Mechanics on coadjoint orbits

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Based on the joint work with D.Bykov (arXiv:2404.15900 [hep-th] + work in progress)

Definitions

Recall the main definition

$$\mathcal{O}^{\mathrm{G}}_{\Lambda} := \{ \, \mathrm{Ad}_g \Lambda := g\Lambda g^{-1} \, , \; g \in \mathrm{G} \, \} = \frac{\mathrm{G}}{\mathrm{Stab}_{\Lambda}} \, , \tag{1}$$

where G is a compact simple Lie group. We are interested in $\mathrm{SU}(n), \mathrm{SO}(n), \mathrm{Sp}(n)$.

We assume $\Lambda \in \mathfrak{t}$, where $\mathfrak{t} \subset \mathfrak{g}$ is a Cartan subalgebra.

The Kirillov-Kostant-Souriau symplectic form:

$$\omega_x(\xi, \zeta) := \langle x, [\xi, \zeta] \rangle , \tag{2}$$

where $\langle \bullet, \bullet \rangle := \mathrm{Tr}\,(\bullet \cdot \bullet)$ is a Killing form on $\mathfrak{g}$; $x \in \mathcal{O}^{\mathrm{G}}_{\Lambda}$; $\xi, \zeta \; \in \; \mathrm{T}_x \mathcal{O}^{\mathrm{G}}_{\Lambda}$.

The moment map $\mu : \; \mathcal{O}^{\mathrm{G}}_{\Lambda} \mapsto \mathfrak{g}^*$ is given by

$$\mu(x) = \langle x, \bullet \rangle \, . \tag{3}$$

Special embedding

Assume the following decomposition of $\Lambda \in \mathfrak{t}$

$$\Lambda = \sum_{i=0}^{r} C_i \Lambda_i \,, \tag{4}$$

where $C_i \in \mathbb{R}$ and $\Lambda_i \in \mathfrak{t}$ such that $\sum_{i=0}^{r} \Lambda_i = 0$.

Consider the map

$$\mathcal{M} : \mathcal{O}^{G}_{\Lambda} \hookrightarrow \prod_{i=0}^{r} \mathcal{O}^{G}_{\Lambda_i} \,, \tag{5}$$

which is defined by

$$\mathrm{Ad}_g \Lambda \mapsto \{\mathrm{Ad}_g \Lambda_i\}_{i=0}^{r} \,. \tag{6}$$

We assume $\mathrm{Stab}_{\Lambda} = \bigcap_{i=0}^{r} \mathrm{Stab}_{\Lambda_i}$.

Properties of $\mathcal{M}$

The moment map for the diagonal G-action on $\prod_{i=0}^{r} \mathcal{O}^{\mathrm{G}}_{\Lambda_i}$:

$$\tilde{\mu}(x) = \sum_{i=0}^{r} \langle x_i, \bullet \rangle \,, \tag{7}$$

where $x_i \in \mathcal{O}^{\mathrm{G}}_{\Lambda_i}$.

Clearly,

$$\mathcal{M}\left(\mathcal{O}^{\mathrm{G}}_{\Lambda}\right) \subset \tilde{\mu}^{-1}(0) \,. \tag{8}$$

Then

$$\iota_\xi \, \iota_\zeta \, \Omega_{\mathrm{prod}}\big|_{\mathcal{O}^{\mathrm{G}}_{\Lambda}} = \nabla_\xi \, \tilde{\mu}[\zeta]\big|_{\mathcal{O}^{\mathrm{G}}_{\Lambda}} = 0 \,, \tag{9}$$

where $\xi, \zeta \in \mathfrak{g}$.

$\mathcal{M}$ is isotropic!

Proposition

Under the conditions described above, $\mathcal{O}_\Lambda^{\mathrm{G}}$ is an isotropic submanifold of $\prod_{i=0}^{r} \mathcal{O}_{\Lambda_i}^{\mathrm{G}}$.

$\mathcal{O}_\Lambda^{\mathrm{G}}$ is a Lagrangian submanifold of $\prod_{i=0}^{r} \mathcal{O}_{\Lambda_i}^{\mathrm{G}}$ iff

$$2 \dim \left(\mathcal{O}_\Lambda^{\mathrm{G}} \right) = \sum_{i=0}^{r} \dim \left(\mathcal{O}_{\Lambda_i}^{\mathrm{G}} \right) . \tag{10}$$

Corollary

The embedding $\mathcal{O}_\Lambda^{\mathrm{G}} \hookrightarrow \mathcal{O}_\Lambda^{\mathrm{G}} \times \mathcal{O}_{-\Lambda}^{\mathrm{G}}$ is Lagrangian.

Cartan subalgebra:

$$\Lambda = \begin{pmatrix} \lambda_0 \cdot \mathbb{1}_{n_0} & & & \\ & \lambda_1 \cdot \mathbb{1}_{n_1} & & \\ & & \ddots & \\ & & & \lambda_r \cdot \mathbb{1}_{n_r} \end{pmatrix}, \tag{11}$$

where $n_0 + \cdots + n_r = n$ and $\mathrm{Tr}\,(\Lambda) = 0$.
The element Λ generates

$$\mathcal{F}_{n_0,n_1,\ldots,n_r} := \mathcal{O}_\Lambda^{\mathrm{SU}(n)} = \frac{\mathrm{SU}(n)}{\mathrm{S}\,(\mathrm{U}(n_0) \times \cdots \times \mathrm{U}(n_r))}\,. \tag{12}$$

Examples: Lagrangian embeddings of flag manifolds

We construct an embedding via $\{\Lambda_i\}_{i=0}^r$

$$\Lambda_i := \begin{pmatrix} -n_i \cdot \mathbb{1}_{d_{i-1}} & & \\ & (n-n_i)\cdot \mathbb{1}_{n_i} & \\ & & -n_i \cdot \mathbb{1}_{n-d_i} \end{pmatrix}, \tag{13}$$

where $d_i = \sum_{k=0}^{i} n_k$
Every Λ_i generates a Grassmannian manifold

$$\mathrm{Gr}(n_i, n) := \mathcal{O}_{\Lambda_i}^{\mathrm{SU}(n)} = \frac{\mathrm{SU}(n)}{\mathrm{S}\left(\mathrm{U}(n_i)\times \mathrm{U}(n-n_i)\right)} \tag{14}$$

We get the Lagrangian embedding (Bykov'13)

$$\mathcal{F}_{n_0,n_1,\dots,n_r} \hookrightarrow \mathrm{Gr}(n_0,n)\times \mathrm{Gr}(n_1,n)\times\cdots\times \mathrm{Gr}(n_r,n)\,. \tag{15}$$

Examples: SO(2n)-orbits

Cartan subalgebra:

$$\Lambda = \begin{pmatrix} \lambda_0 \cdot \mathbb{1}_{n_0} & & & \\ & \lambda_1 \cdot \mathbb{1}_{n_1} & & \\ & & \ddots & \\ & & & \lambda_r \cdot \mathbb{1}_{n_r} \end{pmatrix} \otimes \mathcal{J}_2 \,, \tag{16}$$

$$\mathcal{J}_2 := \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \,,$$

where $n_0 + \cdots + n_r = n$.

Suppose $\lambda_0 = 0$. Then, Λ generates

$$\mathcal{O}_\Lambda^{\mathrm{SO}(2n)} = \frac{\mathrm{SO}(2n)}{\mathrm{SO}(2n_0) \times \mathrm{U}(n_1) \times \cdots \times \mathrm{U}(n_r)} \,. \tag{17}$$

Define

$$\mathrm{OGr}_m := \frac{\mathrm{SO}(2n)}{\mathrm{SO}(2n - 2m) \times \mathrm{U}(m)} \,. \tag{18}$$

Examples: $\mathrm{SO}(2n)$ embeddings

We choose Λ_i's as follows

$$\Lambda_0 = \begin{pmatrix} 0 \cdot \mathbb{1}_{n_0} & \\ & -1 \cdot \mathbb{1}_{n-n_0} \end{pmatrix} \otimes \mathcal{J}_2 \, ,$$
$$\Lambda_i = \begin{pmatrix} 0 \cdot \mathbb{1}_{d_{i-1}} & & \\ & 1 \cdot \mathbb{1}_{n_i} & \\ & & 0 \cdot \mathbb{1}_{n-d_i} \end{pmatrix} \otimes \mathcal{J}_2 \, , \text{ where } i = 1, 2, \ldots, r \, . \tag{19}$$

Then, the isotropic embedding is

$$\mathcal{O}_\Lambda^{\mathrm{SO}(2n)} \hookrightarrow \mathrm{OGr}_{n-n_0} \times \prod_{i=1}^{r} \mathrm{OGr}_{n_i} \, . \tag{20}$$

Examples: SO(2n) Lagrangian embeddings

One can construct the embedding

$$\frac{\mathrm{SO}(6)}{\mathrm{U}(1)^3} \hookrightarrow \left(\frac{\mathrm{SO}(6)}{\mathrm{U}(3)}\right)^{\times 4} . \tag{21}$$

The corresponding orbits are defined by the matrices

$$\Lambda_1 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \otimes \mathcal{J}_2 , \qquad \Lambda_2 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix} \otimes \mathcal{J}_2 ,$$
$$\Lambda_3 = \begin{pmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix} \otimes \mathcal{J}_2 , \quad \Lambda_4 = \begin{pmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \otimes \mathcal{J}_2 . \tag{22}$$

The Lagrangian embedding:

$$\frac{\mathrm{SO}(2n)}{\mathrm{U}(1)\times\mathrm{U}(n-1)} \hookrightarrow \frac{\mathrm{SO}(2n)}{\mathrm{U}(n)} \times \frac{\mathrm{SO}(2n)}{\mathrm{U}(n)} \times \frac{\mathrm{SO}(2n)}{\mathrm{SO}(2n-2)\times\mathrm{U}(1)}\,. \tag{23}$$

The orbits are defined via

$$\Lambda_1 = \begin{pmatrix} 1 & & & \\ & 1 & & \\ & & \ddots & \\ & & & 1 \end{pmatrix} \otimes \mathcal{J}_2\,, \quad \Lambda_2 = \begin{pmatrix} 1 & & & \\ & -1 & & \\ & & \ddots & \\ & & & -1 \end{pmatrix} \otimes \mathcal{J}_2\,,$$

$$\Lambda_3 = \begin{pmatrix} -2 & & & \\ & 0 & & \\ & & \ddots & \\ & & & 0 \end{pmatrix} \otimes \mathcal{J}_2\,. \tag{24}$$

Consider the embedding

$$\mathcal{F}_{n_0,n_1,\dots,n_r} \hookrightarrow \mathrm{Gr}(n_0,n) \times \mathrm{Gr}(n_1,n) \times \cdots \times \mathrm{Gr}(n_r,n)\,. \tag{25}$$

The Grassmannian $\mathrm{Gr}(n_k,n)$ is parameterized by

$$Z_k = \begin{pmatrix} z_1 & z_2 & \dots & z_{n_k} \end{pmatrix} \quad \text{and} \quad Z_k^\dagger Z_k = p_k \mathbb{1}_{n_k}\,, \tag{26}$$

where $z_i \in \mathbb{CP}^{n-1}$ and Z_k is defined up to an $\mathrm{U}(n_k)$ transformation.
The standard Fubini-Study form on $\mathrm{Gr}(n_k,n)$ is

$$\Omega_{\mathrm{Gr}(n_k,n)} = i\,\mathrm{Tr}\left(\mathrm{d}Z_k^\dagger \wedge \mathrm{d}Z_k\right). \tag{27}$$

Define a matrix

$$Z = \begin{pmatrix} Z_0 & Z_1 & \dots & Z_r \end{pmatrix}\,. \tag{28}$$

The symplectic form on the product of Grassmannians is

$$\Omega_{\mathrm{M}} = i\,\mathrm{Tr}\left(\mathrm{d}Z^\dagger \wedge \mathrm{d}Z\right). \tag{29}$$

Magnetic symplectic form

For generic $Z \in \mathcal{X} = \{\det(Z) \neq 0\}$, use polar-decomposition theorem $Z = UH$, where $U = Z\left(Z^\dagger Z\right)^{-1/2}$ and $H = \left(Z^\dagger Z\right)^{1/2}$.
Define $K := H^2$, then

$$\Omega_M = i\,\mathrm{Tr}\left(P\,\mathrm{d}U^\dagger \wedge \mathrm{d}U\right) + i\,\mathrm{d}\left(\sum_{j\neq k} K_{jk}\,\bar{u}_j \mathrm{d}u_k\right), \tag{30}$$

$$\text{where} \quad P = \begin{pmatrix} p_0 \mathbb{1}_{n_0} & & & \\ & p_1 \mathbb{1}_{n_1} & & \\ & & \ddots & \\ & & & p_r \mathbb{1}_{n_r} \end{pmatrix}. \tag{31}$$

Proposition

$\mathcal{X} \subset \left(\prod_{i=0}^{r} \mathrm{Gr}(n_i, n), \Omega_M\right)$ is symplectomorphic to an open subset of $\left(\mathrm{T}^*\mathcal{F}_{n_0,\ldots,n_r}, \Omega_M^0\right)$.

The numbers $q_i := p_i - p_{i-1}$ for $i = 1, \ldots, r$ turn to be magnetic charges.

Magnetic geodesic flow

The complete flag manifold:

$$\mathcal{F}_n := \mathcal{F}_{1,1,\ldots,1} = \frac{\mathrm{SU}(n)}{\mathrm{S}\left(\mathrm{U}(1)^{\times n}\right)}\,, \tag{32}$$

The action for a dynamical system on $\prod_{i=0}^{r} \mathrm{Gr}(1,n) = \left(\mathbb{CP}^{n-1}\right)^{\times n}$

$$\mathcal{S} = \int_0^\tau \mathrm{d}t \left(i\,\mathrm{Tr}\left(\mathrm{Z}^\dagger \dot{\mathrm{Z}}\right) - \sum_{i<j} \alpha_{ij} |\bar{z}_i z_j|^2 \right)\,, \tag{33}$$

where z_i's are columns of the matrix Z and $\bar{z}_i z_i := p_i$.
Using polar decomposition $\mathrm{Z} = \mathrm{UH}$ we get

$$\mathcal{S} = \int_0^\tau \mathrm{d}t \left(i\,\mathrm{Tr}\left(\mathrm{P}\mathrm{U}^\dagger \dot{\mathrm{U}}\right) + \sum_{j \neq k} \mathrm{K}_{jk}\, \bar{u}_j \dot{u}_k - \sum_{i<j} \alpha_{ij} \mathrm{K}_{ij} \mathrm{K}_{ji} \right)\,. \tag{34}$$

Or

$$\mathcal{S} = \int_0^\tau \mathrm{d}t \left(i\,\mathrm{Tr}\left(\mathrm{P}\mathrm{U}^\dagger \dot{\mathrm{U}}\right) + \sum_{i<j} \frac{1}{\alpha_{ij}} \dot{\bar{u}}_i u_j \bar{u}_j \dot{u}_i \right)\,. \tag{35}$$

Examples

- $\mathcal{F}_2 = \mathbb{CP}^1 = \mathcal{S}^2$
  The metric:

$$\mathrm{d}s^2 = \frac{1}{\alpha}\,\mathrm{d}\bar{u}_1 u_2 \bar{u}_2 \mathrm{d}u_1\,. \tag{36}$$

  Magnetic geodesics (Bolsinov, Jovanovic'2006):

$$\mathrm{U}(t) = \mathrm{U}(0) \times \exp\left[-i\,\alpha \begin{pmatrix} p_1 & a \\ \bar{a} & p_2 \end{pmatrix} t\right]\,.$$

- $\mathcal{F}_3$
  The metric:

$$\mathrm{d}s^2 = \frac{1}{\alpha}\mathrm{d}\bar{u}_1 u_2 \bar{u}_2 \mathrm{d}u_1 + \frac{1}{\beta}\left(\mathrm{d}\bar{u}_2 u_3 \bar{u}_3 \mathrm{d}u_2 + \mathrm{d}\bar{u}_1 u_3 \bar{u}_3 \mathrm{d}u_1\right)\,. \tag{37}$$

  Magnetic geodesics (Arvanitoyeorgos, N. Souris'2020):

$$\mathrm{U}(t) = \mathrm{U}(0)\exp\left[-i\beta \begin{pmatrix} p_1 & a_0 & b_0 \\ \bar{a}_0 & p_2 & c_0 \\ \bar{b}_0 & \bar{c}_0 & p_3 \end{pmatrix} t\right] \begin{pmatrix} \exp\left[i\,(\beta-\alpha)\begin{pmatrix} p_1 & a_0 \\ \bar{a}_0 & p_2 \end{pmatrix} t\right] & 0 \\ 0 & 1 \end{pmatrix}\,.$$

The last example

Consider $\mathcal{F}_n$ equipped with the metric

$$\mathrm{d}s^2 = \sum_{k=2}^{n} \frac{1}{\alpha_k} \mathrm{d}\bar{u}_k \sum_{j=1}^{k-1} u_j \bar{u}_j \mathrm{d}u_k \,. \tag{38}$$

Consider the Hermitian matrix $\mathcal{A}$ such that

$$(\mathcal{A})_{ii} := p_i \,. \tag{39}$$

The geodesics have the following form

$$\mathrm{U}(t) = \mathrm{U}(0) \times \overleftarrow{\prod_{k=2}^{n}} \exp\left[i\left(\alpha_{k+1} - \alpha_k\right) t\, \mathrm{Pr}_k \left(\mathcal{A}^0\right)\right] \,,$$

where $\alpha_{n+1} := 0$, $\mathcal{A}^0 := \mathcal{A}(0)$, Pr_k is a projector on a $(k \times k)$ block in the upper left corner of the matrix and the arrow means that we multiply the matrices from right to left.

Conclusion and outlook

- We have found a remarkable class of isotropic embeddings.
- We have found magnetic geodesics for $\mathrm{SU}(n)$ flag manifolds equipped with metrics from a special class.

- ~~SUSY extension?~~
- Infinite-dimensional Lie groups?
- Non-compact groups?
- Geodesics and the Laplace-Beltrami operator in SO, Sp cases?
- Two-dimensional sigma models on flag manifolds?