$\mathbb{CP}^1$ SUSY Sigma Models from Nonlinear Chiral Multiplets
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Plan

1. 1D Supersymmetric sigma models
2. The "harmonic truncation" problem for sigma models
3. The basic geometric idea behind the formalism: the (co)adjoint orbits
4. $\mathcal{N}=2$ "classical spin chain" on $\mathbb{CP}^1 \times \mathbb{CP}^1$ and non-linear chirality, D-model
5. D-model through flat superconnection
6. $\mathcal{N}=4$ generalization, K-model

- Discussion.

Short Recall: 1D SUSY Sigma Models

[Ivanov,Smilga'12], [Hull'99]

- **Geometrical input:**
  Complex $\dim_{\mathbb{C}} = n$ (Kähler) manifold $\mathcal{M}$ with a Hermitian (Kähler) metric $g_{\mu\bar{\nu}}$.
- **Field content:** n chiral fields $\mathcal{Z}^\mu$, n chiral fields Φ^μ and their conjugates.

We are interested in two types of supersymmeric sigma models:

- $\mathcal{N} = 2(b)$ model:

$$\mathcal{S} = \frac{1}{2}\int \mathrm{d}t \int \mathrm{d}^2\theta \Big(g_{\mu\bar{\nu}} D\mathcal{Z}^\mu \overline{D}\,\overline{\mathcal{Z}}^{\bar{\nu}} \Big) . \quad (1)$$

  Math interpretation: Quantum Hamiltonian $\sim$ Dolbeault-Laplace operator Δ_∂.

- $\mathcal{N} = 4(a)$ model:

$$\mathcal{S} = \frac{1}{2}\int \mathrm{d}t \int \mathrm{d}^2\theta \Big(g_{\mu\bar{\nu}} \Big(D\mathcal{Z}^\mu \overline{D}\,\overline{\mathcal{Z}}^{\bar{\nu}} + \Phi^\mu \overline{\Phi}^{\bar{\nu}} \Big)\Big) . \quad (2)$$

  Math interpretation: Quantum Hamiltonian $\sim$ Laplace-de Rham operator Δ_{d}.

Problematics

- **Problem:** find a suitable finite-dimensional analog for sigma models / spectral problem for Laplace operators.
- **Naive way:** lattice regularization,...
- **Modification of the problem:** Find a finite-dimensional analog that exactly reproduces the part of the original ∞-dim spectral problem.
- **Solution:** “harmonic truncation method”.

Geometric idea

[Kirillov'61]

- What is the classical analog of a single irreducible representation ρ_λ? Answer: a (co)adjoint orbit $\mathcal{O}_\lambda$. Nice property: $\mathcal{O}_\lambda$ is naturally symplectic.

[Bykov'12], [Bykov,Kuzovchikov'24]

- Suppose we explore a Laplace spectral problem on $\mathcal{M}$. Classical analog: mechanics with the phase space $\mathrm{T}^*\mathcal{M}$. Suppose $\mathrm{T}^*\mathcal{M}$ is an "almost symplectomorphic" to the $\mathcal{O}_\lambda \times \mathcal{O}_{\lambda'} \times \ldots$ (maybe in some limit in $\lambda, \lambda', \ldots$). Then, after geometric quantization, the harmonics of the Laplace operators factorizes as $\rho_\lambda \otimes \rho_{\lambda'} \otimes \ldots$ in this limit.
- Laplace operator on the sphere $\mathbb{CP}^1$: λ is the spin, $\mathcal{O}_\lambda \simeq \mathbb{CP}^1$ of radius $\sim \lambda$ and ρ_λ are the spherical harmonics. The sphere mechanics on $\mathrm{T}^*\mathbb{CP}^1$ is "almost symplectomorphic" to $\mathbb{CP}^1 \times \mathbb{CP}^1$ in the large spin limit.

[Bykov,Krivorol,Kuzovchikov'24]

- Why supersymmetry? In order to extend the Laplacian to the differencial forms.

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$\mathcal{N}=2$ mechanics on $\mathbb{CP}^1 \times \mathbb{CP}^1$

- Let us consider the following model (D-model):

$$\mathcal{S} = \frac{1}{2}\int \mathrm{d}t \int \mathrm{d}^2\theta \Big(|\mathcal{Z}_1|^2 e^{\mathcal{V}_1} + |\mathcal{Z}_2|^2 e^{\mathcal{V}_2} + |\Psi|^2 e^{\mathcal{V}_1 - \mathcal{V}_2} - \lambda(\mathcal{V}_1 + \mathcal{V}_2) \Big), \tag{3}$$

where $\mathcal{V}_1$, $\mathcal{V}_2$ are real gauge superfields, $\mathcal{Z}_2$ and Ψ are chiral bosonic and fermionic superfields, but $\mathcal{Z}_1$ satisfy a non-linear chirality constraint [Ivanov, Krivonos, Toppan'97]

$$\overline{D}\mathcal{Z}_1 + \varkappa \Psi \mathcal{Z}_2 = 0 . \tag{4}$$

Here $\varkappa$ is a coupling constant (which controls the size of the sphere).

- In components (oscillator variables):

$$\mathcal{S} = \int \mathrm{d}t \Big(i\bar{z}_1 \circ \dot{z}_1 + i\bar{z}_2 \circ \dot{z}_2 + i\bar{\psi}\dot{\psi} - \varkappa^2 (\bar{z}_1 \circ z_2)(\bar{z}_2 \circ z_1) - \varkappa^2 \bar{\psi}\psi(\bar{z}_2 \circ z_2 - \bar{z}_1 \circ z_1) +$$
$$+ \alpha_1(\bar{z}_1 \circ z_1 + \bar{\psi}\psi - \lambda) + \alpha_2(\bar{z}_2 \circ z_2 - \bar{\psi}\psi - \lambda) \Big). \tag{5}$$

In the "spin variables" $S^a_A = \bar{z}_A \sigma^a z_A$ we obtain the spin chain Hamiltonian

$$\mathcal{H} = \varkappa^2 \Big((S^a_1 \otimes S^a_2) + \text{fermions} \Big). \tag{6}$$

Reduction to $\mathcal{N} = 1$ superfields

- The idea is that integration over $\bar{\theta}$ is the same as $\bar{D}$ differentiation (up to a total derivative). This is the reason why $|\mathcal{Z}_1|^2$ in $\mathcal{N} = 2$ gives cubic terms in $\mathcal{N} = 1$ superspace (i.e. non-linear chirality gives the interaction).
- The reduced action in $\mathcal{N} = 1$ superfields is

$$\mathcal{S} = \int \mathrm{d}\theta \Big[- \overline{\mathsf{Z}}_A \circ D\mathsf{Z}_A + \overline{\Psi} D\Psi + \varkappa \Big(\overline{\Psi}\, \overline{\mathsf{Z}}_2 \circ \mathsf{Z}_1 + \Psi \overline{\mathsf{Z}}_1 \circ \mathsf{Z}_2 \Big) + \\ + \mathsf{A}_1 \Big(|\mathsf{Z}_1|^2 + |\Psi|^2 + \xi_1 \Big) + \mathsf{A}_2 \Big(|\mathsf{Z}_2|^2 - |\Psi|^2 + \xi_2 \Big) \Big]. \quad (7)$$

- One can check (see Andrew's talk) that in the limit $\lambda \to \infty$ this model is equivalent $\mathcal{N} = 2(b)$ sigma model. The idea is to decompose the matrix $\mathsf{Z} = (\mathsf{Z}_1, \mathsf{Z}_2)$ as $\mathsf{Z} = \mathsf{U}\mathsf{H}$ via the polar decomposition (U is a unitary and H is a Hermitian positive matrices). One can argue that U realize "good" coordinates along Lagrangian embedding $\mathbb{CP}^1 \hookrightarrow \mathbb{CP}^1 \times \mathbb{CP}^1$ and H gives the "momentum direction". Thus, the connection with the sigma model can be obtained if integrated by H (this is well defined in the above limit).

$\mathcal{N}=2$ mechanics on $\mathbb{CP}^1\times\mathbb{CP}^1$ through flat superconnection

- It is useful to look at the action in $\mathcal{N}=2$ superfields from a slightly different angle. We consider an action

$$\mathcal{S}=\frac{1}{2}\int \mathrm{d}t\Bigg(\int \mathrm{d}^2\theta\Big(|\mathcal{Z}_1|^2+|\mathcal{Z}_2|^2+|\Psi|^2\Big)+\lambda\left(\int \mathrm{d}\theta\left(\Lambda_1+\Lambda_2\right)+\text{c.c.}\right)\Bigg)\,, \qquad (8)$$

where we assume that our $\mathcal{Z}$ fields are “covariantly constant” in the sense

$$\begin{pmatrix}\overline{D}\mathcal{Z}_1\\ \overline{D}\mathcal{Z}_2\end{pmatrix}+\underbrace{\begin{pmatrix}\Lambda_1 & \varkappa\Psi\\ 0 & \Lambda_2\end{pmatrix}}_{:=\mathscr{A}}\begin{pmatrix}\mathcal{Z}_1\\ \mathcal{Z}_2\end{pmatrix}=0\,. \qquad (9)$$

We also assume that the “superconnection” $\overline{D}+\mathscr{A}$ is *flat*, i.e. $\overline{D}\mathscr{A}+\mathscr{A}^2=0$.

- From the flatness condition one can deduce that $\Lambda_{1,2}$ superfiedls are chiral and hence $\Lambda_{1,2}=\overline{D}\mathcal{V}_{1,2}$. Thus, we establish an equivalence with the previous formulation (up to a change of variables).

Generalization to $\mathcal{N} = 4$ (K-model)

- It is easy to generalize this story with superconnection to $\mathcal{N} = 4$. Let us consider the action

$$\mathcal{S} = \frac{1}{2} \int \mathrm{d}t \Bigg(\int \mathrm{d}^2\theta \Big(|\mathcal{Z}_1|^2 + |\mathcal{Z}_2|^2 + |\Psi|^2 + |\Phi|^2 \Big) + \lambda \Big(\int \mathrm{d}\theta \left(\Lambda_1 + \Lambda_2 \right) + \text{c.c.} \Big) \Bigg) , \quad (10)$$

where fields are satisfy the nonlinear chirality constraints of the form

$$\begin{pmatrix} \overline{D}\mathcal{Z}_1 \\ \overline{D}\mathcal{Z}_2 \end{pmatrix} + \underbrace{\begin{pmatrix} \Lambda_1 & \varkappa \Psi \\ \varkappa \Phi & \Lambda_2 \end{pmatrix}}_{:=\mathscr{A}} \begin{pmatrix} \mathcal{Z}_1 \\ \mathcal{Z}_2 \end{pmatrix} = 0 \quad (11)$$

with flateness condition.

- One can check that $\Lambda_1 + \Lambda_2$ is chiral but not Λ_1 and Λ_2 separately. This manifest the fact that we cannot introduce monopoles in this model.
- As in previous case, in the limit $\lambda \to \infty$ this model is equivalent to the $\mathcal{N} = 4(a)$ sigma model.

Discussion

Results:

1. We discuss a finite dimensional "truncations into spin chains" (D-model and K-model) for $\mathcal{N} = 2(b)$ and $\mathcal{N} = 4(a)$ sigma models on the sphere. They provide exact solutions in the level of a finite number of harmonics.
2. These models are constructed *via* non-linear chiral multiplets and flat superconnections.
3. The connection between D/K models and sigma models is provided by the "polar decomposition variables".
4. The whole story can be generalize to the flag manifolds (SU(n) (co)adjoint orbits).

Open questions:

1. Other groups and (co)adjoint orbits: SO, Sp, exceptional, non-compact, ∞-dimensional,...
2. Can we construct "long spin chains" by our method? Are they integrable?
3. Other non-linear chiral multiplets