Lagrangian manifolds and Petrovsky surfaces corresponding to short-wave asymptotics for hyperbolic systems with abruptly varying coefficients.

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Outline

1. Strictly hyperbolic systems with smooth coefficients

2. Strictly hyperbolic systems with singular coefficients
   - Reflection and transmission of Lagrangian manifolds. Ramifying Hamiltonian billiards and Petrovsky surfaces.
   - Amplitudes of transmitted and reflected waves. Discontinuous coefficients.
   - Smoothed discontinuity. Model equation.

Hyperbolic systems

$$(-i\frac{\partial}{\partial t})^m u = A(t, x, -i\frac{\partial}{\partial t}, -i\frac{\partial}{\partial x})u,$$
$$x \in \mathbb{R}^n, \quad u \in \mathbb{C}^l, \quad A(t, x, p_0, p) - l \times l \quad matrix.$$

Matrix $A(t, x, p_0, p)$ is polynomial of degree m in (p_0, p), smooth in (t, x) and does not depend on (x, t) outside a compact.
Hyperbolicity in Petrovsky sense: equation

$$\det(p_0^m - A_m) = 0$$

has ml real roots $p_0 = -H_k(t, x, p)$, distinct if $p \neq 0$.

Examples

1. One-dimensional systems

$$\frac{\partial u}{\partial t} = A(t,x)\frac{\partial u}{\partial x}, \quad x \in \mathbb{R}, \quad H_k = \lambda_k(t,x)p,$$

λ_k — eigenvalues of A.

2. Wave equation

$$\frac{\partial^2 u}{\partial t^2} = c^2(t,x)\Delta u, \quad x \in \mathbb{R}^n, \quad H_{1,2} = \pm c(t,x)|p|.$$

3. Linearized shallow-water equations

$$\frac{\partial u}{\partial t} + (V(t,x),\nabla)u + (u,\nabla)V(t,x) + \nabla\eta = 0,$$
$$\frac{\partial \eta}{\partial t} + (\nabla, \eta V(t,x)) + (\nabla, c^2(t,x)u) = 0, \quad x \in \mathbb{R}^2,$$
$$H_1 = (V,p), \quad H_{2,3} = (V(t,x),p) \pm c(t,x)|p|$$

4. Massless $(2+1)$- Dirac equations

$$i\frac{\partial u}{\partial t} = \begin{pmatrix} 0 & -i\frac{\partial}{\partial x_1} + \frac{\partial}{\partial x_2} \\ -i\frac{\partial}{\partial x_1} - \frac{\partial}{\partial x_2} & 0 \end{pmatrix} u + V(t,x)u, \quad H_{1,2} = \pm|p|$$

Short-wave initial conditions

$$u|_{t=0} = \varphi^0(x)e^{\frac{iS_0(x)}{h}}, \quad (\frac{\partial}{\partial t})^j u|_{t=0} = 0, \quad j = 1, \dots m{-}1, \quad h \to 0.$$

$S_0 \in C^\infty(\mathbb{R}^n)$, $\varphi^0 \in C_0^\infty(\mathbb{R}^n)$. Two cases

- S_0 — real-valued (rapidly oscillating wave packet);
- S_0 — complex-valued, $\Im S_0 \geq 0$, $\Im S_0 = 0 \Leftrightarrow x \in W_0$, W_0 — smooth k-dimensional surface, $d^2 \Im S_0|_{NW} > 0$ (squeezed state).

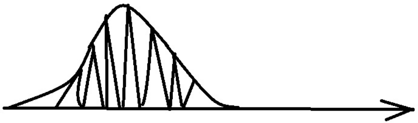

Figure: Wave packet

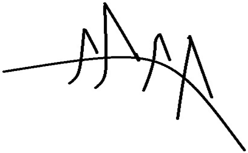

Figure: Squeezed state

Construction of asymptotics. Small times — without focal points ($\Lambda_k^t : p = \frac{\partial S_k(t,x)}{\partial x}$) — WKB-formulae

$$u \sim \sum_{k=1}^{ml} \mathrm{e}^{\frac{iS_k(t,x)}{h}} (\sum_{s=0}^{\infty} h^s \varphi_{k,s}(t,x)),$$

$$\frac{\partial S_k}{\partial t} + H_k(t, x\frac{\partial S_k}{\partial x}) = 0, \quad S_k|_{t=0} = S_0(x).$$

Arbitrary finite time. Phase space $\mathbb{R}^{2n}_{(x,p)}$; initial Lagrangian surface $\Lambda_0 : p = \frac{\partial S_0}{\partial x}$.
Hamiltonan systems

$$\dot{x} = \frac{\partial H_k}{\partial p}, \quad \dot{p} = -\frac{\partial H_k}{\partial x}.$$

Lagrangian surfaces $\Lambda_k^t = g_k^t \Lambda_0$.
Volume forms $\sigma_0 = dx$ on Λ_0, $\sigma_t = (g_k^t)^* dx$ on Λ_k^t.

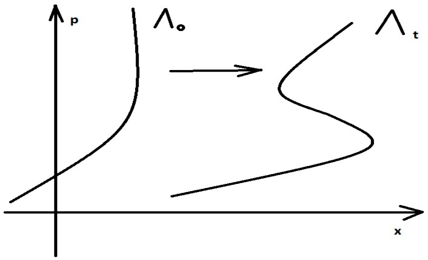

Figure: Lagrangian surface

Theorem

(V.P. Maslov, $\sim$ 1965). Under certain technical conditions the solution $u(x,t,h)$ can be represented as asymptotic series

$$u \sim \sum_{k=1}^{ml} K_{\Lambda_k^t, \sigma_k^t} \Big(\sum_{s=0}^{\infty} h^s \varphi_{k,s} \Big),$$

$K : C_0^\infty(\Lambda_k^t) \to C^\infty(\mathbb{R}_x^n)$ is the Maslov canonical operator, $\varphi_{k,s}$ are smooth functions on Λ_t^t, computed recurrently in terms of Hamiltonian trajectories; $\varphi_{k,0}$ are eigenvectors of $A_m(t,x,-H_k,p)$.

Sqeezed states. Simplest case:

$$S_0 = (p_0, x - x_0) + \frac{1}{2}(x - x_0, Q_0(x - x_0))), \quad p_0 \in \mathbb{R}^n, Q^t = Q, \Im Q > 0.$$

W_0 is the point x_0, $\rho_0 : \xi_p = Q_0 \xi_x$.

$$u(x,t,h) \sim \sum_{k=1}^{ml} e^{\frac{iS_k(x,t)}{h}} \sum_{s=0}^{\infty} (h^s \varphi_{k,s}(x,t)).$$

$$S_k = q_k(t) + (P_k(t), x - X_k(t)) + \frac{1}{2}(x - X_k(t), Q_k(t)(x - X_k(t))),$$

$$\dot{X}_k = \frac{\partial H_k}{\partial p}, \quad \dot{P}_k = -\frac{\partial H_k}{\partial x},$$

Q can be expressed explicitly in terms of solutions of the linearized system.

Solutions, corresponding to complex vector bundles

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Localized ("squeezed") initial state $S_0(x)$ is complex, $\Im S_0 \geq 0$, $\Im S_0 = 0$ on the smooth k-dimensional surface W_0, $d^2 \Im S_0|_{NL_0} > 0$. Consider k-dimensional isotropic surface $\Lambda_0 \subset \mathbb{R}^{2n}$: $x \in W_0, p = \frac{\partial S_0}{\partial x}$ and n-dimensional complex vector bundle ρ_0 over Λ_0 (Maslov complex germ): fiber $\rho(x, p)$ is the plane in ${}^{\mathbb{C}}T_{x,p}\mathbb{R}^{2n}$, $\xi_p = \frac{\partial^2 S_0}{\partial x^2}\xi_x$. Shifted bundle $\Lambda_k^t = g_k^t \Lambda_0$, $\rho_k^t = dg_k^t \rho_0$.

Theorem (V.P. Maslov)

Under certain technical conditions the solution $u(x, t, h)$ can be represented as asymptotic series

$$u \sim \sum_{k=1}^{ml} \hat{K}_{\Lambda_k^t, \rho_k^t} \Big(\sum_{s=0} h^s \varphi_{k,s} \Big),$$

$\hat{K} : C_0^\infty(\Lambda_k^t) \to C^\infty(\mathbb{R}_x^n)$ *is the Maslov canonical operator on the complex germ,* $\varphi_{k,s}$ *are smooth functions on* Λ_k^t.

$$(-i\frac{\partial}{\partial t})^m u = A(t,x,-i\frac{\partial}{\partial t},-i\frac{\partial}{\partial x})u,$$
$$x \in \mathbb{R}^n, \quad u \in \mathbb{C}^l, \quad A(t,x,p_0,p) - l \times l \quad matrix.$$

We consider two situations.

1. A is discontinuous on an orientable hypersurface $M^{n-1} \subset \mathbb{R}^n_x$ and smooth outside M, $A = A^{\pm}(t,x,p_0,p)$ at the positive (negative) side of M.
2. A is rapidly varying near M: $A = A(\frac{\Phi(x)}{h}, t, x, p_0, p)$, $A(y,t,x,p_0,p) = A^{\pm}(t,x,p_0,p)$ as $y \notin [y_-, y_+]$, where M is defined by the equation $\Phi(x) = 0$.

For each side of M we have Hamiltonians $H_k^{\pm}$ and corresponding trajectories. We assume that $\mathrm{supp}\varphi^0$ is small and

$\Phi|_{\mathrm{supp}\varphi^0} < 0.$
For sufficiently small t (until trajectories of Hamiltonan systems reach M) solution is the same as for smooth coefficients.

The main problem: what happens with Lagrangian surfaces and amplitudes $\varphi_{k,s}$ near M?

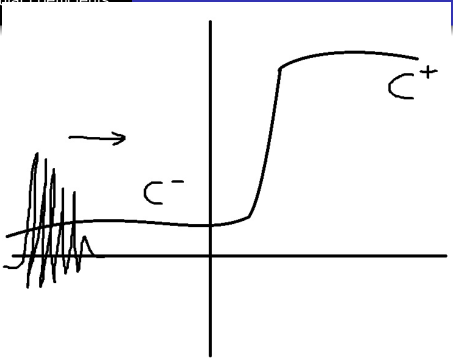

Figure: Scattering

Main effects

1. Many reflected and transmitted waves, defined by different Lagrangian surfaces.

2. Total reflection. Transmitted wave can dissapear.

Figure: Total reflection

Outline

Lagrangian surfaces, corresponding to incident waves
Extended phase space. $\Lambda_k^0 \subset \mathbb{R}^{2n+2}$, $p = \frac{\partial S_0}{\partial x}$, $t = 0$,
$p_0 + H_k^-(t, x, p) = 0$,
Hamiltonian systems

$$\dot{x} = \frac{\partial H_k^-}{\partial p}, \quad \dot{p} = -\frac{\partial H_k^-}{\partial x}, \quad \dot{t} = 1, \quad \dot{p}_0 = -\frac{\partial H_k^-}{\partial t},$$

$\Lambda_k = \cup_s g_\pm^s \Lambda_k^0$

Consider small interval of time, when trajectories of the Hamiltonian H_1^- intersect M (for others Hamiltonians trajectories still stay it the domain $\Phi < 0$).
Surface $\hat{M} \subset \mathbb{R}^{2n+2}$: $x \in M$, t, p_0, p — arbitrary (the lifting of M to the phase space),
$N = \Lambda_1 \bigcap \hat{M}$.
We assume that on the surface N, for some $\delta > 0$, $\frac{\partial H_1^-}{\partial p_n} \geq \delta$. ($p_n$ — normal to M component of the vector p) — trajectories are transversal to M.

In order to describe reflected and transmitted waves, we have to consider roots of the following equations

1. Reflecting roots

$$H_k^-(t,x,p_0,p_\tau,\varkappa) = H_1^-(t,x,p_0,p_\tau,p_n), \quad \frac{\partial H_k^-}{\partial p_n} < 0$$

or

2. Transmitting roots

$$H_k^+(t,x,p_0,p_\tau,\varkappa) = H_1^-(t,x,p_0,p_\tau,p_n), \quad \frac{\partial H_k^+}{\partial p_n} > 0$$

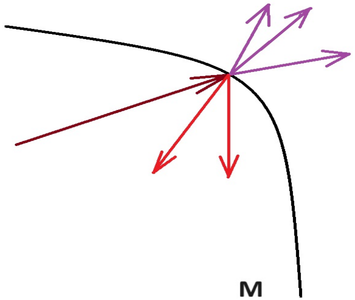

Figure: Ramifying billiard

Lemma

(A.I. Allilueva, A.S.) There exists at least one either reflecting or transmitting root

Consider also complex roots; in the first case we choose $\Im\varkappa < 0$, in the second — $\Im\varkappa > 0$.

Lemma

(A.I. Allilueva, A.S.) $\#$ (complex reflecting roots)+$\#$ (complex transmitting roots)=ml.

Proof is based on the study of intersections of a certain line in $\mathbb{R}P^n$ with the Petrovsky surfaces

$$\Gamma : \det(p_0^m - A_m^{\pm}) = 0$$

Theorem

(I.G. Petrovskii, 1945) $\Gamma = \bigcup_j^{ml/2} \Gamma_j$, *if ml is even,*
$\Gamma = \bigcup_j^{[ml/2]} \Gamma_j \bigcup \Gamma_0$, *if ml is odd.*
$\Gamma_j \cong S^{n-1}, \quad \Gamma_0 \cong \mathbb{R}P^{n-1}$.

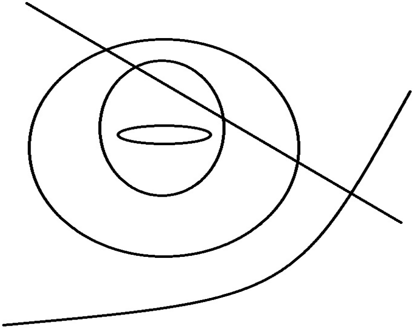

Figure: Petrovsky surface

Reflected and transmitted Lagrangian surfaces
Mappings $Q_k^\pm : \hat{M} \to \hat{M}$:
$Q_k^\pm(t, x, p_0, p_\tau, p_n) = (t, x, p_0, p_\tau, \varkappa_k(t, x, p))$, $\varkappa_k$ — real (transmitting or reflecting) roots.
$N_k^\pm = Q_k^\pm(N)$. We shift $N_k^\pm$ along the trajectories of the Hamiltonian systems with Hamiltonians $H_k^\pm$.

$$\Lambda_k^\pm = \bigcup_{s\in\mathbb{R}} g_{s,k}^\pm N^\pm$$

For complex roots let

$$\Lambda_k^\pm = N \subset T^*(M_x \times \mathbb{R}_t)$$

WKB-case (no focal points). Construction of phases for reflected and transmitted waves.

$$\frac{\partial S_k^{\pm}}{\partial t} + H_k^{\pm}(t, x, \frac{\partial S_k^{\pm}}{\partial x}) = 0,$$

$$S_k^{\pm}|_M = S_1|_M, \quad \frac{\partial S_k^{\pm}}{\partial \nu}|_M = \varkappa_k^{\pm}(t, x, \frac{\partial S_1}{\partial t}, \frac{\partial S_1}{\partial x})|_M$$

ν — unit normal to M.

Outline

Let A be discontinuous.

Theorem

(A.I. Allilueva, A.S.) During certain time interval

$$u \sim \sum_{k=1}^{ml} K_{\Lambda_k}\Big(\sum_{s=0}^{\infty} h^s \varphi_{k,s}\Big) + \\ + \sum_{k'} K_{\Lambda^-_{k'}}\Big(\sum_{s=0}^{\infty} h^s \varphi^-_{k',s}\Big) + \sum_{k''} K_{\Lambda^-_{k''}}\Big(\mathrm{e}^{\frac{i\varkappa^-_k \Phi}{h}} \sum_{s=0}^{\infty} h^s \varphi^-_{k'',s}\Big),$$

on the negative part of M,

$$u \sim \sum_{k'} K_{\Lambda^+_{k'}}\Big(\sum_{s=0}^{\infty} h^s \varphi^+_{k',s}\Big) + \sum_{k''} K_{\Lambda^+_{k''}}\Big(\mathrm{e}^{\frac{i\varkappa^+_k \Phi}{h}} \sum_{s=0}^{\infty} h^s \varphi^+_{k'',s}\Big)$$

on the positive part of M.

Here indexes k' correspond to real and k'' — to complex reflecting and transmitting roots; amplitudes $\varphi^{\pm}_{k',s}$ are computed explicitly in terms of corresponding Hamiltonian trajectories, $\varphi^{\pm}_{k'',s}$ are computed algebraically. On the surface $\hat{M}$ the leading amplitudes have the following form

$$\varphi^{+}_{k,0} = \sigma_1 \tau_k e^{+}_{k}, \quad \varphi^{-}_{k,0} = \sigma_1 r_k e^{-}_{k},$$

where $\varphi_1 = \sigma_1 e^{-}_{1}$, τ_k, r_k are defined by the system of ml linear equations

$$\sum_k \tau_k(\varkappa^{+}_{k})^j e^{+}_{k} - \sum_k r_k(\varkappa^{-}_{k})^j e^{-}_{k} = p^j_n e_1, \quad j = 0, \dots, m-1,$$

$e^{\pm}_{k}$ are eigenvectors of the matrix $A^{\pm}_{m}$, corresponding to eigenvalues $(-H^{\pm}_{k})^m$.

Outline

Let $A = A(\frac{\Phi(x)}{h}, t, x, p_0, p)$. In order to compute amplitudes consider at points of N scattering problem for the model equation

$$(A_m(y, t, x, p_0, p_\tau, -i\frac{d}{dy}) - p_0^m)w = 0.$$

If $y \notin [y_-, y_+]$ we have equations with constant coefficients $A_m^\pm$; let $L^\pm$ are spaces of their solutions and $B : L^+ \to L^-$ — the monodromy operator ($ml \times ml$-matrix). Transmitting and reflecting roots correspond to subspaces $O^\pm \subset L^\pm$, $\dim O^- + \dim O^+ = ml$. We assume that the following condition of general position hold on N

$$B(O^+) \oplus O^- = L^-.$$

We have to consider Lagrangian manifolds, corresponding to real reflecting and transmitting roots only.

Theorem

(A.I. Allilueva, A.S.) During certain time interval

$$u \sim \sum_{k=2}^{ml} K_{\Lambda_k}\Big(\sum_{s=0}^{\infty} h^s \varphi_{k,s}\Big) + K_{\Lambda_1}\Big(\sum_{s=0}^{\infty} h^s f_{1,s}\Big(\frac{\Phi(x)}{h},\cdot\Big)\Big)$$

$$+ \sum_{k} K_{\Lambda_k^-}\Big(\sum_{s=0}^{\infty} h^s f_{k,s}^{-}\Big(\frac{\Phi(x)}{h},\cdot\Big)\Big) +$$

$$+ \sum_{k} K_{\Lambda_k^+}\Big(\sum_{s=0}^{\infty} h^s f_{k,s}^{+}\Big(\frac{\Phi(x)}{h},\cdot\Big)\Big)$$

Functions $f^{\pm}_{k,s}$ are expressed in terms of the model equation.
Leading term.
The incident wave $K_{\Lambda_1}(\varphi_{1,0})$ defines at the points of the surface N vector $\varphi_{1,0}\mathrm{e}^{ip_n y} = \sigma_1\xi \in L^-$; $\xi = e_1^-\mathrm{e}^{ip_n y}$, e_1 — eigenvector of A_m^-. Let $w^+ = B^{-1}\Pi(\xi)$, Π — projection to $B(O^+)$ along O^- and $w^- = (\Pi - 1)\xi$. We fix the solution w of the model equation by the conditions

$$w = \xi + w^-, \quad y < y_-, \quad w = w^+, \quad y > y_+.$$

Asymptotics of w:

$$w \to \xi + \sum_k r_k e_k^- \mathrm{e}^{i\varkappa_k^- y}, \quad y \to -\infty,$$

$$w \to \sum_k \tau_k e_k^+ \mathrm{e}^{i\varkappa_k^+ y}, \quad y \to +\infty,$$

$\varkappa_k^\pm$ — real reflecting and transmitting roots. We shift $\sigma_1 r_k e_k^-$ and $\sigma_1 \tau_k e_k^+$ along corresponding Hamiltonian trajectories, obtaining $\varphi_{k,0}^\pm$.

Let $\eta(y) = \frac{1}{2}(1 + \tanh y)$. We construct $f^{\pm}_{k,0}$, $f_{1,0}$ as follows

$$f^{+}_{k,0} = \eta(y)\varphi^{+}_{k,0}, \quad f^{-}_{k,0} = (1 - \eta(y))\varphi^{-}_{k,0},$$

$$f_{1,0} = \varphi_{1,0}(1 - \eta) + \mathrm{e}^{-ip_1 y}\Big(w - \sum_k f^{-}_{k,0}\mathrm{e}^{i\varkappa^{-}_k y} - \sum_k f^{+}_{k,0}\mathrm{e}^{i\varkappa^{+}_k y}\Big).$$

Reflection and transmission of vector bundles. Complex Lagrangian planes correspond to quadratic forms — matrices $Q^{\pm}$: $\rho : p = Qx$. Rules of reflection:

$$Q^{-}|_{T_M} = Q^{+}|_{T_M} + 2p_n b,$$

b is the second fundamental form of M.

THANK YOU
FOR YOUR ATTENTION!