On isomonodromic deformations given by Painlevé equations.

Babich M.V.

PDMI, SPbSU

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Goal of the report.

The purpose of the report is exclusively methodological, it is about the connection of the Painlevé equations and isomonodromy, about the method of developing the theory of Painlevé equations.

A scheme that allows one to present all six equations and their basic properties in lectures, say, over a semester is proposed.

The derivation of equations is introduced, with all the proofs of the statements, without restrictions on the parameters and without the spell: "Omitting the calculations, we obtain..."
... but with: "We can verify by direct calculation that ...":)

Philosophy:

- "From the Painlevé equations follows Isomonodromy." – the theory is transparent and not cumbersome.
- "From the Isomonodromy follows Painlevé equations." – this is only partially true. It is true in the case of a general position of the parameters only. The theory is both complex and cumbersome.

What is it all about? What is the connection between the Painlevé equations and the differential system (of linear equations).

The Painlevé equation is a nonlinear ODE:

$$q_{tt} = \mathcal{F}(q, q_t, t).$$

The linear differential system is $N \times N$-matrix equation on Ψ

$$d\Psi = A(z; q, q_t, t)dz\Psi.$$

Matrix A is constructed by a set of complex values z, q, q_t, t. Namely. Having a solution $q = q(t)$ one can, in a rational way, construct a set of differential systems $d\Psi = A(z; q, q_t, t)dz\Psi$ such that each of these systems has the same monodromy. The set of systems is parameterized by $t \in \mathbb{C}$.

The source of the principal difficulties.

An honest deductive exposition of "Isomonodromy"⟶"Painlevé" is impossible(!) – there are counterexamples.

In more details. Assuming the constancy of monodromy, as a consequence, the Painlevé equation $\mathrm{PVI}(\alpha_1, \alpha_2, \alpha_3, \alpha_4)$ is obtained only if $2\alpha_{\mathrm{k}} \notin \mathbb{Z}$. If $2\alpha_{\mathrm{k}} \in \mathbb{Z}$, then it is possible to deform the equation, preserving monodromy, differently.

Consequently, it is impossible to say: "... for simplicity, consider $2\alpha_{\mathrm{k}} \notin \mathbb{Z}$, the general case is only technically more complicated" – it is not (only) "technically more complicated", the main implication is not true, Painlevé does not follow from the isomonodromy.

If we restrict ourselves to the general position, then the question immediately arises – why is the condition $2\alpha_{\mathrm{k}} \notin \mathbb{Z}$ not specified when writing the Painlevé equation itself?!

A.A. Bolibruch's counterexample.

It will be shown that the isomonodromy of the deformation is equivalent to the existence of a flat 1-form
$\Omega = \mathrm{A}dz + \mathrm{B}dt$.
It will be shown that the Painlevé VI equation define a deformation form of the following form: $\sum_k A^{(k)} \frac{dz - dz_k}{z - z_k}$.
However, there are other flat forms of deformation, for example $\Omega =$

$$\begin{pmatrix} 1 & 0 \\ \frac{2t}{1-t^2} & 0 \end{pmatrix} \frac{d(z+t)}{z+t} + \begin{pmatrix} 0 & -6t \\ 0 & -1 \end{pmatrix} \frac{dz}{z} + \begin{pmatrix} 2 & 3+3t \\ \frac{1}{t+1} & -1 \end{pmatrix} \frac{d(z-1)}{z-1} +$$
$$+ \begin{pmatrix} -3 & -3+3t \\ \frac{1}{t-1} & 2 \end{pmatrix} \frac{d(z+1)}{z+1} + \underline{\begin{pmatrix} 0 & 0 \\ \frac{-2t}{1-t^2} & 0 \end{pmatrix} \frac{dt}{z+t}} = \mathrm{A}dz + \mathrm{B}dt.$$

What does this example mean?

The example does not affect the
"Painlevé"$\longrightarrow$"Isomonodromy" transition
and destroys the transition
"Isomonodromy"$\longrightarrow$"Painlevé".

The PVI equation with the parameters of this example $(9/2, 1/2, 25/2, 2)$ describes an isomonodromic deformation of the same differential system, but a different deformation.

The deformation that exists in the case of general position (Painlevé) is also isomonodromic for this system.
But, with this choice of the parameters, there are more deformations that do not change the monodromy than in the case of general position. It happens.

It is suggested:

When getting acquainted with the theory of Painlevé equations completely abandon the "Isomonodromy"⟶"Painlevé" transition. Just inform, without going into details, that in the case of a general position, the inverse transition is also possible. It is obtained by a deep analysis of the asymptotic expansions.

The inverse monodromy problem and the Riemann-Hilbert approach shold be presented later, separately – as the most powerful method for studying the already written equations. At the time already equipped with both a Lax pair and Hamiltonian theory.

This is all the more logical since, if we take the point of view of "Painlevé equations are nonlinear special functions", then most scientists need Painlevé equations as a tool, and not at all as an object of investigations.

Advantages:

Simple logic – “one-way implication”.

You can remove the cumbersome calculations and many murky speculations, replacing them with the “Guess-and-check” trick, where desired.

About priorities.

I have never been interested in this, so I can only say whose works I studied, the list will be at the end of the presentation.
Historical references you can find in the number of monographs.

Structure of the report.

I describe the Fuchsian case (PVI) in as much detail as possible first. It is the basis, the foundation.

For the non-Fuchsian case I prove the constancy of the monodromy. It is sufficient for the form of deformation to be rational and for the Stokes multipliers to be checked to be constant.

I explain "on the fingers" how it was possible to guess all this – what manipulations give the answer that is checked. It is so called "confluence procedure".

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True Monodromy. The Deformation Form.

Consider the equation $d_z\Psi = Adz\Psi$ in connected not simply-connected domain of $\overline{\mathbb{C}}_z$. We obtain a connected simply connected domain by removing the system of cuts l_k. For example, connect the poles $A(z)dz$ with some non-singular point P_{start}.

Consider the solution $\Psi : \Psi(P_{start}) = I$ in the domain. On the different sides of each cut $\Psi_-(z) = \Psi_+(z)M^{(k)}_{+-}, z \in l_k$, since any two solutions differ by a constant right-hand factor.

We introduce the parameter $t \in [0,1]$. The cuts will turn into "films" $l_k \times [0,1]$, in the neighborhood of each of which there are two solutions $\Psi_-(z,t)$ and $\Psi_+(z,t)M^{(k)}_{+-}(t)$.

The sewing together (matching) condition $d_t\Psi_-\Psi_-^{-1} = d_t\Psi_+\Psi_+^{-1} =: B$ is the condition of constancy of the monodromy $M^{(k)}_{+-}(t) = \text{const}_k$!

$\Omega := d\Psi\Psi^{-1} = Adz + Bdt$ is the main object of the theory.

Fuchsian case. Schlesinger Ansatz.

Consider $d\Psi/dz = \sum_k A^{(k)} \frac{1}{z-z_k}\Psi$,
in the neighborhood of $z \sim z_k$: $d_z\Psi = \left(\frac{A^{(k)}dz}{z-z_k} + O(1)\right)\Psi$.
The singular part depends only on the difference $z - z_k$,
let's try to find a solution with a symmetric (locally!!)
dependence: $d\Psi = \left(A^{(k)}\frac{dz-dz_k}{z-z_k} + O(1)\right)\Psi$. Let's sum and
<u>voluntaristically</u>(!) throw away $O(1)dz_k$:

$$d\Psi = \sum_k A^{(k)} \frac{dz - dz_k}{z - z_k}\Psi.$$

The compatibility condition, that is, $d^2\Psi = 0$ gives a
dynamical system on $A^{(k)}$, the famous Schlesinger system.

Schlesinger equations on $\mathcal{O}^{(1)} \times \ldots \times \mathcal{O}^{(4)} =: \prod_k \mathcal{O}^{(k)}$

The condition that each of the m residues of the zero differential $d^2\Psi$ is equal to zero is

$$dA^{(k)} + \left[A^{(k)}, \sum_{i \neq k} A^{(i)} \frac{dz_k - dz_i}{z_k - z_i}\right] = 0, \; k = 1, \ldots m.$$

The conjugate class $A^{(k)}$ is preserved since $dA^{(k)} = [A^{(k)}, *]$, therefore $A^{(k)}$ belong to the orbit $\mathcal{O}^{(k)}$ of the (co)adjoint action of the linear group. Consequently each residue $A^{(k)}$ lies in the symplectic space $\mathcal{O}^{(k)}$.

Let us construct a Hamiltonian theory.

Hamiltonian structure of the Schlesinger equations.

A verification shows that the Hamiltonian of the dynamics with respect to “time” (parameter) z_k is

$$H_k = \mathsf{tr}\, A^{(k)} \sum_{i \neq k} A^{(i)} \frac{1}{z_k - z_i}.$$

Symplectic reduction. Projection of equations from $\prod_k \mathcal{O}^{(k)}$ to $\prod_k \mathcal{O}^{(k)} /\!\!/ \mathrm{GL}(2,\mathbb{C})$.

The Hamiltonian does not depend on the simultaneous conjugation of all $A^{(k)}$ by a single matrix, even if it depends on z_k, i.e., on the diagonal action of $\mathrm{GL}(2,\mathbb{C})$. The Hamiltonian system can be projected to the quotient space $\prod_k \mathcal{O}^{(k)}/\mathrm{GL}(2,\mathbb{C})$.
The momentum map is $\sum_k A^{(k)}$, it is the total residue of the differential Adz. It (the residue) is constant, equal to zero. We obtain a system with the same Hamiltonians, but on another symplectic space:

$$\mathcal{O}^{(1)} \times \ldots \times \mathcal{O}^{(4)}\Big|_{\sum_k A^{(k)}=0} /\mathrm{GL}(2,\mathbb{C}) =: \prod_k \mathcal{O}^{(k)} /\!\!/ \mathrm{GL}(2,\mathbb{C})$$

Coordinatization of $\prod_k \mathcal{O}^{(k)} /\!\!/ \mathrm{GL}(N, \mathbb{C})$.

We can take functions (matrix elements $A^{(k)}_{ij}$) from any section of the bundle

$$\prod_k \mathcal{O}^{(k)} \to \prod_k \mathcal{O}^{(k)} /\!\!/ \mathrm{GL}(N, \mathbb{C})$$

as coordinates. Consider the Painlevé case $N = 2, m = 4$. Usually one of $A^{(k)}$ is taken as diagonal, and implicitly, by equations, the diagonal factor is fixed. This is not an optimal choice! Much more efficient:

$$A^{(2)} = \begin{pmatrix} * & * \\ 0 & * \end{pmatrix}, A^{(3)} = \begin{pmatrix} * & 0 \\ * & * \end{pmatrix},$$

and, optionally, we fix either an off-diagonal element or a proper direction of $A^{(4)}$:

$$(A^{(4)})_{12} = -1 \text{ , or } (A^{(4)})_{11} - (A^{(4)})_{12} = (A^{(4)})_{22} - (A^{(4)})_{21}.$$

Coordinatization of $\prod_k \mathcal{O}^{(k)} /\!\!/ \mathrm{GL}(2,\mathbb{C})$ (continuation).

The resulting simplifications:

1. Symplectic form on the quotient space $\omega^{(1)} + \cdots + \omega^{(4)} = \underline{\omega^{(1)}}$ – the canonical coordinates on the space $\prod_k \mathcal{O}^{(k)} /\!\!/ \mathrm{GL}(2,\mathbb{C})$ are the canonical coordinates on one orbit $\mathcal{O}^{(1)}$.
2. The equation $\sum_k A^{(k)} = 0$ is solved elementarily.

Explicit form of the section if $A_{12}^{(4)} = -1$.

$$A^{(1)} = \begin{pmatrix} \lambda_1 - pq & q \\ -p(pq - \Delta_1) & \lambda_1' + pq \end{pmatrix}, A^{(2)} = \begin{pmatrix} \lambda_2 & 1 - q \\ 0 & \lambda_2' \end{pmatrix},$$

$$A^{(3)} = \begin{pmatrix} \lambda_3 & 0 \\ a_{21}^{(3)} & \lambda_3' \end{pmatrix}, A^{(4)} = \begin{pmatrix} -\Sigma_{11} & -1 \\ -\Sigma_{11}\Sigma_{22} + \lambda_4'\lambda_4 & -\Sigma_{22} \end{pmatrix},$$

here λ_k, λ_k' are the eigenvalues of $A^{(k)}$, $\Delta_k = \lambda_k - \lambda_k'$ and Σ_{11}, Σ_{22} are the sums of the corresponding matrix elements

$$\Sigma_{11} := -pq + \lambda_1 + \lambda_2 + \lambda_3, \Sigma_{22} := pq + \lambda_1' + \lambda_2' + \lambda_3',$$

$$a_{21}^{(3)} = p(pq - \lambda_1 + \lambda_1') - (pq - \sum_j \lambda_j + \lambda_4)(pq + \sum_j \lambda_j' - \lambda_4') - \lambda_4'\lambda_4.$$

Section, if the eigen-direction λ_4 of $A^{(4)}$ is constant and equal to $(1,-1)^T$.

$$A^{(1)} = \begin{pmatrix} \lambda_1 + pq & p \\ -q(pq + \Delta_1) & -pq + \lambda_1' \end{pmatrix}, \quad A^{(2)} = \begin{pmatrix} \lambda_2 & a_{12}^{(2)} \\ 0 & \lambda_2' \end{pmatrix},$$

$$A^{(3)} = \begin{pmatrix} \lambda_3 & 0 \\ a_{21}^{(3)} & \lambda_3' \end{pmatrix}, A^{(4)} = \begin{pmatrix} -pq - \lambda_\Sigma + \lambda_4 & -pq - \lambda_\Sigma \\ pq + \lambda_\Sigma - \Delta_4 & pq + \lambda_\Sigma + \lambda_4' \end{pmatrix},$$

$a_{12}^{(2)} = p(q-1) + \lambda_\Sigma$, $a_{21}^{(3)} = pq(q-1) + \Delta_1 q - \lambda_\Sigma + \Delta_4$,
$\lambda_\Sigma := \sum_k \lambda_k$, $\Delta_k := \lambda_k - \lambda_k'$.

Hamiltonian PVI system.

Let $z_1 = 0, z_2 = 1, z_3 = t, z_4 = \infty$. The calculation of the Hamiltonian for the section $A^{(4)}_{12} = -1$ gives:

$$H := \frac{\operatorname{tr} A^{(3)}A^{(1)}}{t} + \frac{\operatorname{tr} A^{(3)}A^{(2)}}{t-1} = \frac{1}{t(t-1)} \operatorname{tr} A^{(3)}((t-1)A^{(1)} + tA^{(2)}) =$$

$$= \frac{q(q-1)(q-t)}{t(t-1)} \left(p^2 - p \left(\frac{\Delta_1}{q} + \frac{\Delta_2}{q-1} + \frac{\Delta_3}{q-t} \right) \right) +$$

$$+ q \frac{\Delta_\Sigma(\Delta_\Sigma - 2\Delta_4)}{4t(t-1)} + *.$$

Here “$*$” does not contain the coordinates p, q, it does not affect the equations of motion.

The PVI equation.

The PVI equation is the Euler-Lagrange equation for the Hamiltonian PVI system just written out. Having written it out, we see that it depends on the following combinations of the eigenvalues $\lambda_{\mathrm{k}} - \lambda'_{\mathrm{k}} =: \Delta_{\mathrm{k}}$ of $\mathrm{A}^{(\mathrm{k})}$:

$$\mathrm{PVI}\left(\frac{\Delta_4^2}{2},\ \frac{\Delta_1^2}{2},\ \frac{\Delta_2^2}{2},\ \frac{(\Delta_3+1)^2}{2}\right).$$

We will demonstrate that the generators of the PVI symmetry groups are easy to obtain. /Advertisement:)/

Schlesinger transform.

Multiplying Ψ from the left by a diagonal matrix of the form $\begin{pmatrix} \frac{z-z_2}{z-z_3} & 0 \\ 0 & 1 \end{pmatrix}$ preserves the Fuchsianity of the system, since the residues at z_2 and z_3 have zeros where a 2nd-order pole could arise:

$$A^{(2)} = \begin{pmatrix} \lambda_2 & * \\ 0 & \lambda_2' \end{pmatrix}, A^{(3)} = \begin{pmatrix} \lambda_3 & 0 \\ * & \lambda_3' \end{pmatrix}.$$

The transformation is regular at the other points.
This action changes $\lambda_2 \to \lambda_2 + 1$ and $\lambda_3' \to \lambda_3' - 1$. Thus we can add any integers to Δ_k, provided that the sum of the added is even.

Okamoto transformation.

Let us calculate the Hamiltonian for the second choice of coordinate section — when one eigen-direction of $A^{(4)}$ is constant:

$$\frac{q(q-1)(q-t)}{t(t-1)}\left(p^2 - p\left(\frac{\Delta_4 - \frac{\Delta_\Sigma}{2}}{q} + \frac{\Delta_2 - \frac{\Delta_\Sigma}{2}}{q-1} + \frac{\Delta_3 - \frac{\Delta_\Sigma}{2}}{q-t}\right)\right) + q\frac{\Delta_\Sigma \Delta_1}{2t(t-1)},$$

here $\{z_1, z_3, z_4, z_2\} = \{0, 1, \infty, t\}$.
We got the same equation, but

$$\Delta_k \to \Delta_k - \frac{\Delta_\Sigma}{2}.$$

It is the famous Okamoto transform.

Permutation of $0 \leftrightarrow \infty$.

In the first normalization, the z-independent transformation $\Psi \to \mathrm{diag}(1/q, -1)\Psi$ swaps $A^{(1)}$ and $A^{(4)}$:

$$\left\{\frac{\Delta_4^2}{2}, \frac{\Delta_1^2}{2}, \frac{\Delta_2^2}{2}, \frac{(\Delta_3+1)^2}{2}\right\} \to \left\{\frac{\Delta_1^2}{2}, \frac{\Delta_4^2}{2}, \frac{\Delta_2^2}{2}, \frac{(\Delta_3+1)^2}{2}\right\}.$$

This leads to the transformation $q_{\mathrm{new}} = 1/q$.

Permutation $0 \leftrightarrow 1$.

Transformation of the direction of the first coordinate axis from the eigen-direction of $\mathrm{A}^{(2)}$ to the eigen-direction of $\mathrm{A}^{(1)}$ swaps $\mathrm{A}^{(1)}$ and $\mathrm{A}^{(2)}$, and also transforms $\mathrm{q}_{\mathrm{new}} = 1 - \mathrm{q}$,

$$\left\{ \frac{\Delta_4^2}{2}, \frac{\Delta_1^2}{2}, \frac{\Delta_2^2}{2}, \frac{(\Delta_3+1)^2}{2} \right\} \to \left\{ \frac{\Delta_4^2}{2}, \frac{\Delta_2^2}{2}, \frac{\Delta_1^2}{2}, \frac{(\Delta_3+1)^2}{2} \right\}.$$

Permutation $1 \leftrightarrow t$.

Here the geometric picture is not clear. However, one can see that the canonical transformation on the fiber $t = \mathrm{const}$

$$t = 1/t_{\mathrm{new}}, \quad q = tq_{\mathrm{new}}, p = p_{\mathrm{new}}/t$$

preserves the form of the quadratic in "p" part of the differential Hdt:

$$p^2 q(q-1)(q-t) d \log \frac{t}{t-1},$$

but swaps the roots $q = 1$ and $q = t$: $q = 1 \longleftrightarrow 1/t = t_{\mathrm{new}}$. It is easy to check that the remaining terms, linear in "p", in the formula $dH \wedge dt$ also preserve their form.

Permutation of $1 \leftrightarrow t$ (continuation).

Comparing the linear terms in "p", we see that:

$$dp \wedge dq - dH(p, q, t; \Delta_1, \Delta_2, \Delta_3, \Delta_4) \wedge dt =$$
$$dp_{new} \wedge dq_{new} - dH(p_{new}, q_{new}, t_{new}; \Delta_1, \Delta_3 + 1, \Delta_2 - 1, \Delta_4) \wedge dt_{new}.$$

This is transformation

$$\left\{ \frac{\Delta_4^2}{2}, \frac{\Delta_1^2}{2}, \frac{\Delta_2^2}{2}, \frac{(\Delta_3+1)^2}{2} \right\} \to \left\{ \frac{\Delta_4^2}{2}, \frac{\Delta_1^2}{2}, \frac{(\Delta_3+1)^2}{2}, \frac{\Delta_2^2}{2} \right\}.$$

Note. The shifts ± 1 in the parameters Δ are a consequence of the explicit dependence of the transformation on time

$$dp \wedge dq = dp_{new} \wedge dq_{new} - d(pq) \wedge dt/t.$$

This also explains the presence of "+1" in the definition of the Painlevé parameter in the moving pole. Namely $\Delta_k^2, k = 1, 2, 4$ and $(\Delta_3 + 1)^2 =: \delta$.

Nonfuchsian case.

Let the differential $A(z)dz$ in a linear differential system

$$d_z\Psi = A(z)dz\Psi$$

has multiple poles.

How to deform such an equation while preserving monodromy?

Multiple poles of a differential system and true monodromy.

The concept of monodromy is preserved, in particular, its constancy, as before, is equivalent to the existence of a flat deformation form

$$\Omega = \mathrm{Adz} + \mathrm{Bdt}.$$

But the concept becomes poor, for example, any(!) deformation of a system with a single pole is isomonodromic – the fundamental group is trivial and, therefore, constant. Analytically:

$$\partial\Psi(z,t)/\partial z = \left(A_1(t) + A_2(t)z + \ldots + A_n(t)z^{n-1}\right)\Psi(z,t).$$

$\Psi(z,t) \in GL(N,\mathbb{C})$ is an entire function of z, so setting $B := \Psi_t\Psi^{-1}$, we obtain the flat deformation form:

$$d\Psi = (\mathrm{Adz} + \mathrm{Bdt})\Psi.$$

Stokes effect:

Let $z \to \infty$,

$$e^z + e^{-z} \sim \begin{matrix} e^z, & \Re z > 0 \\ e^{-z}, & \Re z < 0 \end{matrix}$$

It demonstrates that a function analytic in the punctured neighborhood can have different asymptotics at the point.

Note. This does not happen for rational functions.

Elements of the Theory of Differential Equations.

Consider $d\Psi = \left(\frac{\Theta_n}{z^n} + \frac{A_{n-1}}{z^{n-1}} + \dots \right) \frac{dz}{z} \Psi, z \sim 0,$
where Θ_n is diagonal, with different eigenvalues.

From the Theory of Differential Equations it follows that the general solution has an asymptotics
$\Psi(z) \sim \left(I + \sum_{i=1}^{\infty} g_i z^i \right) \exp\{\Theta(z)\} C_k$, if
$\arg z \in]\theta_0 - \delta + \frac{\pi}{n} k, \theta_0 + \frac{\pi}{n}(k+1)[$, $k = 0, 1, ..., 2n$.

Here $\Theta(z) = \sum_{j=1}^{n} \Theta_j \frac{z^{-j}}{-j} + \Theta_0 \ln z$, where all Θ_j are diagonal, and all g_i, Θ_j are (rationally) determined by A_i (the equation).

It is the dependence on k that is the Stokes effect. The factors $C_k C_{k+1}^{-1}$ are a generalization of the monodromy matrices. They are called Stokes multipliers.

Constancy of the generalized monodromy and rationality of the deformation form.

Let $d\Psi = \Omega\Psi$, $\Omega = Adz + Bdt$ be the deformation form. Comparison of asymptotics Ψ_t and $B\Psi$ gives

$$\sum_{i=0}^{\infty}\left(B(z,t)g_i - g_i\dot{\Theta}(z,t) - \dot{g}_i\right)z^{-i} \sim$$

$$\left(I + \sum_{i=1}^{\infty} g_i z^{-i}\right) e^{\Theta(z,t)}(\dot{C}_k C_k^{-1}(t))e^{-\Theta(z,t)}. \quad (1)$$

If B is rational in z, then $e^{\Theta(z,t)}(\dot{C}_k C_k^{-1}(t))e^{-\Theta(z,t)}$ does not depend neither on z nor on k. It is a diagonal matrix depending, perhaps, only on t.
For $\dot{C}_k$ to be zero, it is sufficient that the coefficient of z^0 in the series on the left-hand side be zero. This can be verified.

Irregular isomonodromic deformation forms as the degenerations of the Schlesinger deformation form

There are limit transitions of the Schlesinger deformation form that yield finite rational limits with multiple poles. For example, let $\epsilon \to 0$:

$$\sum_k A^{(k)} \frac{dz-dz_k}{z-z_k} + (-A_1/\epsilon + A_0)\frac{dz-dz_0}{z-z_0} + \frac{1}{\epsilon}A_1\frac{dz-d(z_0+\epsilon t)}{z-(z_0+\epsilon t)} =$$

$$= \sum_k A^{(k)} \frac{dz-dz_k}{z-z_k} + \left(\frac{tA_1}{(z-z_0)^2} + \frac{A_0}{z-z_0}\right)(dz-dz_0) - \frac{A_1}{z-z_0}dt + o(1)$$

Similarly, setting

$z_1 = z_0 + \epsilon t_1, z_2 = z_0 + \epsilon t_1 + \epsilon^2 t_2, z_3 = z_0 + \epsilon t_1 + \epsilon^2 t_2 + \epsilon^3 t_3 \ldots$

and, selecting the appropriate matrix residues depending on the parameters t_k and $\epsilon \to 0$, we can obtain a deformation form with poles of any multiplicity.

Isomonodromic Painlevé systems PV-PI can be obtained by degenerating the Schlesinger deformation form.

The PV – PI equations correspond to isomonodromic deformations of systems with multiple poles. These systems are obtained by various fusions (confluences) of several poles of the Fuchsian PVI system.

Limit procedures yield rational deformation forms corresponding to the PV – PI equations.

The “Result”, i.e. that the deformation form leads precisely to the Painlevé equation, is verified.
The constancy of the Stokes matrices is also verified.
Consequently these deformations are isomonodromic.

The Hamiltonian structure of an irregular singular point. A truncated loop group and its Lie algebra, the Takiff algebra.

Fuchsian systems are naturally related to the group $GL(N, \mathbb{C})$ and its coadjoint action.
Consider a generalization, the group G_n, whose elements are matrix polynomials of degree at most n:
$g_0 + g_1 z + \ldots + g_n z^n \in G_n, \det g_0 \neq 0$, the group operation is the product of polynomials with degrees higher than n discarded.

Its Lie algebra (Takiff algebra):
$\alpha = \alpha_0 + \alpha_1 z + \ldots + \alpha_n z^n \in \mathfrak{g}_n, \quad \alpha_k \in gl(N, \mathbb{C})$. Coalgebra:
$A = (A_n \frac{1}{z^n} + \ldots + A_1 \frac{1}{z} + A_0)\frac{dz}{z} \in \mathfrak{g}_n^*, \quad A_k \in gl^*(N, \mathbb{C})$.
The group G_n acts coadjointly on differentials with multiple poles.
Pairing $< A, \alpha >= \mathrm{tr}\,\mathrm{Res}\, \alpha A$.

Symplectic reduction to (Hamiltonian) systems PV-PI.

As in the Fuchsian case, the original symplectic space is the Cartesian product of orbits corresponding to each singular point.
The same Hamiltonian reduction by the diagonal action of $GL(2, \mathbb{C})$ at the zero level of the momentum map, equal to the total residue, yields the reduced phase space.

As the pole order increases, the corresponding orbit becomes more and more complicated. The choice of a section convenient for the canonical parametrization of the space also becomes more and more difficult.
The difficulties in obtaining the already known answer are less and less reasonable.

My sources:

General theory, both regular and irregular:

The Japan School: M. Jimbo, T. Miwa, K. Ueno, K. Okamoto.

The Leningrad School: A.R. Its, A. A. Kapaev, A.V. Kitaev and D.A. Korotkin.

A.A. Bolibruch, B.A. Dubrovin and M. Mazzocco.

The theory of the confluence, Lax pairs and the Hamiltonian structure of the irregular case:

V. Rubtsov, M. Mazzocco and, especially, I. Gaiur.

Thank you!

The End = Конец :)