Quantum chaos measures for Floquet systems

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Plan

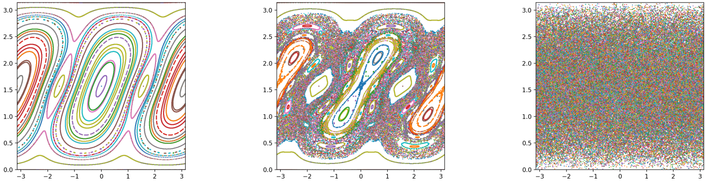
- Classical chaos
- Quantum chaos measures
- Krylov complexity
- Generalization of energy level statistics

Classical chaos

- Deterministic Hamiltonian evolution

$$\dot{q}^i = \partial H / \partial p_i, \qquad \dot{p}_j = -\partial H / \partial q^j$$

- Poincare sections



- Ergodicity and mixing

$$\lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{T} dt\, \mu \left[\left(\mathcal{T}^t A \right) \cap B \right] = \mu(A)\mu(B)$$

- Lyapunov exponent $(z^I = (q^i, p_j))$

$$\lim_{t\to\infty} \lim_{\|\delta \mathbf{z}_0\|\to 0} \frac{1}{\|\delta \mathbf{z}_0\|} \left\| \mathbf{z}(t; \mathbf{z}_0 + \delta \mathbf{z}_0) - \mathbf{z}(t; \mathbf{z}_0) \right\| \sim e^{\lambda t}, \qquad \lambda > 0$$

Quantum Lyapunov exponent

- Response matrix

$$\Phi^{IJ}(t; z_0) \coloneqq \frac{\partial z^I(t; z_0)}{\partial z_0^J}, \qquad \mathrm{tr}(\Phi^T \Phi) \sim e^{2\lambda t}$$

- In terms of canonical variables

$$\Phi^{IJ} = \frac{\partial z^I(t; z_0)}{\partial z_0^J} = \left\{ z^I(t, z_0), z_0^K \right\}_{z_0} (\pi^{-1})^{KJ}$$

- Canonical quantization

$$\left\{ z^I(t, z_0), z_0^K \right\}_{z_0} \quad \mapsto \quad -\frac{i}{\hbar}\left[\hat{z}^I(t), \hat{z}^K(0)\right]$$

- OTOC

$$C(t) = \sum_{I,J} \mathrm{tr}\left(\hat{\rho}\left[\hat{z}_I(t), \hat{z}_J(0)\right]\left[\hat{z}_J(0), \hat{z}_I(t)\right]\right)$$

- “Definition” of quantum chaos

$$C(t) \sim \begin{cases} e^{2\lambda t} & \text{— YES chaos} \\ t & \text{— NO chaos} \end{cases}$$

Matrix models

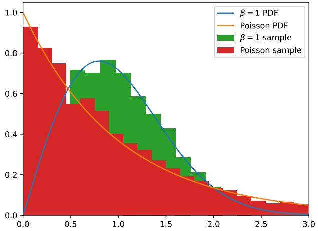
- Energy level statistics

$$H|\psi_k\rangle = \epsilon_k|\psi_k\rangle,$$

$$s_k = \epsilon_{k+1} - \epsilon_k$$

- Distribution for complex (chaotic) systems **[Wigner; Dyson]**

$$p(s) = a\, s^\beta \exp(-bs^2)$$



- This can be obtained in large N limit of Gaussian matrix distribution

$$Z_\beta = \int d^{N^2}X\; e^{-\operatorname{tr} X^2} \propto \int d^N\epsilon\, |\Delta(\epsilon_1, \ldots \epsilon_N)|^\beta\, e^{-\sum \epsilon_k^2}$$

OPRL and Krylov space

- Unitary evolution ($\mathcal{L}^\dagger = \mathcal{L}$)

$$|\Psi(t)\rangle = e^{it\mathcal{L}}|\Psi\rangle = \sum_{n=0}^{\infty} \frac{(it)^n}{n!} \mathcal{L}^n |\Psi\rangle$$

- Gram-Schmidt orthogonalization gives **Krylov basis** $\{|\Psi_n\rangle\}_{n=0}^{\infty}$.

$$\mathcal{L}|\Psi_n\rangle = b_n|\Psi_{n+1}\rangle + a_n|\Psi_n\rangle + b_{n-1}|\Psi_{n-1}\rangle, \quad |\Psi(t)\rangle = \sum_{n=0}^{\infty} \psi_n(t)\,|\Psi_n\rangle,$$

- Coefficients $\psi_n(t)$ satisfy Schroedinger-type equation

$$i\partial_t \psi_n(t) = b_n \psi_{n+1}(t) + a_n \psi_n(t) + b_{n-1}\psi_{n-1}(t)$$

Ergodic classification

- **Krylov entropy** S and **complexity** K

$$S(t) = -\sum_{n=0}^{\infty} |\psi_n(t)|^2 \log |\psi_n(t)|^2, \qquad K(t) = \sum_{n=0}^{\infty} n\, |\psi_n(t)|^2.$$

- Typical asymptotic behavior **[Rabinovici,Barbon,Sonner'21-22]**

$$K(t),\, e^{S(t)} \sim D^{\gamma},\; t > t_E \qquad K(t),\, e^{S(t)} \sim e^{\lambda t},\; t \ll t_E$$

  - $\gamma = 1,\ \lambda > 0$ for chaotic systems
  - $\gamma < 1$ for integrable systems

Distributions for tridiagonal form

- Gaussian matrix distribution in tridiagonal form **[Dumitriu,Edelman'99]**

$$Z_\beta = \int d^{N^2}X \, e^{-\operatorname{tr} X^2} \propto \int d^N a \, d^{N-1} b \left[\prod_k b_k^{\beta(k+1)-1} \right] e^{-\frac{1}{4}\sum a_k^2 - \frac{1}{2}\sum b_k^2}$$

- Distribution for tridiagonal form

$$H_\beta \sim \frac{1}{\sqrt{2}} \begin{pmatrix} N(0,2) & \chi_{(n-1)\beta} & & & \\ \chi_{(n-1)\beta} & N(0,2) & \chi_{(n-2)\beta} & & \\ & \ddots & \ddots & \ddots & \\ & & \chi_{2\beta} & N(0,2) & \chi_\beta \\ & & & \chi_\beta & N(0,2) \end{pmatrix}$$

- Quantum chaotic systems exhibit such behavior!
  [Balasubramian'22]

First attempt: Szego polynomials

- Try to orthogonalize $\{\mathcal{U}^n|\Phi\rangle\}_{n=0}^{\infty}$, for $\mathcal{U}^\dagger = \mathcal{U}^{-1}$,
- Gram-Schmidt orthogonalization leads to **[Szegö'67]**

$$\begin{pmatrix} |\Phi_{n+1}\rangle \\ |\Phi_{n+1}^*\rangle \end{pmatrix} = \frac{1}{\rho_n} \begin{pmatrix} \mathcal{U} & -\bar{\alpha}_n \\ -\alpha_n \mathcal{U} & 1 \end{pmatrix} \begin{pmatrix} |\Phi_n\rangle \\ |\Phi_n^*\rangle \end{pmatrix}, \quad \rho_n = \sqrt{1 - |\alpha_n|^2}$$

- However, this is not 3-term relation

$$\bar{\alpha}_{n-1}\rho_n \, |\Phi_{n+1}\rangle = (\bar{\alpha}_n + \bar{\alpha}_{n-1}\mathcal{U}) \, |\Phi_n\rangle - \bar{\alpha}_n \rho_{n-1}\mathcal{U} \, |\Phi_{n-1}\rangle$$

- Hessenberg matrix

$$\langle \Phi_i | \mathcal{U} | \Phi_j \rangle = \begin{cases} -\alpha_{i-1}\bar{\alpha}_j \prod_{l=i}^{j-1} \rho_l & i < j+1 \\ \rho_{j-1} & i = j+1 \\ 0 & i > j+1 \end{cases}$$

Success: CMV polynomials

- Try orthogonalize $\{|\Phi\rangle, \mathcal{U}|\Phi\rangle, \mathcal{U}^{-1}|\Phi\rangle, \mathcal{U}^{2}|\Phi\rangle, \mathcal{U}^{-2}|\Phi\rangle, \dots\}$,

$$|X_{2k}\rangle = \mathcal{U}^{-k}|\Phi^*_{2k}\rangle, \qquad |X_{2k-1}\rangle = \mathcal{U}^{-k+1}|\Phi_{2k-1}\rangle,$$

- It satisfies **5-term relation** **[Cantero,Moral,Velazquez'03]**

$$C_{kl} = \langle X_k|\mathcal{U}|X_l\rangle = \begin{pmatrix} \bar\alpha_0 & \bar\alpha_1\rho_0 & \rho_1\rho_0 & 0 & 0 & \dots \\ \rho_0 & -\bar\alpha_1\alpha_0 & -\rho_1\alpha_0 & 0 & 0 & \dots \\ 0 & \bar\alpha_2\rho_1 & -\bar\alpha_2\alpha_1 & \bar\alpha_3\rho_2 & \rho_3\rho_2 & \dots \\ 0 & \rho_2\rho_1 & -\rho_2\alpha_1 & -\bar\alpha_3\alpha_2 & -\rho_3\alpha_2 & \dots \\ 0 & 0 & 0 & \bar\alpha_4\rho_3 & -\bar\alpha_4\alpha_3 & \dots \\ \dots & \dots & \dots & \dots & \dots & \dots \end{pmatrix}$$

- CMV matrix C can be obtained as $C = L\,M$

$$L = 1 \oplus \Theta_1 \oplus \Theta_3 \oplus \cdots \qquad M = \Theta_0 \oplus \Theta_2 \oplus \Theta_4 \cdots \qquad \Theta_k = \begin{pmatrix} \bar\alpha_k & \rho_k \\ \rho_k & -\alpha_k \end{pmatrix}$$

- Matrix distribution counterpart **[Killip,Nenciu'04]**

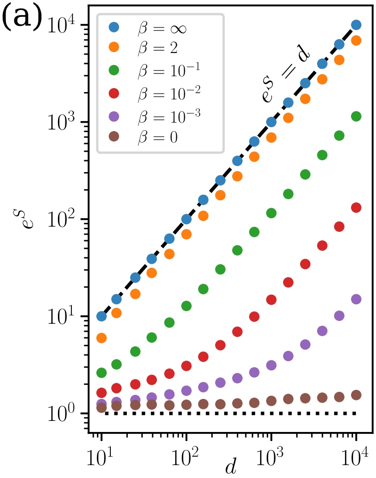
$$Z_\beta = \int [d\mathcal{U}_\beta]_{\text{Haar}} \propto \prod_{k=0}^{n-2} \left(1 - |\alpha_k|^2\right)^{\frac{\beta}{2}(n-k-1)-1} d^2\alpha_0 \cdots d^2\alpha_{n-2} d\phi$$

Classification suggestion

- Conjecture for classification based on **[Killip,Nenciu'06]**
  - (i) *Degenerate case*: $\langle|\alpha_n|^2\rangle < A/(d-n)^{1+\epsilon}$,
  - (ii) *Chaotic case*: $\langle|\alpha_n|^2\rangle \sim A/\left[\beta(d-n)/2+1\right]$,
  - (iii) *Integrable case*: $\langle|\alpha_n|^2\rangle > A/(d-n)^{1-\epsilon}$,
- Representative distributions

$$p_n^{(i)}(\alpha_n) \sim \left(1-|\alpha_n|^2\right)^{\left[(d-n)^{(1+\epsilon)}-1\right]},$$
$$p_n^{(ii)}(\alpha_n) \sim \left(1-|\alpha_n|^2\right)^{\left[\beta(d-n)/2-1\right]},$$
$$p_n^{(iii)}(\alpha_n) \sim \left(1-|\alpha_n|^2\right)^{\left[(d-n)^{(1-\epsilon)}-1\right]}.$$

Numerical tests

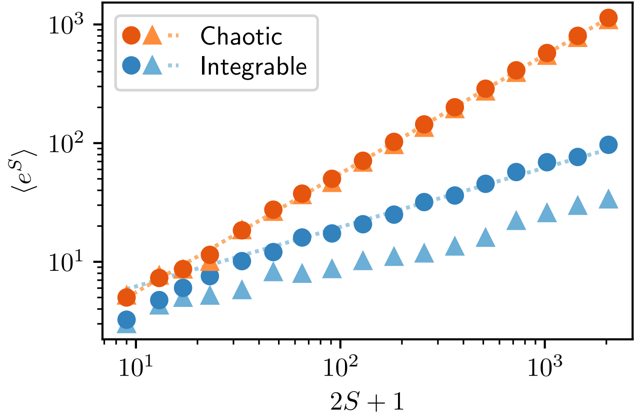


Application: Kicked Top

- Instructive example:

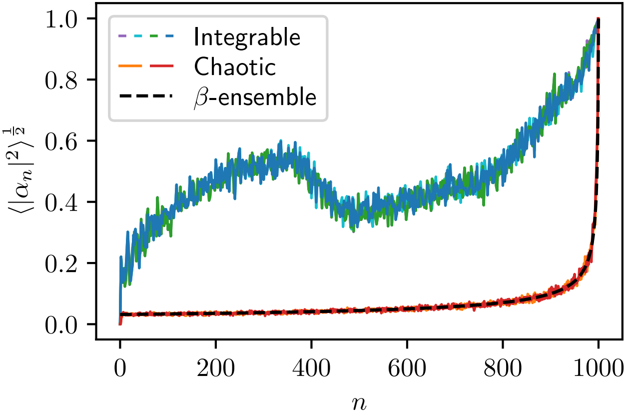
$$U_{\text{top}} = \exp\left(-i\frac{\kappa_x}{2J}J_x^2\right)\exp\left(-i\frac{\kappa_z}{2J}J_z^2\right)\exp[-ib\,(\mathbf{n}\cdot\mathbf{J})]$$

- Krylov entropy



Statistics of Verblunsky coefficients

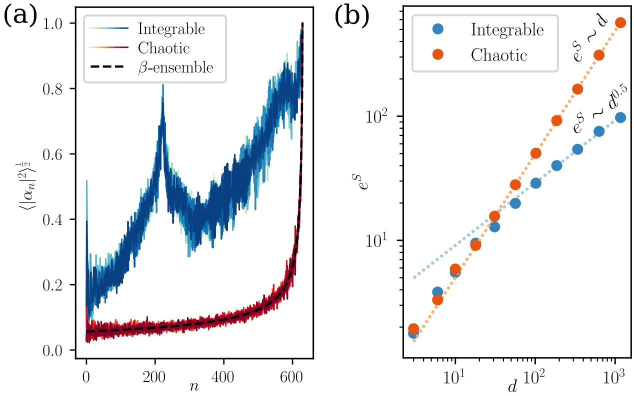
- Distribution of Verblunsky coefficients for Kicked top



Kicked Ising

- Evolution operator

$$U_{\mathrm{KI}} = \exp\left(-iJ\sum_{k=0}^{L-1}\sigma_k^z\sigma_{k+1}^z\right)\exp\left[-i\sum_{k=0}^{L-1}(\mathbf{b}\cdot\sigma_k)\right]$$



CMV integrability: Ablowitz-Ladik system

- Like OPRL, OPUC also admit integrable measure modifications. Consider the following one

$$d\nu(z) \quad \mapsto \quad d\nu_\tau(z) = e^{\tau(z+z^{-1})}\, d\nu(z)$$

- Then, Verblunsky coefficients satisfy Ablowitz-Ladik equation **[Killip,Nenciu'05]**

$$\frac{d}{d\tau}\alpha_n(\tau) = \left(1 - |\alpha_n(\tau)|^2\right)\left(\alpha_{n+1}(\tau) - \alpha_{n-1}(\tau)\right)$$

which can be rewritten in Lax form $\dot{C} = [A, C]$.

- In terms of operators, this measure modification corresponds to

$$O \quad \mapsto \quad O_\tau = e^{\tau\,\mathcal{U}} O,$$

whose physical meaning and importance for spectrum investigation is still not clear.

Time evolution and small-time limit I

- It would be interesting to examine the reduction of OPUC case to OPRL one in the following sense. Consider $\mathcal{U}_t = e^{it\mathcal{L}}$ for some time-independent Hermitian $\mathcal{L}$
- Knowing Lanczos coefficients a_n, b_n, constructed by $\mathcal{L}$, can we reconstruct Verblunsky coefficients $\alpha_n(t)$?
- This problem reduces to the following problem, namely, one should reduce the exponential of Jacobi matrix

$$U(t) = e^{itJ}, \qquad J = \begin{pmatrix} b_1 & a_1 & 0 & \dots \\ a_1 & b_2 & a_2 & \dots \\ 0 & a_2 & b_3 & \dots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix}$$

and reduce to the CMV form $C = QUQ^\dagger$ by uniqely defined unitary matrix Q. Then, recover Verblunsky coefficients from C.

Time evolution and small-time limit II

- We can solve problem perturbatively in powers of t

$$\bar{\alpha}_n(t) = (-)^n \Big(1 - it\,\bar{\alpha}_n^{(1)} - \frac{1}{2}t^2\,\bar{\alpha}_n^{(2)} + \dots\Big)$$

The result reads

$$\bar{\alpha}_n^{(1)} = \sum_{k=0}^{n} a_k, \qquad \bar{\alpha}_n^{(2)} = b_n^2 + (\bar{\alpha}_n^{(1)})^2$$

The calculation of next coefficients is technically extremely hard.

- Such defined CMV matrix obey the equation

$$\partial_t C = [B, C] + t\,C \log C$$

which is not of the Lax form.

- For particular OPRL the problem can be solved explicitly, e.g. Hermite OPRL lead to Szego-Rogers OPUC with $q = e^{-t^2/2}$.

Conclusions and future directions

Done:

- Probes of chaos for Floquet systems
  - Quantum entropy and complexity
  - Statistics of Verblunski coefficients

Not done:

- Application to mixed phase-space systems and time crystals

Thank you for your attention!

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