Determinant anomalies and the Wodzicki residue

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Definition

Multiplicative (determinant) anomaly for elliptic operators $\mathcal{O}_1$, $\mathcal{O}_2$ ($\mathcal{O}_{12} \stackrel{\text{def}}{=} \mathcal{O}_1\mathcal{O}_2$) [Wodzicki (1984); Bytsenko et al (2003); Kontsevich, Vishik (1994)]

$$\mathcal{A}(\mathcal{O}_1, \mathcal{O}_2) \stackrel{\text{def}}{=} \log\,\mathrm{Det}\,\mathcal{O}_{12} - \log\,\mathrm{Det}\,\mathcal{O}_1 - \log\,\mathrm{Det}\,\mathcal{O}_2. \tag{1}$$

Relation to surface terms

Determinant (order-preserving) deformation

$$\begin{aligned} \delta \log\,\mathrm{Det}\,[\mathcal{O}_{12}] \equiv \delta[\,\mathrm{Tr}\,\log \mathcal{O}_{12}] &= \mathrm{Tr}\,[\mathcal{O}_{12}^{-1}\delta\mathcal{O}_{12}] \\ &= \mathrm{Tr}\left[\mathcal{O}_2^{-1}\mathcal{O}_1^{-1}\left(\delta\mathcal{O}_1\mathcal{O}_2 + \mathcal{O}_1\delta\mathcal{O}_2\right)\right]. \end{aligned} \tag{2}$$

Cyclic permutations under Tr involve integration by parts

$$\delta \log\,\mathrm{Det}\,[\mathcal{O}_1\mathcal{O}_2] = \delta \log\,\mathrm{Det}\,\mathcal{O}_1 + \delta \log\,\mathrm{Det}\,\mathcal{O}_2 + \int_{\mathcal{M}} d^d x\, \partial_\mu [\ldots]^\mu. \tag{3}$$

$\Rightarrow$ expect only surface terms in $\mathcal{A}(\mathcal{O}_1, \mathcal{O}_2)$.

Method

For ord $\hat{F}_{(2)} = 2$, ord $\hat{F}_{(4)} = 4$

$$\mathcal{A}_{12}^{d\to 2,4}\Big|_d^{\mathrm{div}} = \frac{1}{\omega - d/2}\int d^d x\, g^{1/2}\,\mathrm{tr}\,\Big[2\hat{E}_d^{\hat{F}_{12}} - \hat{E}_d^{\hat{F}_1} - \hat{E}_d^{\hat{F}_2}\Big],\ \omega \to d/2 - 0, \quad (4)$$

where $\hat{E}_{2m}^{\hat{F}_{(2)}}$ and $\hat{E}_{2m}^{\hat{F}_{(4)}}$ are Gilkey–Seeley coefficients.

Minimal operators

$$\hat{F}_1 = \Box\hat{1} + \hat{A}_\alpha\nabla^\alpha + \hat{Q}, \qquad \hat{F}_2 = \Box\hat{1} + \hat{P}, \quad (5)$$

$$\mathcal{A}_{12}^{d\to 2}\Big|^{\mathrm{div}} = -\frac{1}{\omega - 1}\int \frac{d^2x\, g^{1/2}}{8\pi}\nabla_\alpha\,\mathrm{tr}\,\hat{A}^\alpha, \quad (6)$$

$$\begin{aligned}\mathcal{A}_{12}^{d\to 4}\Big|^{\mathrm{div}} = \frac{1}{\omega - 2}\int \frac{d^4x\, g^{1/2}}{16\pi^2}\nabla_\alpha\,\mathrm{tr}\,\Big\{ &-\frac{1}{4}\Big(\hat{A}^\alpha(\hat{P}+\hat{Q})\Big) - \frac{1}{12}(\hat{A}^\alpha R + \hat{A}_\beta R^{\alpha\beta}) \\ &- \frac{1}{9}\nabla_\alpha\nabla_\beta\hat{A}^\beta - \frac{7}{36}\Box\hat{A}^\alpha + \frac{2}{9}\nabla_\beta\nabla^\alpha\hat{A}^\beta + \frac{11}{72}\hat{A}^\alpha\nabla_\beta\hat{A}^\beta \\ &- \frac{1}{72}\Big(\nabla^\alpha\hat{A}^\beta\hat{A}_\beta + \nabla_\beta\hat{A}^\alpha\hat{A}^\beta\Big) + \frac{1}{24}(\hat{A}^\alpha\hat{A}_\beta\hat{A}^\beta)\Big\}.\end{aligned} \quad (7)$$

Nonminimal operators

$$F_1{}^{\alpha}_{\beta}(\lambda) = \Box\delta^{\alpha}_{\beta} - \frac{\lambda}{\lambda-1}\nabla^{\alpha}\nabla_{\beta} + X^{\alpha}_{\beta}, \quad F_2{}^{\alpha}_{\beta}(\lambda) = \Box\delta^{\alpha}_{\beta} - \lambda\nabla^{\alpha}\nabla_{\beta} + Y^{\alpha}_{\beta}. \tag{8}$$

$$\mathcal{A}_{12}^{d\to 2}\big|^{\rm div} = 0, \tag{9}$$

$$\begin{aligned}\mathcal{A}_{12}^{d\to 4}\Big|^{\rm div} = \frac{1}{\omega-2}\int\frac{d^4x\,g^{1/2}}{16\pi^2}\nabla_{\alpha}\Big\{&\frac{7(\lambda-2)((\lambda-2)\lambda+2(\lambda-1)\log(1-\lambda))}{24(\lambda-1)\lambda}\nabla^{\alpha}R\\ &+\frac{(5\lambda-4)((\lambda-2)\lambda+2(\lambda-1)\log(1-\lambda))}{12\lambda^2}\nabla_{\beta}X^{\alpha\beta}\\ &+\frac{(\lambda+1)((\lambda-2)\lambda+2(\lambda-1)\log(1-\lambda))}{12\lambda^2}\nabla^{\alpha}X\\ &-\frac{(\lambda+4)((\lambda-2)\lambda+2(\lambda-1)\log(1-\lambda))}{12(\lambda-1)\lambda^2}\nabla_{\beta}Y^{\alpha\beta}\\ &-\frac{(2\lambda-1)((\lambda-2)\lambda+2(\lambda-1)\log(1-\lambda))}{12(\lambda-1)\lambda^2}\nabla^{\alpha}Y\Big\}.\end{aligned} \tag{10}$$

Only total-derivative terms, as expected.

Kontsevich-Vishik formula

Alternative approach: explicit formula for determinant anomaly via the Wodzicki (non-commutative) residue

$$\mathcal{A}(A,B) = \int_0^1 dt\, \mathrm{Wres}\left\{ \log\eta \left(\frac{\log[A_t]}{\mathrm{ord}\, A} - \frac{\log[A_t B]}{\mathrm{ord}\, A + \mathrm{ord}\, B} \right) \right\}, \tag{11}$$

$$\begin{aligned} \eta &\equiv A B^{-\frac{\mathrm{ord}\, A}{\mathrm{ord}\, B}}, \qquad && \mathrm{ord}\, \eta = 0, \\ A_t &\equiv \eta^t B^{\frac{\mathrm{ord}\, A}{\mathrm{ord}\, B}}, \qquad && \mathrm{ord}\, A_t = \frac{\mathrm{ord}\, A}{\mathrm{ord}\, B}. \end{aligned} \tag{12}$$

The rest of the talk: what is Wres and how to compute it.

Pseudodifferential operator (ΨDO)

F is ΨDO of order $\mathrm{ord}\, F$ if

$$F\varphi(x) = \int \frac{d^d\xi}{(2\pi)^d} \int d^d y\, e^{i\xi(x-y)} a(x,y,\xi)\varphi(y), \tag{13}$$

$$a(x,y,\xi) \in S^{\mathrm{ord}\, F} \quad \Leftrightarrow \quad \exists C : \left| \partial_\xi^\alpha \partial_x^\beta a(x,\xi) \right| \le C_{\alpha,\beta} \left(\sqrt{1+\xi^2} \right)^{\mathrm{ord}\, F + |\beta|}. \tag{14}$$

Total symbol

Equivalently, a ΨDO via *total symbol* $s_F(x,\xi) \in S^{\operatorname{ord} F}$

$$F\varphi(x) = \int \frac{d^d\xi}{(2\pi)^d} \int d^d y\, e^{i\xi(x-y)} s_F(x,\xi)\varphi(y), \quad s_F(x,\xi) = e^{i\xi x} F e^{-i\xi x}. \qquad (15)$$

$$s_{F\circ G}(x,\xi) = \sum_{k=0}^{\infty} \frac{i^{-k}}{k!} \partial_\xi^k s_F(x,\xi)\, \partial_x^k s_G(x,\xi). \qquad (16)$$

Classical total symbol has a decomposition into *homogeneous components* $F_n(x, t\cdot\xi) = t^n F_n(x,\xi)$.

$$s_F(x,\xi) \sim \sum_{j=0}^{\infty} F_{\operatorname{ord} F - j}(x,\xi), \qquad |\xi| \to \infty. \qquad (17)$$

Wodzicki (or noncommutative) *residue*

$$\operatorname{Wres} F \stackrel{\text{def}}{=} \operatorname{ord} T \operatorname*{Res}_{z=-1} \left\{ \frac{\partial}{\partial u} \operatorname{Tr}\left[(T + u\,F)^{-z}\right]\Big|_{u\to 0} \right\}, \qquad (18)$$

where T is an arbitrary elliptic *test* operator of $\operatorname{ord} T \geq \operatorname{ord} F$ [Wodzicki (1984, 1987)].

Other relations

As a log-coefficient in early time expansion

$$\mathrm{Tr}\left[F\,e^{\tau T}\right] \simeq -\sum_{j=0}^{\infty} A_j\,\tau^{\frac{2j-d}{\mathrm{ord}\,T}-1} - \frac{\mathrm{Wres}\,F}{\mathrm{ord}\,T}\log\tau + O(\tau\log\tau). \tag{19}$$

Via the total symbol

$$\mathrm{Wres}\,F = \int_{\mathcal{M}} d^d x\, g^{1/2} \int_{|\xi|=1} \frac{d^d\xi}{(2\pi)^d}\; F_{-d}(x,\xi). \tag{20}$$

$\Rightarrow$ we need to compute homogeneous components of total symbol on $\mathcal{M}$.

Covariant total symbol (CTS)

$$s_F(x,\xi) = e^{i\xi x} F\, e^{-i\xi x} \Leftrightarrow s_F(x',\xi') = F_x\, e^{i\xi'(x'-x)}\Big|_{x\to x'}, \tag{21}$$

hence the manifestly *covariant total symbol (CTS)* should read

$$\varsigma_F(x',\xi') \overset{\mathrm{def}}{=} F(\nabla_x)\, e^{i\xi_{\alpha'}\sigma^{\alpha'}(x,x')}\Big|_{x\to x'}, \tag{22}$$

$\sigma_{\alpha'} = \nabla^{x'}_{\alpha'}\sigma(x,x')$, where $\sigma(x,x') = \frac{1}{2}[\mathrm{dist}(x,x')]^2$ is a Synge world function.

Operator map ("quantization")

$$F(\nabla_x)\varphi(x) = \int \frac{d^d\xi}{(2\pi)^d} \int d^d x' |\det \sigma_{\alpha\beta'}(x,x')|\, \varsigma_F(x,\xi)\, e^{-i\xi^\alpha \sigma_\alpha(x,x')} \varphi(x'), \tag{23}$$

where $\sigma_{\alpha\beta'}(x,x') = \nabla_\alpha \nabla_{\beta'} \sigma(x,x')$.

Covariant total symbol of composition

Expanding into covariant Taylor series

$$\begin{aligned} A_x\, \varphi(x)\, e^{i\xi_{\alpha'}\sigma^{\alpha'}}\Big|_{x\to x'} &= \sum_{k=0}^{\infty} \frac{1}{k!} \nabla_{\alpha'_1} \dots \nabla_{\alpha'_k} \varphi(x')\, A_x\, \sigma^{\alpha'_1} \dots \sigma^{\alpha'_k}\, e^{i\xi_{\alpha'}\sigma^{\alpha'}}\Big|_{x\to x'} \\ &= \sum_{k=0}^{\infty} \frac{i^{-k}}{k!} \nabla_{(\alpha'_1} \dots \nabla_{\alpha'_k)}\, \varphi(x')\, \partial_\xi^{\alpha'_1\dots\alpha'_k}\, \varsigma_A(x',\xi'). \end{aligned} \tag{24}$$

$$\begin{aligned} \varsigma_{A\circ B}(x',\xi') \equiv A_x\, B_x\, e^{i\xi_{\alpha'}\sigma^{\alpha'}}\Big|_{x\to x'} &= A_x\, e^{i\xi_{\alpha'}\sigma^{\alpha'}}\, e^{-i\xi_{\alpha'}\sigma^{\alpha'}}\, B_x\, e^{i\xi_{\alpha'}\sigma^{\alpha'}}\Big|_{x\to x'} \\ &= \sum_{k=0}^{\infty} \frac{i^{-k}}{k!} \Psi^B_{(\alpha'_1\dots\alpha'_k)}(x',\xi')\, \partial_\xi^{\alpha'_1\dots\alpha'_k} \varsigma_A(x',\xi'), \end{aligned} \tag{25}$$

$$\Psi^B_{(\alpha_1\dots\alpha_k)}(x,\xi) \equiv \nabla_{(\alpha_1} \dots \nabla_{\alpha_k)}\, e^{-i\xi_{\alpha'}\sigma^{\alpha'}}\, B_x\, e^{i\xi_{\alpha'}\sigma^{\alpha'}}\Big|_{x'\to x}. \tag{26}$$

Wres $f(A)$ can be found from total symbol of the resolvent

$$\varsigma\{f(A)\}(x,\xi) = -\frac{1}{2\pi i}\int d\lambda\, f(\lambda)\,\varsigma\left\{(A-\lambda)^{-1}\right\}(x,\xi), \tag{27}$$

$\Rightarrow$ need symbols of inverse operators.

Covariant total symbol of the inverse operator.

Schematically we have

$$\varsigma\{A^{-1}\circ A\} = 1 = \sum_k \frac{i^{-k}}{k!}\partial_\xi^k \varsigma_{A^{-1}}\,\Psi_k^A \tag{28}$$

Since $k=0$ term is just $\varsigma_{A^{-1}}\varsigma_A$, we can use this to iteratively solve for $\varsigma_{A^{-1}}$

$$\varsigma_{A^{-1}} = \sum_{m=0}^{\infty} Q_m, \quad Q_0 = \frac{1}{\varsigma_A}, \quad Q_m = \sum_{k=1}^{m}\frac{i^{-k}}{k!}\partial_\xi^k Q_{m-k}\,\Psi_k^A. \tag{29}$$

Let us use this to calculate Wres $1/\Box$ at $d=4$, which requires the knowledge of $\varsigma_{-4}\left\{\Box^{-1}\right\}(x,\xi)$. Power counting tells that we only need Q_m up to $m=2$.

Wres $\Box^{-1}$

Using the fact that for $B = B(\nabla_\alpha)$

$$\Psi^B_{\alpha_1\ldots\alpha_k}(x,\xi) \equiv \nabla_{\alpha_1}\ldots\nabla_{\alpha_k}\, e^{-i\xi_{\alpha'}\sigma^{\alpha'}} B_x\, e^{i\xi_{\alpha'}\sigma^{\alpha'}}\Big|_{x'\to x} \\ = \nabla_{\alpha_1}\ldots\nabla_{\alpha_k} B(\nabla_\alpha + i\xi_{\mu'}\sigma^{\mu'}_\alpha)\, 1\Big|_{x'\to x}, \tag{30}$$

we find Ψ's in terms of coincidence limits of covariant derivatives of the Synge world function $\left[\sigma_{\alpha\beta\ldots}{}^{\mu'\nu'\ldots\lambda'}\right] \equiv \nabla_\alpha\nabla_\beta\ldots\nabla^{\mu'}\nabla^{\nu'}\ldots\sigma^{\lambda'}(x,x')\Big|_{x\to x'}$

$$\Psi^\Box_\emptyset = -\xi^2,\ \Psi^\Box_\alpha = i\xi_\mu\left[\sigma_\alpha{}^{\lambda\mu'}{}_\lambda\right],\ \Psi^\Box_{(\alpha\beta)} = i\xi_\mu\left[\sigma_{(\alpha\beta)}{}^{\lambda\mu'}{}_\lambda\right] - \xi_\mu\xi_\nu\left[\sigma_{(\alpha\beta)}{}^{\mu'\nu} + \sigma_{(\alpha\beta)}{}^{\mu\nu'}\right], \tag{31}$$

whence

$$Q_0 = -\xi^{-2},\quad Q_1 = \frac{4}{3}\frac{\xi_\alpha\xi_\beta}{\xi^6}R^{\alpha\beta},\quad Q_2 = -\frac{2}{3}\frac{\xi_\alpha\xi_\beta}{\xi^6}R^{\alpha\beta} + O\left(\frac{1}{\xi^5}\right), \tag{32}$$

$$\varsigma_{-4}\left\{\Box^{-1}\right\}(x,\xi) = \frac{2}{3}\frac{\xi_\alpha\xi_\beta}{\xi^6}R^{\alpha\beta},\qquad \mathrm{Wres}\,\Box^{-1} = \frac{1}{6}\int\frac{g^{1/2}d^4x}{(2\pi)^4}R, \tag{33}$$

which is in agreement with literature [Kalau, Waltze (1995), Ackermann (1996)].

$\varsigma_B(x,\xi)$ is not enough for $\Psi^B_{\alpha_1,\dots\alpha_n} = \nabla_{\alpha_1\dots\alpha_n} e^{-i\xi_{\alpha'}\sigma^{\alpha'}} B_x\, e^{i\xi_{\alpha'}\sigma^{\alpha'}}\Big|_{x\to x'}$.
Can we close the covariant symbol algebra? Yes [Widom (1978, 1979, 1980)].

Algebra of total symbols

$$e^{-it\varphi(x)} A_x\, f(x)\, e^{it\varphi(x)} = \sum_{\ell,k=0}^{\infty} \frac{i^{k-\ell}}{k!} \sum_{\substack{m_0+\dots+m_k=\ell \\ m_1,\dots,m_k>1}} \frac{t^k}{m_0!\dots m_k!} \nabla^{m_0} f(x) \times \nabla^{m_1}\varphi(x)\dots\nabla^{m_k}\varphi(x)\,\partial^\ell_\xi\,\varsigma_A(x,\xi)\Big|_{\xi_\alpha\to t\partial_\alpha\varphi(x)}, \tag{34}$$

apply to calculating $\Psi^B_{\alpha_1\dots\alpha_k}$

$$\varsigma_{A\circ B}(x',\xi') = \sum_{k=0}^{\infty} \sum_{\substack{m_1,\dots m_k\geq 2 \\ n_0,n_1\dots n_k\geq 0}} \frac{1}{k!} \frac{i^{k-n_0-\sum_{j=1}^k(n_j+m_j)}}{n_0!\,n_1!\dots n_k!\,m_1!\dots m_k!} \times \xi_{\alpha'_1}\,\nabla_x^{n_1+m_1}\sigma^{\alpha'_1}(x,x')\dots\xi_{\alpha'_k}\,\nabla_x^{n_k+m_k}\sigma^{\alpha'_k}(x,x') \times \partial_{\xi'}^{n_1+\dots+n_k}\varsigma_A(x',\xi')\,\nabla_x^{n_0}\left[\partial_\eta^{m_1+\dots+m_k}\varsigma_B(x,\eta)\Big|_{\eta_\alpha\to\xi_{\alpha'}\sigma^{\alpha'}_\alpha(x,x')}\right]\Bigg|_{x\to x'}. \tag{35}$$

All of the above is for scalar operators. Generalization to vector bundles:

Operators on vector bundles

Delta function contains parallel displacement propagator
$\hat{a}_0(x,x') = \hat{\mathcal{P}}(x,x')$: $\sigma^\alpha \nabla_\alpha \hat{\mathcal{P}}(x,x') = 0$ [Barvinsky, Wachowski (2022)]

$$\hat{\delta}(x,x') \equiv \delta^A_{B'}(x,x') = \int \frac{d^d\xi}{(2\pi)^d} e^{i\xi_{\alpha'}\sigma^{\alpha'}(x,x')} \mathcal{P}^A_{B'}(x,x'), \tag{36}$$

So the covariant total symbol should read

$$\hat{\varsigma}_F(x',\xi') \overset{\text{def}}{=} \hat{F}_x \, e^{i\xi_{\alpha'}\sigma^{\alpha'}(x,x')} \, \hat{\mathcal{P}}(x,x')\Big|_{x\to x'}. \tag{37}$$

All formulas modify slightly, now containing $\hat{\mathcal{P}}(x,x')$ derivatives:

$$\begin{aligned}
\hat{\varsigma}_{\hat{A}\circ\hat{B}}(x',\xi') = \sum_{k=0}^{\infty} \sum_{\substack{m_1,\dots m_k \geq 2 \\ n,m_0,n_0,\dots n_k \geq 0}} \frac{1}{k!\,n!} \frac{i^{k-n-\sum_{j=0}^{k}(n_j+m_j)}}{n_0!\dots n_k!\,m_0!\dots m_k!} \\
\times\ \xi_{\alpha'_1} \nabla_x^{n_1+m_1} \sigma^{\alpha'_1}(x,x') \dots \xi_{\alpha'_k} \nabla_x^{n_k+m_k} \sigma^{\alpha'_k}(x,x') \partial_{\xi'}^{n+n_1+\dots+n_k} \hat{\varsigma}_A(x',\xi') \\
\times\ \nabla_x^n \left[\partial_\eta^{m_1+\dots+m_k} \hat{\varsigma}_B(x,\eta)\Big|_{\eta_\alpha \to \xi_{\alpha'}\sigma^{\alpha'}_\alpha(x,x')} \right] \nabla_x^{m_0+n_0} \hat{\mathcal{P}}(x,x')\Bigg|_{x\to x'}.
\end{aligned} \tag{38}$$

Summary

- Shown, that the determinant anomaly is proportional to a total-derivative term.
- Verified by explicit calculations using the heat kernel method for minimal and nonminimal anomalies.
- Manifestly covariant total symbol calculus used to calculate Wres $\frac{1}{\Box}$

To do

- Apply covariant total symbol calculus to traces of square roots of operators.
- Employ Kontsevich-Vishik formula to anomaly calculations in physically relevant examples.

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Thank you for your attention!