Lyapunov-Krasovskii functional with prescribed derivative and the Lyapunov matrix for a partial differential equation with delay

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Problems of the Modern Mathematical Phisics (PMMP'25)
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Problem statement

Consider system

$$\begin{cases} u_t(x,\ t) = au_{xx}(x,\ t) + bu(x, t-h), \quad x \in (0, l),\ t > 0, \\ u(0,\ t) = u(l,\ t) = 0, \quad t \geq 0, \\ u(x,\ \theta) = \varphi(x,\ \theta), \quad \theta \in [-h, 0],\ x \in [0, l], \end{cases} \tag{1}$$

where φ is an initial function.
The solution can be presented as a series

$$u(x, t) = \sum_{i=1}^{\infty} g_i(t) \sin(\pi i x), \tag{2}$$

where g_i satisfy the system

$$\begin{cases} \dot{g}_i(t) = -\mu_i g_i(t) + b g_i(t-h), \\ g_i(\theta) = \varphi_i(\theta), \quad \theta \in [-h, 0], \end{cases} \tag{3}$$

where $\varphi_i(\theta)$ are coefficients of the Fourier series expansion of the function φ, $\mu_i = -a(\pi i)^2$ are coefficient.

Solution of the differential-difference equation

Theorem (Cauchy formula)

By the initial functions φ_i we construct the solutions of the system (3):

$$g_i(t) = K_i(t)\varphi_i(0) + b\int_{-h}^{0} K_i(t-h-\theta)\varphi_i(\theta)d\theta, \quad t > 0, \tag{4}$$

where the explicit form of the coefficients of the Fourier series expansion for the function φ is as follows

$$\varphi_i(\theta) = 2\int_{0}^{1} \sin(\pi i x)\varphi(x,\theta)dx, \quad i \in \mathbb{N}, \quad \theta \in [-h, 0], \tag{5}$$

$K_i(t)$ *is the fundamental solution and it satisfies the equation:*

$$\frac{dK_i(t)}{dt} = -\mu_i K_i(t) + bK_i(t-h),\ t \geq 0, \tag{6}$$

and $K_i(t) = 0$ *for all* $t \in [-h, 0)$, $K_i(0) = 1$.

Solution of the original system

Turn back to system (1)

$$\begin{cases} u_t(x,\ t) = au_{xx}(x,\ t) + bu(x, t-h), \quad x \in (0,1),\ t > 0, \\ u(0,\ t) = u(1,\ t) = 0, \quad t \geq 0, \\ u(x,\ \theta) = \varphi(x,\ \theta), \quad \theta \in [-h, 0],\ x \in [0,1]. \end{cases}$$

The solution can be represented as

$$u(x,t,\varphi) = \sum_{i=1}^{\infty} \sin(\pi i x) \left[K_i(t)\varphi_i(0) + b \int_{-h}^{0} K_i(t-h-\theta)\varphi_i(\theta)d\theta \right]. \qquad (7)$$

We consider a function $u(\cdot, t, \varphi)$ as an element of the Lebesgue space $\mathcal{L}_2((0,1))$.

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Construction of functional v_0

Theorem

Let system (1) be exponentially stable. If a non-positive quadratic derivative along the solutions of the system (1) is given

$$\frac{d}{dt}v_0(\varphi) = -\left\|\varphi(\cdot,0)\right\|^2_{\mathcal{L}_2}, \tag{8}$$

then the functional has the form

$$\begin{aligned} v_0(\varphi) = &\int_0^1 \varphi(y_2,0)\int_0^1 U(0,y_1,y_2)\varphi(y_1,0)dy_1dy_2 \\ &+ 2b\int_0^1 \varphi(y_1,0)\int_0^1\int_{-h}^0 U(-h-\theta,y_1,y_2)\varphi(y_2,\theta)d\theta dy_2dy_1 \\ &+ b^2\int_0^1\int_{-h}^0 \varphi(y_1,\theta_1)\int_0^1\int_{-h}^0 U(\theta_1-\theta_2,y_1,y_2)\varphi(y_2,\theta_2)d\theta_2dy_2d\theta_1dy_1. \end{aligned} \tag{9}$$

The Lyapunov matrix in this functional can be represented as a series

$$U(\tau, y_1, y_2) = 2\sum_{i=1}^{\infty} \sin(\pi i y_1)\sin(\pi i y_2)\int_0^{\infty} K_i(t)K_i(t+\tau)dt, \tag{10}$$

$$\tau \in \mathbb{R}, \quad y_1 \in [0,1], \quad y_2 \in [0,1].$$

The idea of the proof. We want to choose a functional with a given non-positive derivative of the form

$$\frac{d}{dt}v_0(\varphi) = -\left\|\varphi(\cdot,0)\right\|^2_{\mathcal{L}_2}. \tag{11}$$

The functional can be defined by the formula

$$v_0(\varphi) = \int_0^{\infty}\int_0^1 \Big(u(x,t,\varphi)\Big)^2 dxdt, \tag{12}$$

where $u(x,t,\varphi)$ is the solution of the original system(1).

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Lyapunov matrix

Computation of the Lyapunov matrix using series:

$$U(\tau, y_1, y_2) = 2 \sum_{i=1}^{\infty} \sin(\pi i y_1) \sin(\pi i y_2) \int_0^{\infty} K_i(t) K_i(t+\tau) dt, \qquad (13)$$

$$\tau \in \mathbb{R}, \quad y_1 \in [0,1], \quad y_2 \in [0,1].$$

Lyapunov matrix

An alternative method for computing the Lyapunov matrix:

Theorem

The Lyapunov matrix (13) *is represented as absolutely and uniformly convergent on the set* $\{\tau \in [-1,1],\ y_1, y_2 \in [0,1]\}$ *series*

$$U(\tau, y_1, y_2) = \sum_{i=1}^{\infty} \sin(\pi i y_1) \sin(\pi i y_2)\, u_i(|\tau|). \tag{14}$$

Here each of the functions $u_i(\tau)$, $\tau \in [0,1]$, $i \in \mathbb{N}$, *is defined by the corresponding formula from Theorem 2 in article* [1] . *In particular, for sufficiently large* i

$$u_i(\tau) = \frac{b \sinh \lambda_i (1-\tau) + \mu_i \sinh \lambda_i \tau - \lambda_i \cosh \lambda_i \tau}{\lambda_i (b \cosh \lambda_i - \mu_i)}, \quad \tau \in [0,1],$$

where $\lambda_i = \sqrt{\mu_i^2 - b^2}$, $\mu_i = -a(\pi i)^2$.

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[1] Egorov A. V., Mondie S. A stability criterion for the single delay equation in terms of the Lyapunov matrix // V. of St. Petersburg Univ., S. 10., App. math. Comp. science. Control proc., 2013.

Lyapunov matrix

The idea of the proof.

$$2\int_0^{\infty} K_i(t)K_i(t+\tau)\,dt = u_i(|\tau|) \tag{15}$$

It remains to prove the uniform absolute convergence of series (14)

$$\max_{\tau\in[0,1]} u_i(\tau) = O(i^{-2}). \tag{16}$$

It is necessary to prove that the expression is uniform in τ and in i

$$H_i(\tau) = \frac{b\sinh\lambda_i(1-\tau)+\mu_i\sinh\lambda_i\tau-\lambda_i\cosh\lambda_i\tau}{b\cosh\lambda_i-\mu_i}, \tag{17}$$

where H_i determined from the formula $u_i = \frac{H_i}{\lambda_i}$.
For sufficiently large values of i the following estimate is valid:

$$|H_i(\tau)| \leq 8.$$

Lyapunov matrix

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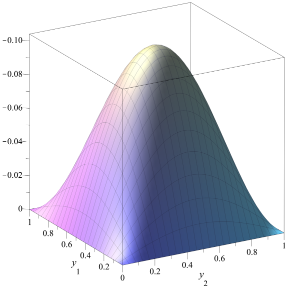
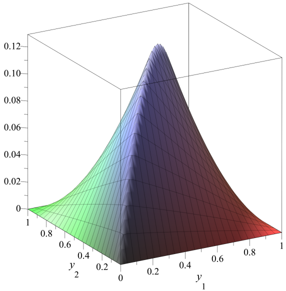
System parameters: $a = 1, b = 3, N = 50.$



(a) $\tau = 0.8$



(b) $\tau = 0$

$$\sup_{y_1, y_2 \in [0,1], \tau \in [-1,1]} \left| U(\tau, y_1, y_2) - \sum_{i=1}^{N} \sin(\pi i y_1) \sin(\pi i y_2)\, u_i(|\tau|) \right| \leq \int_{N+1}^{\infty} \frac{1}{a\pi t^2} dt \approx 0.0065.$$

Conclusion

1. This paper presents a method of constructing functional for a delay equation with distributed parameters.
2. In the case of exponential stability of the system, an explicit form of this functional based on the Lyapunov matrix is presented.
3. An approximate formula is obtained for Lyapunov matrix for $a > 0$; the case $a < 0$ requires further investigation.

Bibliography

1. Egorov A. V., Mondie S. A stability criterion for the single delay equation in terms of the Lyapunov matrix // V. of St. Penersburg Univ. S. 10. App. math. Computer science. Control processes, 2013, issue 1, pp. 106–115.
2. Kharitonov V. L. Lyapunov functionals with a given derivative. I. Functionals of the full type // Vestnik of SPbU. 2005. T. 10, No 1.
3. Shuxia P. Asymptotic behavior of travelling fronts of the delayed Fisher equation // Nonlinear Analysis: Real World Applications. 2009. No 10. P. 1173-1182. C. 110-117.
4. Kharitonov V. L., Zhabko A. P. Lyapunov–Krasovskii approach to the robust stability analysis of time-delay systems // Automatica, 2003. Vol. 39, No 1. P. 15–20.