Neutrino equation – symmetries and conservation laws

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Standard Model Equations

The field equations of quantum physics included in the Standard Model:

- Maxwell equations 1861-1864 (spin 1).
- Klein-Gordon-(Schrodinger) equation 1926 (spin 0).
- Dirac equation 1928 (spin 1/2).
- Weyl equation 1929 (spin 1/2).
- Yang-Mills equations 1954 (spin 1).

Are all these equations satisfactory to physicists?

Until 1998, the answer was "yes".

About neutrino

The existence of the neutrino was predicted by W. Pauli in 1930. The neutrino was discovered in 1956 by a team of experimenters led by C. Cowan and F. Reines. It was discovered that the neutrino is a very light (possibly massless) left-chiral particle, and the antineutrino is a right-chiral particle. In 1957, Landau, Salam, Lee and Yang proposed describing the neutrino with the Weyl equation. It is this equation for the neutrino that was included in the Standard Model. In 1962, in addition to the electron neutrino, the muon neutrino was discovered, and in 2000, the tau neutrino was discovered.

Neutrino oscillations

In 1998, neutrino oscillations were discovered (and later confirmed by many experiments) in an experiment on the Super-Kamiokande detector. The theoretical justification for the possibility of neutrino oscillations was given by B. Pontecorvo back in 1957. Subsequently, the theory of neutrino oscillations was developed by many authors, including those from Pontecorvo's group. Interpretation of experimental data using the theory of neutrino oscillations indicates the possibility that some (or all) of the three neutrino flavors ν_e, ν_μ, ν_τ have non-zero masses and, in this case, cannot be described by the Weyl equation. In this regard, the question of an equation for describing neutrinos with a non-zero mass became relevant.

Known candidates for the equation for non-zero mass neutrinos

In the literature (see, for example, the review [1]), the list of equations currently considered as candidates for the equation for neutrinos with nonzero mass consists of the Dirac equation (1928) and the Majorana equation (1937). Equations for Elko spinors are also discussed (see, for example, [2]).

- M.S.Athar and others, Status and Perspectives of Neutrino Physics, Prog. Part. Nucl. Phys. 124 (2022) 103947, DOI: https://doi.org/10.1016/j.ppnp.2022.103947, arXiv:2111.07586
- D. V. Ahluwalia and D. Grumiller, JCAP 07 (2005) 012; arXiv:0412.080.

New candidate for the neutrino equation

We propose to supplement the list of candidates with another equation for the neutrino (and a corresponding equation for the antineutrino). The new equation is a modification of the biquaternion equation of Lanczos (1929) ([1], formula (63)).

- C. Lanczos, Z. f. Phys. 57 (1929) 447-473, 474-483, 484-493. Reprinted and translated in W.R. Davis et al., eds., Cornelius Lanczos Collected Published PapersWith Commentaries (NorthCarolina StateUniversity,Raleigh, 1998) pages 21132 to 21225. arXiv:physics/0508012, arXiv:physics/0508002, arXiv:physics/0508009.
- Н. Г. Марчук, Класс полевых уравнений для нейтрино с ненулевой массой, ТМФ, 219:3 (2024), 422-439; N. G. Marchuk, A class of field equations for neutrinos with nonzero masses, Theoret. and Math. Phys., 219:3 (2024), 897-912.

Quaternions and biquaternions

$\mathbb{H}$ – algebra of quaternions (1843)

$$q = q_0 + q_1 I + q_2 J + q_3 K, \quad q_\mu \in \mathbb{R}$$

with multiplication

$$I^2 = J^2 = K^2 = IJK = -1.$$

Quaternion conjugation operation

$$q \to \tilde{q} = q_0 - q_1 I - q_2 J - q_3 K, \quad q\tilde{q} = \tilde{q}q = {q_0}^2 + {q_1}^2 + {q_2}^2 + {q_3}^2.$$

$\mathbb{C} \otimes \mathbb{H}$ – algebra of biquaternions (1844)

$$q_\mu \in \mathbb{C}.$$

Isomorphism

$$\mathbb{C} \otimes \mathbb{H} \simeq \mathrm{Mat}(2, \mathbb{C}).$$

Second-order matrices from $\mathrm{Mat}(2,\mathbb{C})$ and conjugation operations $\dagger, \tilde{}, *$

In the following presentation, all physical constants (except for the mass of a particle, which we further denote as $m \in \mathbb{R}$) are taken to be equal to one.

Using conjugation operations

- Hermitian conjugation
$$V \to V^\dagger;$$
- Quaternion conjugation
$$V \to \tilde{V} = (\mathrm{tr}\, V)e - V, \quad V\tilde{V} = \tilde{V}V = (\det\, V)e;$$
- Superposition of Hermitian conjugation and quaternion conjugation
$$V \to V^* = \tilde{V}^\dagger = \widetilde{V^\dagger}.$$

Left and right 2x2-spinors from (ℓ) and (ℓ^*) in Minkowski space $\mathbb{R}^{1,3}$

Let $\sigma^0 = e$ and $\sigma^1, \sigma^2, \sigma^3$ be the Pauli matrices

$$\sigma^0 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \quad \sigma^1 = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \quad \sigma^2 = \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix}, \quad \sigma^3 = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}.$$

A left 2x2-spinor is a matrix function $\mathbb{R}^{1,3} \to \mathrm{Mat}(2, \mathbb{C})$ such that the first and second columns of this matrix are left Weyl spinors (spinor fields). The set of left 2x2-spinors is denoted by (ℓ). The set of right 2x2-spinors is denoted by (ℓ^*).

First order differential operators

$$\tilde{\nabla} = \tilde{\sigma}^\mu \partial_\mu : (\ell) \to (\ell^*), \quad \nabla = \sigma^\mu \partial_\mu : (\ell^*) \to (\ell).$$

Comment. There is a connection between the differential operators ∇, $\tilde{\nabla}$ and the conjugation operation $*$. Namely, for any smooth spinor fields $\Theta \in (\ell^*)$, $\Phi \in (\ell)$

$$(\nabla\Theta)^* = \tilde{\nabla}(\Theta^*), \quad (\tilde{\nabla}\Phi)^* = \nabla(\Phi^*).$$

Left and right Lanczos equations (1929)

Let $\Phi \in (\ell)$, $\Theta \in (\ell^*)$. Equations

$$\tilde{\nabla}\Phi + im\Phi^* N = 0, \quad \nabla\Theta + im\Theta^* N^{-1} = 0, \qquad (1)$$

where a constant matrix $N \in \mathrm{Mat}(2, \mathbb{C})$ satisfies equality

$$N^* N = -e, \qquad (2)$$

are called the *left Lanczos equation* and *right Lanczos equation* respectively (in Lanczos' work $N = N^{-1} = \sigma^3$).

If $\Phi \in (\ell)$, $\Theta \in (\ell^*)$ are twice continuously differentiable solutions of the equations (1), then they satisfy the Klein–Gordon equations

$$(\tilde{\nabla}\nabla + m^2)\Phi = 0, \quad (\nabla\tilde{\nabla} + m^2)\Theta = 0. \qquad (3)$$

where $\nabla\tilde{\nabla} = \tilde{\nabla}\nabla = \square$ is the D'Alembert operator.

Theorem

The matrix $N \in \mathrm{Mat}(2, \mathbb{C})$ satisfies equality (2) if and only if

$$N = cH,$$

where $c \in \mathbb{C}$, $|c| = 1$ and $H = H^\dagger$, det $H = -1$.

Conservation laws for Lanczos equations

Conservation laws for field equations in Minkowski space are expressions of the form

$$\partial_\mu j^\mu = 0, \tag{4}$$

that is, the 4-divergence of some real vector field with components j^μ is equal to zero in some region of space-time.
Let $\Phi = \Phi(x) \in (\ell)$ be a solution to the left Lanczos equation (1), (2). Multiply equation (1) on the left by the matrix $\Phi^\dagger$ and add the result to the Hermitian conjugate expression

$$\Phi^\dagger \tilde{\sigma}^\mu \partial_\mu \Phi + im\Phi^\dagger \Phi^* N = 0, \quad (\partial_\mu \Phi^\dagger)\tilde{\sigma}^\mu \Phi - imN^\dagger \tilde{\Phi}\Phi = 0,$$

$$\partial_\mu(\Phi^\dagger \tilde{\sigma}^\mu \Phi) + im(\Phi^\dagger \Phi^* N - N^\dagger \tilde{\Phi}\Phi) = 0. \tag{5}$$

If $\mathrm{tr}\, N = 0$, then we obtain the conservation law (4) with a real vector field

$$j^\mu = \mathrm{tr}(\Phi^\dagger \tilde{\sigma}^\mu \Phi).$$

Key condition

If

$$\Phi^\dagger \Phi^* N - N^\dagger \tilde{\Phi} \Phi = 0, \quad (6)$$

then formula (5)

$$\partial_\mu (\Phi^\dagger \tilde{\sigma}^\mu \Phi) + im(\Phi^\dagger \Phi^* N - N^\dagger \tilde{\Phi} \Phi) = 0$$

gives the conservation law

$$\partial_\mu (i \Phi^\dagger \tilde{\sigma}^\mu \Phi) = 0. \quad (7)$$

Components of the vector with values in the Lie algebra $u(2)$

$$J^\mu = i \Phi^\dagger \tilde{\sigma}^\mu \Phi \in u(2) \mathrm{T}^1$$

expand in the Pauli basis

$$J^\mu = i j_a^\mu \sigma^a.$$

As a result, from (7) we obtain four real conservation laws

$$\partial_\mu j_a^\mu = 0, \quad a = 0, 1, 2, 3. \quad (8)$$

Conservative 2x2-equation

Let us consider two modifications of the Lanczos equation (1), (2) that ensure the fulfillment of the key condition (6):

- Lanczos equation with an additional condition

$$\det\Phi = 0;$$

- Lanczos equation with $N = cH$ and with phase factor $c \in \mathbb{C}$, $|c| = 1$, depending on spinor Φ by formula

$$c = \frac{\det\,\Phi}{|\det\,\Phi|} \tag{9}$$

In this case we obtain *left conservative* 2x2*-equation*

$$\tilde{\nabla}\Phi + im\hat{\Phi}H = 0, \tag{10}$$

where H is a constant matrix from $Herm(2)$ with condition $\det\,H = -1$ and

$$\hat{\Phi} = \frac{\det\,\Phi}{|\det\,\Phi|}\Phi^{*}. \tag{11}$$

The equation ($\Theta = \Theta(x) \in (\ell^{*})$, $H \in Herm(2)$, $\det\,H = -1$)

$$\nabla\Theta + im\hat{\Theta}H^{-1} = 0, \tag{12}$$

will be called the *right conservative* 2x2*-equation*.

Gauge $U(2)$ symmetry of conservative 2x2-equations

Let $A_\mu = A_\mu(x)$, $(x \in \mathbb{R}^{1,3})$ be a smooth covector field with values in the Lie algebra $u(2)$.

Theorem

Let ρ be a real constant. The equation

$$\tilde{\sigma}^\mu(\partial_\mu \Phi + \Phi A_\mu) + im\hat{\Phi} H = 0, \tag{13}$$

where

$$\Phi = \Phi(x) \in (\ell), \quad H = H(x) \in Herm(2), \quad \det H = -1, \quad \operatorname{tr} H = \rho \tag{14}$$

is invariant under the following gauge transformation with unitary matrix $V = V(x) \in U(2)$

$$\begin{aligned} \Phi &\to \acute{\Phi} = \Phi V, \\ A_\mu &\to \acute{A}_\mu = V^{-1} A_\mu V - V^{-1}\partial_\mu V, \\ H &\to \acute{H} = V^{-1} H V. \end{aligned} \tag{15}$$

Combining the Left Conservative $2\text{x}2$-Equation with the Yang–Mills System of Equations we get

$$
\begin{aligned}
&\tilde{\sigma}^\mu(\partial_\mu \Phi + \Phi A_\mu) + im\hat{\Phi} H = 0,\\
&\partial_\mu A_\nu - \partial_\nu A_\mu - [A_\mu, A_\nu] = F_{\mu\nu},\\
&\partial_\mu F^{\mu\nu} - [A_\mu, F^{\mu\nu}] = i\Phi^\dagger \tilde{\sigma}^\nu \Phi,\\
&\det\, \Phi \neq 0, \quad \det\, H = -1, \quad \mathrm{tr}\, H = \rho,
\end{aligned} \tag{16}
$$

invariant under the gauge transformation (15) with the unitary Lie group $U(2)$ (with $F_{\mu\nu} \in u(2)\mathrm{T}_2$ and $F_{\mu\nu} \to \acute{F}_{\mu\nu} = V^{-1} F_{\mu\nu} V$).
We consider the system of equations (16) as a system of equations for a (left-chiral) neutrino interacting with the Yang–Mills field $(A_\mu, F_{\mu\nu})$ with $U(2)$ gauge symmetry.
For the antineutrino we have the system of equations

$$
\begin{aligned}
&\sigma^\mu(\partial_\mu \Theta - \Theta \tilde{A}_\mu) + im\hat{\Theta} H^{-1} = 0,\\
&\partial_\mu \tilde{A}_\nu - \partial_\nu \tilde{A}_\mu + [\tilde{A}_\mu, \tilde{A}_\nu] = \tilde{F}_{\mu\nu},\\
&\partial_\mu \tilde{F}^{\mu\nu} + [\tilde{A}_\mu, \tilde{F}^{\mu\nu}] = i\Theta^\dagger \sigma^\nu \Theta,\\
&\det\, \Theta \neq 0, \quad \det(H^{-1}) = -1, \quad \mathrm{tr}(H^{-1}) = -\rho,
\end{aligned} \tag{17}
$$

Polar gauge of solutions of the left conservative 2x2-equation

By the polar matrix factorization theorem, any non-singular matrix $A \in \mathrm{Mat}(2, \mathbb{C})$ can be uniquely represented as a product of two matrices

$$A = QW,$$

where Q is a Hermitian positive definite matrix and $V \in U(2)$. If $\Phi = \Phi(x)$ is a solution to the left conservative equation (13), (14) in the domain Ω, then at each fixed point $\acute{x} \in \Omega$ the matrix $\Phi(\acute{x})$ has a polar factorization

$$\Phi(\acute{x}) = Q(\acute{x})W(\acute{x}). \tag{18}$$

Thank you!