{134}R_{124}R_{123} = R_{123}R_{124}R_{134}R_{234},$$
$$[r_{123}, r_{124}] + [r_{123}, r_{134}] + [r_{124}, r_{134}] + [r_{123}, r_{234}] + [r_{134}, r_{234}] + [r_{124}, r_{234}] = 0. \quad (38)$$

[Zamolodchikov (1981); Frenkel,Moore (1991)]

- Not clear how to do classical limit;
- Not clear whether this has anything to do with generalised YB equation.

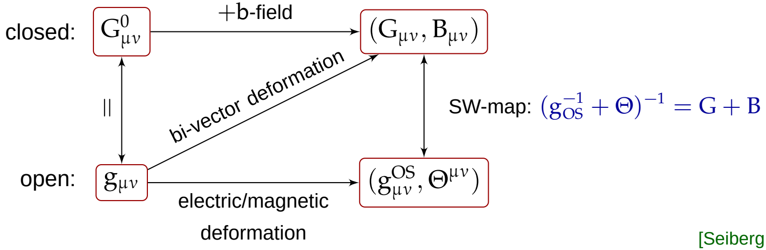
Non-commutativity of open strings

Open string in background fields $G_{\mu\nu}, B_{\mu\nu}$ (closed string fields)

$$S_{OS} = \int_\Sigma d\tau d\sigma \Big(G_{\mu\nu}\, \partial_\alpha X^\mu \partial^\alpha X^\nu + B_{\mu\nu}\, \partial_0 X^\mu \partial_1 X^\nu \Big) \tag{39}$$

Correlator for open string ends

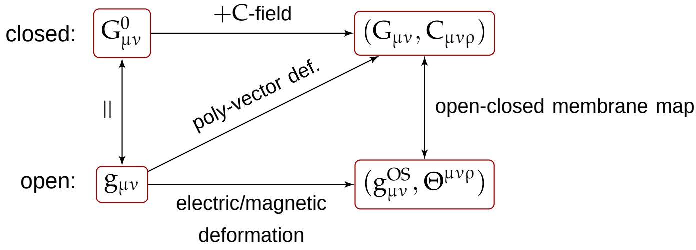
$$\langle X^\mu(\tau), X^\nu(0)\rangle = -g_{OS}^{\mu\nu} \log\tau + i\pi\Theta^{\mu\nu}\varepsilon(\tau) \tag{40}$$



[Seiberg, Witten (1999)]

[Sundell (2000), Berman, Campos, Cederwall, Gran, Larsson, Nielsen, Nilsson, Sundell (2000)]

Non-commutativity of open (mem)branes?



- loop non-commutativity
- deformation of Dirac bracket (only in the flat 3D)

$$[X^\mu(\sigma), X^\nu(\sigma')]_D = \Theta^{\mu\nu\rho} X^\rho(\sigma)\delta(\sigma - \sigma') \tag{41}$$

- other deformations of the algebra of functions

[Sundell (2000), Berman, Campos, Cederwall, Gran, Larsson, Nielsen, Nilsson, Sundell (2001),
Bergshoeff, Berman, van der Schaar, Sundell (2001)]

Summary

- We observe hints that polyvector deformations of the Type II string preserve integrability;
- Some indications that genCYB is related to integrable systems (2d and/or 3d):
  1. fundamental identity of Nambu bracket;
  2. exceptional Drinfeld algebra generalized classical Drinfeld double;
  3. natural connection to loop non-commutativity of the membrane
  4. KAM tori
- Further analysis includes:
  1. Explicit construction of Lax connection for a polyvector deformed string;
  2. Construction of string solitonic solution and mapping them to a known integrable system;
  3. Generalization of qt Hopf algebras (ternary, non-associative etc.)

Thank you!