# "Metric-affine-like" generalization of YM

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# On notations

- I consider purely classical theory everywhere. Space-time is $d$-dimensional with signature $(-, +, \ldots, +)$.
- Space-time indices are denoted throughout by lowercase Latin letters $a, b, c, \ldots$, and internal color indices are denoted by lowercase Greek letters $\alpha, \beta, \gamma, \ldots$
- I do not use bases or component notation anywhere, so indices should be understood as abstract throughout. But if this is unusual, they can be understood simply as components with respect to some basis.
- When moving to abbreviated indexless notation, we will use the sign $\cong$, for example, $\boldsymbol{\varphi} \cong \varphi^{\alpha}$.
- Of course, all the same can be written in other notations (using tetrads, differential forms, etc.). But do not ask me to rewrite something during the report in the notations that you like — I will explain my definitions in detail.

# Motivation

| | Similarities | Differences |
|---|---|---|
| YM | $\nabla_a \boldsymbol{\varphi} = (\partial_a - ie\boldsymbol{A}_a)\boldsymbol{\varphi}$, where $\boldsymbol{\varphi} \cong \varphi^\alpha$, $\boldsymbol{A}_a \cong A_{a\alpha}{}^\beta$, $\boldsymbol{F}_{ab} = \partial_a \boldsymbol{A}_b - \partial_b \boldsymbol{A}_a - ie\left[\boldsymbol{A}_a, \boldsymbol{A}_b\right]$. | DoFs in $\boldsymbol{A}_a$, $S_{\mathrm{YM}} = -\frac{1}{4}\int d^d x\,\sqrt{g}\,\mathrm{tr}\left(\boldsymbol{F}_{ab}\boldsymbol{F}^{ab}\right)$. |
| EG | $\nabla_a v^b = \partial_a v^b + \Gamma_{ac}{}^b v^c$, $R_{abc}{}^d = \partial_a \Gamma_{bc}{}^d - \partial_b \Gamma_{ac}{}^d + \Gamma_{ah}{}^d \Gamma_{bc}{}^h - \Gamma_{bh}{}^d \Gamma_{ac}{}^h$. | DoFs not in $\Gamma_{ac}{}^b$, but in $g_{ab}$, $S_{\mathrm{HE}} = \frac{M_P^2}{2}\int d^d x\,\sqrt{g}\,R$. |

- The Levi-Civita connection is defined by the torsion-free and covariant constancy of metric conditions:
  $T_{ab}{}^c = 0, \quad \nabla_a g_{bc} = 0 \quad \Rightarrow \quad \Gamma_{ac}{}^b = \frac{1}{2} g^{bd}\left(\partial_a g_{cd} + \partial_c g_{ad} - \partial_d g_{ac}\right).$
- If we want to make the two theories even more similar, we should treat the connection $\nabla_a$ and the metric $g_{ab}$ as two independent variables, i.e. $\nabla_a g_{bc} \neq 0$. This approach is well and long known — metric-affine gravity (MAG).
- But we are now interested in a simpler case: how to construct a "metric-affine-like" generalization of YM? Who is the "partner" of the potential $\boldsymbol{A}_a$ in this case?

# Who is the “partner” of the potential $\boldsymbol{A}_a$?

Let $V$ be the internal color space (or typical fiber of the bundle) on which the fundamental representation of the group acts.
For definiteness, we will consider $U(n)$ throughout (the generalization to other groups is straightforward). Then $V$ is a complex $n$-dimensional space.

## A subtle point:

The complex conjugation cannot map $V \to V$. Instead, it is an antilinear bijection into the complex conjugate space $\bar{V}$. Let us construct a tensor algebra from the spaces $V$ and $\bar{V}$: $V$ is assigned to the unprimed indices $\alpha, \beta, \gamma, \ldots$, and $\bar{V}$ is assigned to the primed indices $\alpha', \beta', \gamma', \ldots$. These are different types of indices, so:

- they cannot be contracted,
- they can be rearranged,
- complex conjugation changes unprimed indices into primed ones and vice versa: $\overline{H^{\gamma\ldots\delta'\ldots}_{\alpha\ldots\beta'\ldots}} = \bar{H}^{\gamma'\ldots\delta\ldots}_{\alpha'\ldots\beta\ldots}$.

(This is a direct analogue of the undotted and dotted indices in 2-spinors.)

# How to construct a real action from $\varphi^\alpha$?

## Hermitian form

For this, we need not only the conjugate scalar $\bar{\varphi}^{\alpha'}$, but also the form $g_{\alpha\beta'}$, which is:

- Hermitian $\bar{g}_{\alpha\beta'} = g_{\alpha\beta'}$ (analogous to the symmetry of the metric),
- non-degenerate $g_{\alpha\beta'}g^{\beta\beta'} = \delta^\beta_\alpha$ (allows us to raise and lower indices, with primed ones becoming unprimed and vice versa, e.g., $\bar{\varphi}_\alpha = g_{\alpha\beta'}\bar{\varphi}^{\beta'}$).

## Lagrangian of a charged scalar:

$$|\varphi|^2 = g_{\alpha\beta'}\varphi^\alpha\bar{\varphi}^{\beta'},$$
$$\mathcal{L}_\varphi = -\frac{1}{2}g_{\alpha\beta'}g^{ab}\nabla_a\varphi^\alpha\nabla_b\bar{\varphi}^{\beta'} - P\big(|\varphi|^2\big),$$

where $P\big(|\varphi|^2\big)$ is a self-interaction potential.

# Summary of main claims and results

- In the standard Yang-Mills theory, it is always implicitly assumed that the structure in the fibers is covariantly constant $\nabla_a g_{\alpha\beta'} = 0$.
- Accordingly, the "metric-affine-like" generalization of YM consists in dropping this condition $\nabla_a g_{\alpha\beta'} \neq 0$. Then the connection $\nabla_a$ and the Hermitian form $g_{\alpha\beta'}$ act as two independent variables.
- Any geometrically defined theory always has a general $GL(n,\mathbb{C})$ gauge symmetry. The Hermitian form $g_{\alpha\beta'}$ plays the role of a "Higgs field", spontaneously breaking this symmetry to $U(n)$.
- If the connection respects the structure in fibers, the potential and the curvature take values in the corresponding Lie algebra. In our case this is not so, then along with the usual Yang-Mills fields $\boldsymbol{A}_a$ and $\boldsymbol{F}_{ab}$, they have new Hermitian parts $\boldsymbol{B}_a$ and $\boldsymbol{G}_{ab}$.
- The fields $\boldsymbol{A}_a$ and $\boldsymbol{B}_a$ interact non-trivially. The field $\boldsymbol{A}_a$ is massless, and the field $\boldsymbol{B}_a$ can be given a mass $M$. The limit $M \to \infty$ restores the usual YM.

# Outline

- Motivation and summary of results
- Connection $\nabla_a$ (without $g_{\alpha\beta'}$ yet)
- $+$ Hermitian form $g_{\alpha\beta'}$
- Gauge symmetry and Noether identities
- Action and equations of motion
- Problems with propagators and gauge fixing

# Connection

## Definitions of potentials and curvatures

Let us define $\mathcal{A}_a[\tilde{\nabla} - \nabla] \cong \mathscr{A}_{a\alpha}{}^{\beta}$ and $\mathcal{F}_{ab}[\nabla] \cong \mathscr{F}_{ab\alpha}{}^{\beta}$ as

$$(\tilde{\nabla}_a - \nabla_a)\psi^\alpha = \mathscr{A}_{a\beta}{}^{\alpha}\psi^\beta, \qquad [\nabla_a, \nabla_b]\psi^\alpha = \mathscr{F}_{ab\beta}{}^{\alpha}[\nabla]\psi^\beta.$$

We assume everywhere that the space is flat:
$T^c_{ab} \equiv 0$, $R_{abc}{}^{d} \equiv 0$ (and $g_{ab} = \eta_{ab}$).

## Curvatures transformations and Bianchi identities

$$\mathcal{F}_{ab}[\tilde{\nabla}] - \mathcal{F}_{ab}[\nabla] = \nabla_a \mathcal{A}_b - \nabla_b \mathcal{A}_a + [\mathcal{A}_a, \mathcal{A}_b], \qquad \nabla_{[a}\mathcal{F}_{bc]} = 0.$$

## Important!

Except the anti-symmetry in the first pair of indices and the Bianchi identities, no additional conditions are imposed on the curvature $\mathcal{F}_{ab}[\nabla]$, this is an arbitrary tensor. In particular, it is not an anti-Hermitian matrix (without $g_{\alpha\beta'}$ it is impossible to even define this concept!)

# General $GL(n,\mathbb{C})$ gauge symmetry

Let $\boldsymbol{u} \cong u_\alpha^\beta$ and $\boldsymbol{U} \cong U_\alpha^\beta$ be two arbitrary mutually inverse matrices: $\boldsymbol{uU} = \boldsymbol{Uu} = \boldsymbol{1}$.
Consider invertible linear transformations of the internal color space:

$$H_{\alpha_1\dots\alpha_p}^{\beta_1\dots\beta_q} \mapsto \tilde{H}_{\alpha_1\dots\alpha_p}^{\beta_1\dots\beta_q} = U_{\gamma_1}^{\beta_1}\cdots U_{\gamma_q}^{\beta_q}\, H_{\delta_1\dots\delta_p}^{\gamma_1\dots\gamma_q}\, u_{\alpha_1}^{\delta_1}\cdots u_{\alpha_p}^{\delta_p}.$$

In particular, for a matrix $\boldsymbol{M}$ we will have $\tilde{\boldsymbol{M}} = \boldsymbol{UMu}$.
This transformation preserves contractions (e.g. $\tilde{\chi}_\alpha\tilde{\psi}^\alpha = \chi_\alpha\psi^\alpha$).
Hence, this is a symmetry of *of any* action.
But in order for $\nabla_a\psi^\alpha$ to transform in the same way as $\psi^\alpha$, we must also transform the connection $\nabla_a \mapsto \tilde{\nabla}_a$:

$$\mathcal{A}_a[\tilde{\nabla} - \nabla] = \boldsymbol{U}\nabla_a\boldsymbol{u}.$$

It is easy to show that such a transformation is self-consistent. In particular,

$$\mathcal{F}_{ab}[\tilde{\nabla}] = \boldsymbol{U}\mathcal{F}_{ab}[\nabla]\boldsymbol{u}.$$

## Infinitesimal form:

$$\boldsymbol{u} = \boldsymbol{1} + \boldsymbol{\epsilon}, \qquad \boldsymbol{U} = \boldsymbol{1} - \boldsymbol{\epsilon},$$

$$\mathcal{A}_a[\tilde{\nabla} - \nabla] = \nabla_a\boldsymbol{\epsilon}, \qquad \delta\boldsymbol{M} = \tilde{\boldsymbol{M}} - \boldsymbol{M} = [\boldsymbol{M}, \boldsymbol{\epsilon}].$$

# + Hermitian form $g_{\alpha\beta'}$

The connection $\nabla_a$ is real in the following sense:

$$\overline{\nabla_a \varphi^\alpha} = \nabla_a \bar{\varphi}^{\alpha'} \quad \Rightarrow \quad (\tilde{\nabla}_a - \nabla_a)\bar{\varphi}^{\alpha'} = \bar{\mathscr{A}}_{a\beta'}{}^{\alpha'}\bar{\varphi}^{\beta'}.$$

## Hermitian conjugation

We call the "matrix" $\boldsymbol{M} \cong M^\beta_\alpha$ the map $V \to V$. The matrix product = contraction $\boldsymbol{MN} \cong M^\beta_\gamma N^\gamma_\alpha$.

Note that the complex conjugate of $\bar{M}^{\beta'}_{\alpha'}$ is a mapping $\bar{V} \to \bar{V}$, it is not a "matrix" in this sense.

To define an operation from matrices to matrices, we must, along with the complex conjugate, use the "transposition", i.e. the contractions with the Hermitian form $g_{\alpha\beta'}$:

$$\boldsymbol{M}^\dagger \cong \bar{M}^\beta_\alpha = g_{\alpha\alpha'} g^{\beta\beta'} \bar{M}^{\alpha'}_{\beta'}.$$

## Split into Hermitian and anti-Hermitian parts:

$$\boldsymbol{M} = \boldsymbol{b} - i\boldsymbol{a}, \qquad \boldsymbol{b} = \mathsf{Hrm}\,\boldsymbol{M} = \frac{1}{2}(\boldsymbol{M} + \boldsymbol{M}^\dagger), \qquad \boldsymbol{a} = \mathsf{aHrm}\,\boldsymbol{M} = \frac{i}{2}(\boldsymbol{M} - \boldsymbol{M}^\dagger),$$

$$\mathcal{A}_a = \boldsymbol{B}_a - i\boldsymbol{A}_a, \qquad \boldsymbol{B}_a = \mathsf{Hrm}\,\mathcal{A}_a = \frac{1}{2}(\mathcal{A}_a + \mathcal{A}^\dagger_a), \qquad \boldsymbol{A}_a = \mathsf{aHrm}\,\mathcal{A}_a = \frac{i}{2}(\mathcal{A}_a - \mathcal{A}^\dagger_a),$$

$$\mathcal{F}_{ab} = \boldsymbol{G}_{ab} - i\boldsymbol{F}_{ab}, \qquad \boldsymbol{G}_{ab} = \mathsf{Hrm}\,\mathcal{F}_{ab} = \frac{1}{2}(\mathcal{F}_{ab} + \mathcal{F}^\dagger_{ab}), \qquad \boldsymbol{F}_{ab} = \mathsf{aHrm}\,\mathcal{F}_{ab} = \frac{i}{2}(\mathcal{F}_{ab} - \mathcal{F}^\dagger_{ab}).$$

# YM-deviation vector $\boldsymbol{N}_a$

## Definition

$$\boldsymbol{N}_a \cong N_a{}^{\beta}_{\alpha} = -\frac{1}{2} g^{\beta\beta'} \nabla_a g_{\alpha\beta'}.$$

This is a Hermitian vector — the analogue of non-metricity in MAG.

## Hermiticity and derivatives

If $\boldsymbol{N}_a \neq 0$, then the operations of raising/lowering indices and covariant derivative no longer commute with each other, e.g., $\nabla_a \psi_{\alpha'} \neq g_{\alpha\alpha'} \nabla_a \psi^{\alpha}$.
It is especially important that derivative does not commute with the Hermitian conjugation:

$$\nabla_a \left( \boldsymbol{M}^{\dagger} \right) = (\nabla \boldsymbol{M})^{\dagger} + 2 \left[ \boldsymbol{N}_a, \boldsymbol{M}^{\dagger} \right].$$

## The key relation

The Hermitian part of the curvature of $\boldsymbol{G}_{ab}$ is completely expressed in terms of the YM-deviation vector $\boldsymbol{N}_a$:

$$\boldsymbol{G}_{ab} = \nabla_a \boldsymbol{N}_b - \nabla_b \boldsymbol{N}_a - 2 \left[ \boldsymbol{N}_a, \boldsymbol{N}_b \right].$$

Proof: $[\nabla_a, \nabla_b] g_{\alpha\beta'} = -\mathscr{F}_{ab}{}^{\gamma}_{\alpha} g_{\gamma\beta'} - \bar{\mathscr{F}}_{ab}{}^{\gamma'}_{\beta'} g_{\alpha\gamma'} = -2 G_{ab\alpha\beta'} = -2(\nabla_a N_{b\alpha\beta'} - \nabla_b N_{a\alpha\beta'})$.

# Hermitian form transformations

We know how to describe the transformation $\nabla_a \mapsto \tilde{\nabla}_a$. But how to describe the transformation $g_{\alpha\beta'} \mapsto \tilde{g}_{\alpha\beta'}$?

## Matrices $\boldsymbol{\omega}$ and $\boldsymbol{\Omega}$

Let us define

$$\boldsymbol{\omega} \cong \omega^{\beta}_{\alpha} = \tilde{g}_{\alpha\beta'} g^{\beta\beta'}, \qquad \boldsymbol{\Omega} \cong \Omega^{\beta}_{\alpha} = g_{\alpha\beta'} \tilde{g}^{\beta\beta'}, \qquad \tilde{g}_{\alpha\beta'} = \omega^{\beta}_{\alpha} g_{\beta\beta'}.$$

It is easy to show that these matrices are Hermitian and mutually inverse

$$\boldsymbol{\omega}^{\dagger} = \boldsymbol{\omega}, \qquad \boldsymbol{\Omega}^{\dagger} = \boldsymbol{\Omega}, \qquad \boldsymbol{\omega}\boldsymbol{\Omega} = \boldsymbol{\Omega}\boldsymbol{\omega} = \mathbf{1}.$$

We can expand them in terms of small perturbations

$$\boldsymbol{\omega} = \mathbf{1} + \boldsymbol{h}, \qquad \boldsymbol{\Omega} = (\mathbf{1} + \boldsymbol{h})^{-1} = \sum_{k=0}^{\infty} (-\boldsymbol{h})^{k}.$$

## YM-deviation vector transformation

$$\boldsymbol{N}_a[\tilde{\nabla}, \tilde{g}] = \boldsymbol{\Omega} \boldsymbol{N}_a[\nabla, g] \boldsymbol{\omega} - \frac{1}{2} \boldsymbol{\Omega} \nabla_a \boldsymbol{\omega} + \frac{1}{2} \big( \mathcal{A}_a + \boldsymbol{\Omega} \mathcal{A}^{\dagger}_a \boldsymbol{\omega} \big),$$

$$\delta_g \boldsymbol{N}_a = -\frac{1}{2} \nabla_a \boldsymbol{h} + [\boldsymbol{N}_a, \boldsymbol{h}], \qquad \delta_B \boldsymbol{N}_a = \boldsymbol{B}_a, \qquad \delta_A \boldsymbol{N}_a = 0.$$

# General field transformations of $\boldsymbol{F}_{ab}$ and $\boldsymbol{G}_{ab}$

## Transformations with Hermitian form

The total curvature $\mathcal{F}_{ab}[\nabla]$ does not depend on $g_{\alpha\beta'}$ at all, and for its Hermitian conjugation the transformation is simple:

$$\mathcal{F}^{\dagger}_{ab}[\nabla, \tilde{g}] = \boldsymbol{\Omega}\mathcal{F}^{\dagger}_{ab}[\nabla, g]\boldsymbol{\omega}.$$

## Transformations with connection:

$$\boldsymbol{G}_{ab}[\tilde{\nabla}] - \boldsymbol{G}_{ab}[\nabla] = \check{\boldsymbol{D}}_{ab} + i\check{\boldsymbol{K}}_{ab} - \hat{\boldsymbol{K}}_{ab} - \boldsymbol{C}_{ab},$$
$$\boldsymbol{F}_{ab}[\tilde{\nabla}] - \boldsymbol{F}_{ab}[\nabla] = \hat{\boldsymbol{D}}_{ab} + i\hat{\boldsymbol{K}}_{ab} + \check{\boldsymbol{K}}_{ab} + \check{\boldsymbol{C}}_{ab} - \hat{\boldsymbol{C}}_{ab}.$$

Where we introduce auxiliary quantities:

$$\check{\boldsymbol{D}}_{ab} = \nabla_a \boldsymbol{B}_b - \nabla_b \boldsymbol{B}_a, \qquad \check{\boldsymbol{K}}_{ab} = i[\boldsymbol{N}_a, \boldsymbol{B}_b] - i[\boldsymbol{N}_b, \boldsymbol{B}_a],$$
$$\hat{\boldsymbol{D}}_{ab} = \nabla_a \boldsymbol{A}_b - \nabla_b \boldsymbol{A}_a, \qquad \hat{\boldsymbol{K}}_{ab} = i[\boldsymbol{N}_a, \boldsymbol{A}_b] - i[\boldsymbol{N}_b, \boldsymbol{A}_a],$$
$$\check{\boldsymbol{C}}_{ab} = i[\boldsymbol{B}_a, \boldsymbol{B}_b], \qquad \hat{\boldsymbol{C}}_{ab} = i[\boldsymbol{A}_a, \boldsymbol{A}_b], \qquad \boldsymbol{C}_{ab} = i[\boldsymbol{A}_a, \boldsymbol{B}_b] - i[\boldsymbol{A}_b, \boldsymbol{B}_a].$$

# $GL(n,\mathbb{C}) \to U(n)$ spontaneous symmetry breaking

## Gauge transformations of the Hermitian form

$GL(n,\mathbb{C})$ gauge transformations, generally speaking, change the Hermitian form:

$$g_{\alpha\alpha'} \mapsto \tilde{g}_{\alpha\alpha'} = u_{\alpha}^{\beta}\bar{u}_{\alpha'}^{\beta'} g_{\beta\beta'} \quad \Rightarrow \quad \boldsymbol{\omega} = \boldsymbol{u}^{\dagger}\boldsymbol{u}, \quad \boldsymbol{\Omega} = \boldsymbol{U}\boldsymbol{U}^{\dagger}.$$

Hence, $g_{\alpha\beta'}$ does not change if the transformations are unitary $\boldsymbol{U} = \boldsymbol{u}^{\dagger}$.
In infinitesimal form:

$$\boldsymbol{\epsilon} = \boldsymbol{\beta} - i\boldsymbol{\alpha}, \quad \boldsymbol{\beta} = \mathsf{Hrm}\,\boldsymbol{\epsilon}, \quad \boldsymbol{\alpha} = \mathsf{aHrm}\,\boldsymbol{\epsilon} \quad \Rightarrow \quad \boldsymbol{h} = 2\boldsymbol{\beta}.$$

Therefore, $g_{\alpha\beta'}$ is a “Higgs field”, breaking $GL(n,\mathbb{C})$ to $U(n)$.

## Infinitesimal transformations:

$$\boldsymbol{A}_a = \nabla_a\boldsymbol{\alpha} - [\boldsymbol{N}_a,\boldsymbol{\alpha}] + i[\boldsymbol{N}_a,\boldsymbol{\beta}], \quad \boldsymbol{B}_a = \nabla_a\boldsymbol{\beta} - [\boldsymbol{N}_a,\boldsymbol{\beta}] - i[\boldsymbol{N}_a,\boldsymbol{\alpha}].$$

In this case, all matrices are transformed simply by similarity transformations:

$$\delta\boldsymbol{N}_a = [\boldsymbol{N}_a,\boldsymbol{\epsilon}], \qquad \delta\boldsymbol{F}_{ab} = [\boldsymbol{F}_{ab},\boldsymbol{\epsilon}], \qquad \delta\boldsymbol{G}_{ab} = [\boldsymbol{G}_{ab},\boldsymbol{\epsilon}].$$

Note that if $\boldsymbol{N}_a \neq 0$ or $\boldsymbol{G}_{ab} \neq 0$ they cannot be removed by gauge transformations.

# Field sources and Noether identities

## Field sources:

$$\mathcal{J}^a = -2\frac{\delta S}{\delta \mathcal{A}_a} = \boldsymbol{\Lambda}^a - i\boldsymbol{J}^a,$$

$$\boldsymbol{\Lambda}^a = -\frac{\delta S}{\delta \boldsymbol{B}_a}, \qquad \boldsymbol{J}^a = \frac{\delta S}{\delta \boldsymbol{A}_a}, \qquad \boldsymbol{E} = -2\frac{\delta S}{\delta \boldsymbol{h}}.$$

## Charged scalar

$$\mathcal{L}_\varphi = -\frac{1}{2} g_{\alpha\beta'} \nabla_a \varphi^\alpha \nabla^a \bar{\varphi}^{\beta'} - P\left(|\varphi|^2\right) \quad \Rightarrow \quad \boldsymbol{E} \cong E^\beta_\alpha = g_{\alpha\beta'} \nabla_a \varphi^\beta \nabla^a \bar{\varphi}^{\beta'} + 2P' \bar{\varphi}_\alpha \varphi^\beta,$$

$$\boldsymbol{J}_a \cong J_{a\alpha}{}^\beta = \frac{i}{2} g_{\alpha\beta'} \left( \varphi^\beta \nabla_a \bar{\varphi}^{\beta'} - \bar{\varphi}^{\beta'} \nabla_a \varphi^\beta \right), \qquad \boldsymbol{\Lambda}_a \cong \Lambda_{a\alpha}{}^\beta = \frac{1}{2} g_{\alpha\beta'} \nabla_a (\bar{\varphi}^{\beta'} \varphi^\beta).$$

## Noether identities (pure mal-YM without matter):

If the theory has a gauge symmetry, the sources are not independent, but are related by Noether identities.

$$\nabla_a \boldsymbol{J}^a - [\boldsymbol{N}_a, \boldsymbol{J}^a] + i[\boldsymbol{N}_a, \boldsymbol{\Lambda}^a] = 0,$$

$$\nabla_a \boldsymbol{\Lambda}^a - [\boldsymbol{N}_a, \boldsymbol{\Lambda}^a] - i[\boldsymbol{N}_a, \boldsymbol{J}^a] = \boldsymbol{E}.$$

# The action we will consider

$$\mathcal{L}_1[\nabla] = -\frac{1}{8}\,\mathrm{tr}\left(\mathcal{F}_{ab}\mathcal{F}^{ab} + \bar{\mathcal{F}}_{ab}\bar{\mathcal{F}}^{ab}\right), \quad \mathcal{L}_{F^2} = -\frac{1}{4}\,\mathrm{tr}\left(\boldsymbol{F}_{ab}\boldsymbol{F}^{ab}\right) = \frac{1}{2}(\mathcal{L}_2 - \mathcal{L}_1),$$

$$\mathcal{L}_2[\nabla, g] = -\frac{1}{4}\,\mathrm{tr}(\mathcal{F}^{\dagger}_{ab}\mathcal{F}^{ab}), \qquad \mathcal{L}_{G^2} = -\frac{1}{4}\,\mathrm{tr}\left(\boldsymbol{G}_{ab}\boldsymbol{G}^{ab}\right) = \frac{1}{2}(\mathcal{L}_2 + \mathcal{L}_1).$$

In addition, we would also like to introduce a term that would allow us to restore the usual YM in some limit (i.e. $\boldsymbol{N}_a = 0$). It is natural to do this as follows:

$$\mathcal{L}_3[\nabla, g] = \mathcal{L}_{N^2} = -\frac{1}{2}\,\mathrm{tr}\left(\boldsymbol{N}_a\boldsymbol{N}^a\right).$$

As a result, we will have the following total Lagrangian of mal-YM:

$$\mathcal{L}_{\mathrm{malYM}} = c_1\mathcal{L}_1 + c_2\mathcal{L}_2 + c_3\mathcal{L}_3 = \frac{1}{e^2}\mathcal{L}_{F^2} + \frac{1}{\tilde{e}^2}\mathcal{L}_{G^2} + \frac{M^2}{\tilde{e}^2}\mathcal{L}_{N^2},$$

$$\frac{1}{e^2} = c_2 - c_1, \qquad \frac{1}{\tilde{e}^2} = c_1 + c_2, \qquad M^2 = \frac{c_3}{c_1 + c_2}.$$

Of course, other terms can be introduced into the action (for example, $\mathrm{tr}(\boldsymbol{F}_{ab}\boldsymbol{G}^{ab})$, $\mathrm{tr}(\boldsymbol{F}_{ab}[\boldsymbol{N}^a, \boldsymbol{N}^b]$, etc.). We will not do this for the sake of simplicity, in order not to clutter the presentation.

# Equations of Motion (EoMs)

## EoMs for background fields

$$c_1 \nabla^b \mathcal{F}_{ab} + c_2 \nabla^b \mathcal{F}^\dagger_{ab} + c_3 \boldsymbol{N}_a = -\mathcal{J}^{\text{ext}}_a,$$
$$c_3 \nabla_a \boldsymbol{N}^a + i c_2 [\boldsymbol{G}_{ab}, \boldsymbol{F}^{ab}] = -\boldsymbol{E}^{\text{ext}}.$$

It is easy to verify that, as it should be, the second equation is not independent, but is a differential consequence of the first. If we split the first equation into a Hermitian and anti-Hermitian part, we get:

$$\nabla^b \boldsymbol{F}_{ab} - [\boldsymbol{N}^b, \boldsymbol{F}_{ab}] - i \frac{e^2}{\tilde{e}^2} [\boldsymbol{N}^b, \boldsymbol{G}_{ab}] = e^2 \boldsymbol{J}^{\text{ext}}_a,$$
$$\nabla^b \boldsymbol{G}_{ab} - [\boldsymbol{N}^b, \boldsymbol{G}_{ab}] + i \frac{\tilde{e}^2}{e^2} [\boldsymbol{N}^b, \boldsymbol{F}_{ab}] + M^2 \boldsymbol{N}_a = -\tilde{e}^2 \boldsymbol{\Lambda}^{\text{ext}}_a.$$

## Linearized equations for small perturbations

For simplicity, we write them on a trivial background $\boldsymbol{N}_a = 0$, $\boldsymbol{G}_{ab} = \boldsymbol{F}_{ab} = 0$ (here $\Box = -g^{ab} \nabla_a \nabla_b$):

$$(\delta^b_a \Box + \nabla^b \nabla_a) \boldsymbol{A}_b = 0,$$
$$(\delta^b_a \Box + \nabla^b \nabla_a) \boldsymbol{B}_b + M^2 (\boldsymbol{B}_a - \frac{1}{2} \nabla_a \boldsymbol{h}) = 0,$$
$$\Box \boldsymbol{h} + 2 \nabla_a \boldsymbol{B}^a = 0.$$

# Analysis of EoMs

- Consequences of gauge symmetry:
  - The gauge transformations $\boldsymbol{h} = 2\boldsymbol{\beta}$, $\boldsymbol{A}_a = \nabla_a \boldsymbol{\alpha}$ and $\boldsymbol{B}_a = \nabla_a \boldsymbol{\beta}$ are solutions (for a nontrivial background this is also true).
  - The third equation is a differential consequence.
  - The wave operator is degenerate, which requires a gauge fixing procedure.
- If we consider only the term $\mathcal{L}_{F^2}$, then we have

$$\nabla^b \boldsymbol{F}_{ab} = ie^2 \mathcal{J}_a^{\text{ext}}, \qquad [\boldsymbol{N}^b, \boldsymbol{F}_{ab}] = ie^2 \boldsymbol{\Lambda}_a^{\text{ext}}.$$

  That is, the field $\boldsymbol{G}_{ab}$ will not be dynamic, but only modifies the interaction with external fields. If we consider only the term $\mathcal{L}_{G^2}$, it will be the opposite.
- If we consider both terms $\mathcal{L}_{F^2}$ and $\mathcal{L}_{G^2}$, then we have two interacting massless gauge fields. If we add the third term $\mathcal{L}_{N^2}$, then the second field $\boldsymbol{B}_a$ acquires the mass $M$.
- If we seek solutions of pure mal-YM with $\boldsymbol{N}_a = 0$ (so $\boldsymbol{G}_{ab} = 0$), then we obtain $\nabla^b \boldsymbol{F}_{ab} = 0$. Thus, any solution of pure YM is also a solution of pure mal-YM.
- However, if $\boldsymbol{\Lambda}_a^{\text{ext}} \neq 0$, then also $\boldsymbol{N}_a \neq 0$.
- Finally, in the limit $M \to \infty$ we obtain $\boldsymbol{N}_a = 0$, i.e. we restore the usual YM.

# Absorption of Goldstone bosons — $\boldsymbol{h} = 0$ gauge

If the Hermitian form $g_{\alpha\beta'}$ is a Higgs field, then its small perturbation $\boldsymbol{h}$ is a Goldstone boson. However, it is well known that one can always use spontaneously broken gauge symmetry to completely eliminate Goldstone bosons by redefining all other fields.
The same can be done in our case: for an arbitrary transformation of the Hermitian form $\boldsymbol{\omega}$, one must extract its Hermitian square root $\boldsymbol{\omega} = \boldsymbol{u}\boldsymbol{u}^\dagger$. Then the gauge transformation given by the matrix $\boldsymbol{u}$ will exactly reproduce the variation of $g_{\alpha\beta'}$, and the transformations of all other fields must be subtracted from their variations. The action will then depend only on these differences, not on $\boldsymbol{h}$.
In the infinitesimal case, this simply amounts to redefining $\boldsymbol{B}_a \mapsto \boldsymbol{B}_a - \nabla_a \boldsymbol{h}/M$, which leads to the equations

$$(\delta^b_a \Box + \nabla^b \nabla_a)\boldsymbol{A}_b = 0, \qquad (\delta^b_a(\Box + M^2) + \nabla^b \nabla_a)\boldsymbol{B}_b = 0,$$

i.e. to the massless field $\boldsymbol{A}_a$ + the massive Proca field $\boldsymbol{B}_a$.

## The problem with the non-decreasing propagator

However, there is a well-known problem with the Proca field — its propagator is non-decreasing as $k^2 \to \infty$:

$$G^b_a(\boldsymbol{k}) = \frac{1}{k^2 + M^2}\left(\delta^b_a + \frac{k_a k^b}{M^2}\right).$$

This leads to the fact that the Proca field has no renormalizable interactions. But this can be circumvented using the Higgs mechanism. Perhaps our case is the same?

# Feynman-like gauge — solution to the problem?

Let us introduce the following gauge-fixing term:

$$\mathcal{L}_{\rm gf} = -\frac{1}{2}\,\mathrm{tr}\left((\nabla_a \boldsymbol{A}^a)^2 + (\nabla_a \boldsymbol{B}^a)^2 + m^2 \boldsymbol{h}^2\right).$$

Then we obtain the following equations of motion:

$$\Box \boldsymbol{A}_a = 0, \qquad (\Box + M^2)\boldsymbol{B}_a - M\nabla_a \boldsymbol{h} = 0, \qquad (\Box + m^2)\boldsymbol{h} + M\nabla_a \boldsymbol{B}^a = 0.$$

After moving to the momentum representation we get:

$$\begin{pmatrix} \delta_a^b (k^2 + M^2) & -iMk_a \\ iMk^b & k^2 + m^2 \end{pmatrix} \begin{pmatrix} \boldsymbol{B}_b(k) \\ \boldsymbol{h}(k) \end{pmatrix} = 0.$$

Matrix inversion yields the following propagator:

$$\hat{G} = \frac{1}{k^4 + m^2(k^2 + M^2)} \begin{pmatrix} \frac{\delta_a^b \left(k^4 + m^2(k^2+M^2)\right) + M^2 k_a k^b}{k^2 + M^2} & iMk_a \\ -iMk^b & k^2 + M^2 \end{pmatrix}$$

$$\xrightarrow[m\to 0]{} \frac{1}{k^4} \begin{pmatrix} \frac{\delta_a^b k^4 + M^2 k_a k^b}{k^2 + M^2} & iMk_a \\ -iMk^b & k^2 + M^2 \end{pmatrix}.$$