Perfect fluid dynamics with conformal Newton-Hooke symmetries

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Plan

1. Symmetries of non-relativistic (NR) perfect fluid equations
   - Free perfect fluid (Galilei $\rightarrow$ Schrodinger)
   - Perfect fluid in harmonic trap (Newton-Hooke $\rightarrow$ conformal NH)
2. Conformal extensions of Galilei and Newton-Hooke algebras
3. Generalized conformal perfect fluid dynamics
4. Niederer's transformation

The talk is based on [TS, arXiv:2501.16781].

NR perfect fluid and its symmetries

Perfect fluid equations

In non-relativistic space-time $(t, x_i),\ i = 1, ..., d$ a compressible fluid is characterized by a density $\rho(t, x)$ and the velocity $v_i(t, x)$. The evolution over time is described by the continuity equation

$$\partial_0 \rho + \partial_i (\rho v_i) = 0.$$

and the Euler equation

$$\mathcal{D} v_i = -\frac{1}{\rho} \partial_i p + \frac{f_i}{\rho}, \quad \text{where} \quad \mathcal{D} = \partial_0 + v_i \partial_i$$

where pressure $p(t, x)$ is assumed to be related to $\rho(t, x)$ via an equation of state

$$p = p(\rho).$$

and $f_i = \mathbf{f}$ designate external forces. For example:

- $\mathbf{f} = \rho \mathbf{g}$ – gravitational force with acceleration $\mathbf{g}$
- $\mathbf{f} = \frac{1}{c} \mathbf{j} \times \mathbf{H}$ – Lorentz force in magnetic field $\mathbf{H}$

Free NR perfect fluid and its symmetries, $f_i = 0$

Hamiltonian formulation

The Hamiltonian=energy reads

$$H = \int dx \left(\frac{1}{2} \rho \upsilon_i \upsilon_i + V \right), \quad p = \rho V' - V$$

It generates the continuity equation and the Euler equation in the usual way

$$\partial_0 \rho = \{\rho, H\} = -\partial_i (\rho \upsilon_i), \qquad \partial_0 \upsilon_i = \{\upsilon_i, H\} = -\upsilon_j \partial_j \upsilon_i - \frac{1}{\rho} \partial_i p$$

provided the non-canonical Poisson brackets for ρ and υ_i are chosen [P. Morrison, J. Greene, 1980]

$$\begin{aligned} \{\rho(x), \upsilon_i(y)\} &= -\partial_i \delta(x, y), \\ \{\upsilon_i(x), \upsilon_j(y)\} &= \frac{1}{\rho} \left(\partial_i \upsilon_j - \partial_j \upsilon_i \right) \delta(x, y). \end{aligned}$$

NR perfect fluid and its symmetries, $f_i = 0$

Stress-energy tensor $T^{\mu\nu}$

For a specific equation of state and $f_i = 0$ the symmetry group coincides with the Schrodinger group. One way to see this is to make recourse to the NR energy-momentum tensor (see e.g. [R.Jackiw, V.Nair, S.Pi, A.Polychronakos, 2004])

$$T^{00} = \frac{1}{2}\rho v_i v_i + V, \qquad T^{i0} = \rho v_i(\frac{1}{2}v_j v_j + V')$$
$$T^{0i} = \rho v_i, \qquad T^{ji} = \rho v_i v_j + \delta_{ij} p,$$

The components satisfy the continuity equations

$$\partial_0 T^{00} + \partial_i T^{i0} = 0, \qquad \partial_0 T^{0i} + \partial_j T^{ji} = 0,$$

as well as the algebraic condition

$$2T^{00} = \delta_{ij} T^{ij}, \qquad V = \frac{1}{2} dp. \tag{1}$$

Firstly, $T^{i0} \neq T^{0i}$ because the theory is not Lorentz-invariant but $T^{ij} = T^{ji}$ because it is invariant under spatial rotations. Secondly, the condition (1) is satisfied only for $p \sim \rho^{1+\frac{2}{d}}$ and it is the analogue of the tracelessness condition characterizing a relativistic conformal field theory.

NR perfect fluid and its symmetries, $f_i = 0$

Conserved charges

Denoting conserved charges associated with the temporal translation, spatial translation, spatial rotations, Galilei boost, dilation and special conformal transformation by H, P_i, M_{ij}, C_i, D, and K, respectively, one readily finds

$$
\begin{aligned}
H &= \int dx T^{00} = \int dx(\frac{1}{2}\rho \upsilon_i \upsilon_i + V), \\
P_i &= \int dx T^{0i} = \int dx \rho \upsilon_i, \\
C_i &= \int dx(T^{0i}t - \rho x_i) = tP_i - \int dx \rho x_i, \\
M_{ij} &= \int dx(T^{0i}x_j - T^{0j}x_i) = \int dx(\rho \upsilon_i x_j - \rho \upsilon_j x_i), \\
D &= \int dx(T^{00}t - \frac{1}{2}T^{0i}x_i) = tH - \frac{1}{2}\int dx \rho \upsilon_i x_i, \\
K &= \int dx(T^{00}t^2 - T^{0i}tx_i + \frac{1}{2}\rho x_i x_i) = -t^2 H + 2tD + \frac{1}{2}\int dx \rho x_i x_i.
\end{aligned}
$$

Motion of the center of mass $mX_i = \int dx \rho x_i$

$$mX_i = tP_i - C_i, \quad m = \int dx \rho$$

NR perfect fluid and its symmetries, $f_i = 0$

Algebra of conserved charges

Within the Hamiltonian formulation the conserved charges do satisfy the structure relations of the Schrodinger algebra under the Poisson brackets

$$\{H, P_i\} = 0, \qquad \{P_i, M_{jk}\} = \delta_{ij} P_k - \delta_{ik} P_j,$$
$$\{H, C_i\} = P_i, \qquad \{C_i, M_{jk}\} = \delta_{ij} C_k - \delta_{ik} C_j,$$
$$\{P_i, C_j\} = \delta_{ij} m, \qquad \{M_{ij}, M_{ab}\} = \delta_{i[a} M_{b]j} - \delta_{j[a} M_{b]i},$$

$$[H, D] = H, \qquad [D, P_i] = -\frac{1}{2} P_i,$$
$$[H, K] = 2D, \qquad [D, C_i] = \frac{1}{2} C_i,$$
$$[D, K] = K, \qquad [K, P_i] = -C_i.$$

where H, D, K form the conformal $so(2,1)$ subalgebra and total mass $m = \int dx \rho$ is central charge.

NR perfect fluid and its symmetries

Galilei and Newton-Hooke algebras

As is well known, the Galilei algebra can be considered as a contraction of the Newton-Hooke algebra [H.Bacry, J.-M.Levy-Leblond, 1967] in which the cosmological constant tends to zero (the flat space limit)

$$\{H, P_i\} = -\frac{1}{R^2} C_i, \qquad \{P_i, M_{jk}\} = \delta_{ij} P_k - \delta_{ik} P_j,$$
$$\{H, C_i\} = P_i, \qquad \{C_i, M_{jk}\} = \delta_{ij} C_k - \delta_{ik} C_j,$$
$$\{P_i, C_j\} = \delta_{ij} m, \qquad \{M_{ij}, M_{ab}\} = \delta_{i[a} M_{b]j} - \delta_{j[a} M_{b]i},$$

The Newton-Hooke algebra follows from the (anti) de Sitter algebra in the non-relativistic limit in much the same way as the Galilei algebra results from the Poincaré algebra.
A natural question arises as to how to formulate perfect fluid equations in non-relativistic spacetime with cosmological constant.

- to analyze the non-relativistic limit of the relativistic hydrodynamics equations formulated in (anti) de Sitter space [Y.Tian, H.Guo, C.Huang, Z.Xu, B.Zhou, 2004]
- One of the subjects: to start with the non-relativistic hydrodynamics equations and accommodate the Newton-Hooke symmetry there.

NR perfect fluid and its symmetries, $f_i = -\omega^2 \rho x_i$

Perfect fluid equations in harmonic trap

Let us consider a perfect fluid in the harmonic trap specified by $f_i = -\omega^2 \rho x_i$, where ω^2 is a positive constant of dimension $[\omega] = t^{-1}$, which is assumed to be small. The Euler equation takes on the form

$$\mathcal{D}\upsilon_i + \omega^2 x_i = -\frac{1}{\rho}\partial_i p.$$

Together with the continuity equation it can be represented in the Hamiltonian form

$$\partial_0 \rho = \{\rho, H\} = -\partial_i(\rho \upsilon_i), \quad \partial_0 \upsilon_i = \{\upsilon_i, H\} = -\upsilon_j \partial_j \upsilon_i - \omega^2 x_i - \frac{1}{\rho}\partial_i p$$

where

$$H = \frac{1}{2}\rho \upsilon_i \upsilon_i + \frac{1}{2}\omega^2 \rho x_i x_i + V, \quad p = \rho V' - V,$$

and the Poisson brackets are the same as in free case.

NR perfect fluid and its symmetries, $f_i = -\omega^2 \rho x_i$

Conserved charges

Similarly to the harmonic oscillator, one can construct integrals of motion that link to spatial translations, the Galilei boost and spatial rotations

$$\begin{aligned} P_i &= \int dx(\rho\upsilon_i \cos\omega t + \omega\rho x_i \sin\omega t), \quad \delta x_i = \cos\omega t a_i \\ C_i &= \frac{1}{\omega}\int dx(\rho\upsilon_i \sin\omega t - \omega\rho x_i \cos\omega t), \quad \delta x_i = \frac{1}{\omega}\sin\omega t b_i \\ M_{ij} &= \int dx(\rho\upsilon_i x_j - \rho\upsilon_j x_i) \end{aligned}$$

Motion of the center of mass $mX_i = \int dx\rho x_i$

$$mX_i = \frac{P_i}{\omega}\sin\omega t - C_i\cos\omega t = A_i \sin(\omega t + \varphi_i)$$

$$\tan\varphi_i = -\frac{\omega C_i}{P_i}, \quad A_i^2 = \frac{P_i^2}{\omega^2} + C_i^2$$

NR perfect fluid and its symmetries, $f_i = -\omega^2 \rho x_i$

Algebra of conserved charges

Conserved charges jointly with H satisfy the Newton-Hooke algebra [H.Bacry, J.-M.Levy-Leblond, 1967] with a negative cosmological constant $\Lambda = -\frac{1}{R^2}$ with respect to the Poisson bracket

$$\{H, P_i\} = -\frac{1}{R^2} C_i, \qquad \{P_i, M_{jk}\} = \delta_{ij} P_k - \delta_{ik} P_j,$$
$$\{H, C_i\} = P_i, \qquad \{C_i, M_{jk}\} = \delta_{ij} C_k - \delta_{ik} C_j,$$
$$\{P_i, C_j\} = \delta_{ij} m, \qquad \{M_{ij}, M_{ab}\} = \delta_{i[a} M_{b]j} - \delta_{j[a} M_{b]i},$$

where we identified $\omega^2 = \frac{1}{R^2}$.

- The case of a positive cosmological constant $\Lambda > 0$ is obtained by a formal replacement $R \to iR$
- The Galilei algebra is reproduced in the flat limit $\Lambda \to 0$ $(R \to \infty)$

NR perfect fluid and its symmetries, $f_i = -\omega^2 \rho x_i$

Conformal extention

Like the Galilei algebra, the Newton-Hooke algebra admits a conformal extension [J.Negro, M.del Olmo, A.Rodriguez-Marco, 1997] by the generators of dilatation D and special conformal transformation K. Additional structure relations read [A. Galajinsky, 2010]

$$
\begin{aligned}
&[H, D] = H \mp \frac{2}{R^2} K, && [D, P_i] = -\frac{1}{2} P_i, \\
&[H, K] = 2D, && [D, C_i] = \frac{1}{2} C_i, \\
&[D, K] = K, && [K, P_i] = -C_i.
\end{aligned}
\qquad (2)
$$

NR perfect fluid and its symmetries, $f_i = -\omega^2 \rho x_i$

Extra conserved charges

Let us construct conserved charges that realize extra conformal symmetries

$$J = \int dx(\beta_1(t)\rho v_i v_i + \beta_2(t)\rho v_i x_i + \beta_3(t)\rho x_i x_i + \beta_4(t)V).$$

From the condition $\partial_0 J = 0$ a system of equations arises

$$\dot{\beta}_1 + \beta_2 = 0, \quad \dot{\beta}_2 + 2(\beta_3 - \beta_1\omega^2) = 0, \quad \dot{\beta}_3 - \beta_2\omega^2 = 0, \quad 2\beta_1 - \beta_4 = 0,$$

and the same condition on the potential $V = \frac{1}{2}dp$ as in the free case. The general solution is easily found

$$\begin{aligned} \beta_1 &= \frac{1}{2}\beta_4 = c_1 + c_2\cos 2\omega t + c_3 \sin 2\omega t, \\ \beta_2 &= 2\omega(c_2 \sin 2\omega t - c_3 \cos 2\omega t), \\ \beta_3 &= \omega^2(c_1 - c_2\cos 2\omega t - c_3 \sin 2\omega t), \end{aligned} \qquad (3)$$

which contains three arbitrary constants $c_{1,2,3}$ meaning that there are three independent integrals of motion.

$$J|_{c_1=\frac{1}{2}, c_2=c_3=0} = H, \quad J|_{c_1=c_2=0, c_3=\frac{1}{4\omega}} = D, \quad J|_{c_1=-c_2=\frac{1}{4\omega^2}, c_3=0} = K$$

Conformal extensions of Galilei and Newton-Hooke algebras

The ℓ-conformal Galilei algebra

There is a one-parameter family of finite-dimensional conformal extensions for Galilei algebra [J.Negro, M.del Olmo, A.Rodriguez-Marco, 1997]

$$
\begin{aligned}
&[H,D]=H, && [H,C_i^{(k)}]=kC_i^{(k-1)},\\
&[H,K]=2D, && [D,C_i^{(k)}]=(k-\ell)C_i^{(k)},\\
&[D,K]=K, && [K,C_i^{(k)}]=(k-2\ell)C_i^{(k+1)},\\
&[C_i^{(k)},M_{ab}]=\delta_{ia}C_b^{(k)}-\delta_{ib}C_a^{(k)}, && [M_{ij},M_{ab}]=\delta_{i[a}M_{b]j}-\delta_{j[a}M_{b]i},
\end{aligned}
\tag{4}
$$

where $k=0,1,...,2\ell$ and the parameter ℓ is an arbitrary integer or half-integer number. Generators H, D, K, M_{ij} correspond to time translation, dilation, special conformal transformation, spatial rotations, while the vector generators $C_i^{(k)}$ correspond to spatial translation and Galilei boost for $k=0,1$ and constant accelerations for $k>1$.

- Under dilatation temporal and spatial coordinates scale differently, $t'=\lambda t$, $x_i'=\lambda^\ell x_i$. The quantity $z=1/\ell$ is known as a critical dynamical exponent.
- The case $\ell=\frac{1}{2}$ is the Schrodinger algebra
- The case $\ell=1$ is the NR contraction of conformal algebra $so(2,d+1)$.

Conformal extensions of Galilei and Newton-Hooke algebras

The ℓ-conformal Newton-Hooke algebra

There is a one-parameter family of finite-dimensional conformal extensions of Newton-Hooke algebra [A.Galajinsky, I.Masterov 2011]

$$
\begin{aligned}
&[H, D] = H \mp \frac{2}{R^2} K, && [H, C_i^{(k)}] = k C_i^{(k-1)} \pm \frac{(k-2\ell)}{R^2} C_i^{(k+1)}, \\
&[H, K] = 2D, && [D, C_i^{(k)}] = (k-\ell) C_i^{(k)}, \\
&[D, K] = K, && [K, C_i^{(k)}] = (k-2\ell) C_i^{(k+1)}, \\
&[C_i^{(k)}, M_{ab}] = \delta_{ia} C_b^{(k)} - \delta_{ib} C_a^{(k)}, && [M_{ij}, M_{ab}] = \delta_{i[a} M_{b]j} - \delta_{j[a} M_{b]i},
\end{aligned}
\tag{5}
$$

In arbitrary dimension and for half-integer ℓ, conformal Newton-Hooke and Galilei algebra admits a central extension

$$[C_i^{(k)}, C_j^{(m)}] = (-1)^k k! m! \delta_{(k+m)(2\ell)} \delta_{ij} m$$

- Constant R is the characteristic time which links to the negative/positive cosmological constant $\Lambda = \mp \frac{1}{c^2 R^2}$
- The flat limit $R \to \infty$ yields the ℓ-conformal Galilei algebra.

Conformal extensions of Galilei and Newton-Hooke algebras

The ℓ-conformal Newton-Hooke algebra

a) Realization in space-time (t, x_i) with $\Lambda < 0$ is

$$
\begin{aligned}
&H = \partial_0, \quad D = \frac{1}{2} R \left(\sin \frac{2t}{R} \right) \partial_0 + \ell \left(\cos \frac{2t}{R} \right) x_i \partial_i, \\
&K = \frac{1}{2} R^2 \left(1 - \cos \frac{2t}{R} \right) \partial_0 + \ell R \left(\sin \frac{2t}{R} \right) x_i \partial_i, \\
&C_i^{(k)} = R^k \left(\tan \frac{t}{R} \right)^k \left(\cos \frac{t}{R} \right)^{2\ell} \partial_i, \quad M_{ij} = x_i \partial_j - x_j \partial_i,
\end{aligned} \tag{6}
$$

b) Realization in space-time (t, x_i) with $\Lambda > 0$ is given by replacement $R \to iR$

c) Flat limit $\Lambda \to 0$ $(R \to \infty)$ gives realization for ℓ-conformal Galilei algebra

$$
H = \partial_0, \quad D = t\partial_0 + \ell x_i \partial_i, \quad K = t^2 \partial_0 + 2\ell t x_i \partial_i, \quad C_i^{(k)} = t^k \partial_i, \tag{7}
$$

d) Dynamical realizations of the ℓ-conformal Newton-Hooke algebra have been extensively studied in the past [C.Duval, P.Horvathy, 2011; A.Galajinsky, I.Masterov, 2013; K.Andrzejewski, A.Galajinsky, J.Gonera, I.Masterov, 2014; S.Krivonos, O.Lechtenfeld, A.Sorin, 2016]

Generalized conformal perfect fluid dynamics

Perfect fluid dynamics with the ℓ-conformal Galilei symmetries

Generalized perfect fluid equations which hold invariant under the action of the ℓ-conformal Galilei group were formulated by [A.Galajinsky, 2022]

$$\partial_0 \rho + \partial_i(\rho \upsilon_i) = 0, \quad \mathcal{D}^{2\ell} \upsilon_i = -\frac{1}{\rho}\partial_i p, \quad p = \nu \rho^{1+\frac{1}{\ell d}}.$$

- Group-theoretic approach [A.Galajinsky, 2022]
- Hamiltonian and Lagrangian formulation [T.S, 2023,2024]
- Example $\ell = \frac{1}{2}$ reproduces Euler fluid with Schrödinger symmetry.

$$\mathcal{D}\upsilon_i = -\frac{1}{\rho}\partial_i p \quad \rightarrow \quad \mathcal{D}\upsilon_i + \frac{1}{R^2}x_i = -\frac{1}{\rho}\partial_i p$$

- Example $\ell = \frac{3}{2}$

$$\mathcal{D}^3 \upsilon_i = -\frac{1}{\rho}\partial_i p \quad \rightarrow \quad ?$$

The objective: to extend these equation to include a cosmological constant.

Generalized conformal perfect fluid dynamics

Perfect fluid dynamics with the ℓ-conformal Newton-Hooke symmetries

It appears natural to deform only the generalized Euler equation and leave the continuity equation and the equation of state unchanged. Focusing in what follows on the case of $\ell = \frac{3}{2}$ we modify the generalized third-order Euler equation as follows

$$\mathcal{D}^3 \upsilon_i + (\omega_1^2 + \omega_2^2)\mathcal{D}\upsilon_i + \omega_1^2\omega_2^2 x_i = -\frac{1}{\rho}\partial_i p,$$

where $\omega_2^2 > \omega_1^2 > 0$ are two arbitrary parameters of dimension $[\omega_1] = [\omega_2] = t^{-1}$. With this choice of the parameters, the left-hand side of the equation is an analogue of the Pais-Uhlenbeck oscillator [A.Pais, G.Uhlenbeck, 1950] in classical mechanics

$$\frac{d^4}{dt^4}x_i + (\omega_1^2 + \omega_2^2)\frac{d^2}{dt^2}x_i + \omega_1^2\omega_2^2 x_i = 0.$$

Generalized conformal perfect fluid dynamics

Perfect fluid dynamics with the ℓ-conformal Newton-Hooke symmetries

Introducing the Ostrogratsky-like auxiliary field variables $\upsilon_i^0, \upsilon_i^1, \upsilon_i^2$ with $\upsilon_i^0 = \upsilon_i$ the equation (8) can be derived from the Hamiltonian

$$H = \int dx \left[\rho\left(\upsilon_i^0\upsilon_i^2 - \frac{1}{2}\upsilon_i^1\upsilon_i^1 - \frac{1}{2}(\omega_1^2+\omega_2^2)\upsilon_i^0\upsilon_i^0 + \frac{1}{2}\omega_1^2\omega_2^2 x_i x_i\right) + V\right],$$

where the potential V links to the pressure via the Legendre transform $p = \rho V' - V$, provided the Poisson brackets [T.S., 2023]

$$\{\rho(x), \upsilon_i^2(y)\} = -\partial_i\delta(x-y), \quad \{\upsilon_i^0(x), \upsilon_j^2(y)\} = -\tfrac{1}{\rho}\partial_j\upsilon_i^0\delta(x-y),$$

$$\{\upsilon_i^0(x), \upsilon_j^1(y)\} = -\tfrac{1}{\rho}\delta_{ij}\delta(x-y), \quad \{\upsilon_i^1(x), \upsilon_j^2(y)\} = -\tfrac{1}{\rho}\partial_j\upsilon_i^1\delta(x-y),$$

$$\{\upsilon_i^2(x), \upsilon_j^2(y)\} = \tfrac{1}{\rho}\left(\partial_i\upsilon_j^2 - \partial_j\upsilon_i^2\right)\delta(x-y),$$

are used. Indeed, the dynamical equations have the form

$$\begin{aligned}
&\partial_0\rho = \{\rho, H\} = -\partial_i(\rho\upsilon_i^0),\\
&\partial_0\upsilon_i^0 = \{\upsilon_i^0, H\} = -\upsilon_j^0\partial_j\upsilon_i^0 + \upsilon_i^1,\\
&\partial_0\upsilon_i^1 = \{\upsilon_i^1, H\} = -\upsilon_j^0\partial_j\upsilon_i^1 - (\omega_1^2+\omega_2^2)\upsilon_i^0 + \upsilon_i^2,\\
&\partial_0\upsilon_i^2 = \{\upsilon_i^2, H\} = -\upsilon_j^0\partial_j\upsilon_i^2 - \omega_1^2\omega_2^2 x_i - \partial_i V',
\end{aligned}$$

Generalized conformal perfect fluid dynamics

Perfect fluid dynamics with the ℓ-conformal Newton-Hooke symmetries

We start with vector generators $C_i^{(0)}, C_i^{(1)}, C_i^{(2)}, C_i^{(3)}$ and choose them as linear expressions in the field variables $\upsilon_i^0, \upsilon_i^1, \upsilon_i^2$ and spatial coordinate x_i multiplied by the density ρ

$$I_i = \int dx \big(\alpha_1(t)\rho\upsilon_i^2 + \alpha_2(t)\rho\upsilon_i^1 + \alpha_3(t)\rho\upsilon_i^0 + \alpha_4(t)\rho x_i\big),$$

The conservation condition $\partial_0 I_i = 0$ gives a system of differential equations

$$\dot{\alpha}_1 + \alpha_2 = 0, \quad \dot{\alpha}_2 + \alpha_3 = 0, \quad \dot{\alpha}_3 + \alpha_4 - (\omega_1^2 + \omega_2^2)\alpha_2 = 0, \quad \dot{\alpha}_4 - \alpha_1\omega_1^2\omega_2^2 = 0$$

which has the general solution

$$\begin{aligned}
\alpha_1 &= c_1\cos\omega_1 t + c_2\sin\omega_1 t + c_3\cos\omega_2 t + c_4\sin\omega_2 t,\\
\alpha_2 &= c_1\omega_1\sin\omega_1 t - c_2\omega_1\cos\omega_1 t + c_3\omega_2\sin\omega_2 t - c_4\omega_2\cos\omega_2 t,\\
\alpha_3 &= -c_1\omega_1^2\cos\omega_1 t - c_2\omega_1^2\sin\omega_1 t - c_3\omega_2^2\cos\omega_2 t - c_4\omega_2^2\sin\omega_2 t,\\
\alpha_4 &= c_1\omega_1\omega_2^2\sin\omega_1 t - c_2\omega_1\omega_2^2\cos\omega_1 t + c_3\omega_2\omega_1^2\sin\omega_2 t - c_4\omega_2\omega_1^2\cos\omega_2 t.
\end{aligned}$$

It is satisfied for arbitrary $\omega_2^2 > \omega_1^2$ and contains four integration constants $c_{1,2,3,4}$ such that there are four functionally independent integrals of motion.

Generalized conformal perfect fluid dynamics

Perfect fluid dynamics with the ℓ-conformal Newton-Hooke symmetries

Let us turn to the construction of conserved charges associated with the dilatation D and special conformal transformation K. We search for them as quadratic combinations involving $\upsilon_i^0, \upsilon_i^1, \upsilon_i^2$ and x_i multiplied by the density ρ. The most general expression with arbitrary time-dependent coefficients β_i reads

$$J = \int dx \left(\beta_1(t)\rho\upsilon_i^0\upsilon_i^2 + \beta_2(t)\rho\upsilon_i^1\upsilon_i^1 + \beta_3(t)\rho\upsilon_i^2 x_i + \beta_4(t)\rho\upsilon_i^1\upsilon_i^0 + \beta_5(t)\rho\upsilon_i^0\upsilon_i^0 + \beta_6(t)\rho\upsilon_i^1 x_i + \beta_7(t)\rho\upsilon_i^0 x_i + \beta_8(t)\rho x_i x_i + \beta_9(t)V\right),$$

where we also included a term with the potential V. From the conservation condition $\partial_0 J = 0$ one obtains the restrictions

$$\beta_1 + 2\beta_2 = 0, \qquad \beta_1 - \beta_9 = 0, \qquad \dot\beta_4 - 2\beta_2(\omega_1^2+\omega_2^2) + 2\beta_5 + \beta_6 = 0,$$

$$\dot\beta_1 + \beta_3 + \beta_4 = 0, \qquad \dot\beta_6 + \beta_7 = 0, \qquad \dot\beta_5 - \beta_4(\omega_1^2+\omega_2^2) + \beta_7 = 0,$$

$$\dot\beta_2 + \beta_4 = 0, \qquad \dot\beta_8 - \beta_3\omega_1^2\omega_2^2 = 0, \qquad \dot\beta_7 - \beta_1\omega_1^2\omega_2^2 - \beta_6(\omega_1^2+\omega_2^2) + 2\beta_8 = 0,$$

$$\dot\beta_3 + \beta_6 = 0, \qquad \beta_9' V + \beta_3 dp = 0,$$

Generalized conformal perfect fluid dynamics

Perfect fluid dynamics with the ℓ-conformal Newton-Hooke symmetries

They prove compatible provided the extra restrictions

$$\omega_2 = 3\omega_1, \quad V = \frac{3}{2} d p \quad \rightarrow \quad p \sim \rho^{1+\frac{2}{3d}}$$

are imposed. Then the coefficients β acquire the form

$$\begin{aligned}
\beta_1 &= -2\beta_2 = \beta_9 = c_1 + c_2 \cos 2\omega_1 t + c_3 \sin 2\omega_1 t, \\
\beta_4 &= -\frac{1}{3}\beta_3 = -\omega_1 (c_2 \sin 2\omega_1 t - c_3 \cos 2\omega_1 t), \\
\beta_5 &= -\omega_1^2 (5c_1 + c_2 \cos 2\omega_1 t + c_3 \sin 2\omega_1 t), \\
\beta_6 &= -6\omega_1^2 (c_2 \cos 2\omega_1 t + c_3 \sin 2\omega_1 t), \\
\beta_7 &= -12\omega_1^3 (c_2 \sin 2\omega_1 t - c_3 \cos 2\omega_1 t), \\
\beta_8 &= \frac{9\omega_1^4}{2} (c_1 - 3c_2 \cos 2\omega_1 t - 3c_3 \sin 2\omega_1 t),
\end{aligned}$$

which contain three constants of integration $c_{1,2,3}$. Identifying $\omega_1^2 = \frac{1}{R^2}$ we get

$$J|_{c_1=1, c_2=c_3=0} = H, \quad J|_{c_1=c_2=0, c_3=\frac{1}{2\omega_1}} = D, \quad J|_{c_1=-c_2=\frac{1}{2\omega_1^2}, c_3=0} = K$$

Niederer's transformation

Isomorphism of ℓ-conformal Newton-Hooke and Galilei algebras

[J.Negro, M.del Olmo, A.Rodriguez-Marco, 1997]

The ℓ-conformal Newton-Hooke algebra

$$
\begin{aligned}
&[H,D]=H\mp\frac{2}{R^2}K, && [H,C_i^{(k)}]=kC_i^{(k-1)}\pm\frac{(k-2\ell)}{R^2}C_i^{(k+1)},\\
&[H,K]=2D, && [D,C_i^{(k)}]=(k-\ell)C_i^{(k)},\\
&[D,K]=K, && [K,C_i^{(k)}]=(k-2\ell)C_i^{(k+1)},
\end{aligned}
$$

The ℓ-conformal Galilei algebra

$$
\begin{aligned}
&[H,D]=H, && [H,C_i^{(k)}]=kC_i^{(k-1)},\\
&[H,K]=2D, && [D,C_i^{(k)}]=(k-\ell)C_i^{(k)},\\
&[D,K]=K, && [K,C_i^{(k)}]=(k-2\ell)C_i^{(k+1)},
\end{aligned}
$$

They are isomorphic by making a linear change of the basis $H\to H\mp\frac{1}{R}K$.

Niederer's transformation

Isomorphism of ℓ-conformal Newton-Hooke and Galilei algebras

Realization of the ℓ-conformal Newton-Hooke algebra

$$H = \partial_0, \quad D = \frac{1}{2} R \left(\sin \frac{2t}{R} \right) \partial_0 + \ell \left(\cos \frac{2t}{R} \right) x_i \partial_i,$$

$$K = \frac{1}{2} R^2 \left(1 - \cos \frac{2t}{R} \right) \partial_0 + \ell R \left(\sin \frac{2t}{R} \right) x_i \partial_i,$$

$$C_i^{(k)} = R^k \left(\tan \frac{t}{R} \right)^k \left(\cos \frac{t}{R} \right)^{2\ell} \partial_i$$

Realization of the ℓ-conformal Galilei algebra

$$H = \partial_0, \quad D = t\partial_0 + \ell x_i \partial_i, \quad K = t^2 \partial_0 + 2\ell t x_i \partial_i, \quad C_i^{(k)} = t^k \partial_i,$$

There exists a coordinate transformation [A. Galajinsky, I. Masterov, 2011] which links them

$$t' = R \tan \frac{t}{R}, \quad x_i' = \left(\frac{\partial t'}{\partial t} \right)^\ell x_i = (\cos \frac{t}{R})^{-2\ell} x_i,$$

where coordinates with prime parameterize the flat space. For $\ell = \frac{1}{2}$ these transformations were first introduced by Niederer [U. Niederer, 1973]

$$\ddot{x} = 0 \quad \rightarrow \quad \ddot{x} + \frac{1}{R^2} x = 0$$

Niederer's transformation

Let us apply Niederer's transformation to generalized perfect fluid equations in flat space

$$\partial_0 \rho + \partial_i(\rho \upsilon_i) = 0, \quad \mathcal{D}^{2\ell} \upsilon_i = -\frac{1}{\rho}\partial_i p, \quad p = \nu \rho^{1+\frac{1}{\ell d}}.$$

Transformations of ρ and υ_i under Niderer's

The density transformation is obtained by requiring the mass to be invariant

$$\int_{V'} dx' \rho'(t', x') = \int_V dx \rho(t, x) \quad \rightarrow \quad \rho'(t', x') = (\cos\frac{t}{R})^{2\ell d} \rho(t, x).$$

To obtain the transformation law for $\upsilon_i(t, x)$, consider the orbit of a fluid particle $x_i(t)$ and take into account that

$$\frac{dx_i(t)}{dt} = \upsilon_i(t, x(t)) \quad \rightarrow \quad \upsilon_i'(t', x') = (\cos\frac{t}{R})^{-2\ell+2}\left(\upsilon_i(t, x) + \frac{2\ell}{R}\tan\frac{t}{R} x_i\right).$$

One needs to take into account the identities

$$\frac{\partial}{\partial t} = (\frac{\partial t'}{\partial t})\frac{\partial}{\partial t'} + (\frac{\partial x_i'}{\partial t})\frac{\partial}{\partial x_i'}, \quad \frac{\partial}{\partial x_i} = (\frac{\partial t'}{\partial x_i})\frac{\partial}{\partial t'} + (\frac{\partial x_j'}{\partial x_i})\frac{\partial}{\partial x_j'},$$

Niederer's transformation

The generalized Niederer transformation does not alter the continuity equation and the equation of state, while it modifies the Euler equation

$$\prod_{k=1}^{n-1}(\mathcal{D}^2+\frac{(2k+1)^2}{R^2})(D\upsilon_i+\frac{1}{R^2}x_i)=-\frac{1}{\rho}\partial_i p \tag{8}$$

for a half-integer $\ell=n-\frac{1}{2}$ and

$$\prod_{k=1}^{n}(\mathcal{D}^2+\frac{(2k)^2}{R^2})\upsilon_i=-\frac{1}{\rho}\partial_i p \tag{9}$$

for an integer $\ell=n$.

- By construction, the equations hold invariant under the ℓ-conformal Newton-Hooke transformations.
- In the particular cases $\ell=\frac{1}{2}$ and $\ell=\frac{3}{2}$ the equations reproduce the previously obtained results.

Conclusion

- We formulated perfect fluid equations which enjoy the ℓ-conformal Newton-Hooke symmetry
- For $\ell = \frac{1}{2}$, the symmetries are naturally realized by the harmonic trap potential and imposing a suitable equation of state
- For higher values of ℓ, the symmetries demand a higher derivative generalization of the Euler equation which is an analogue of the Pais-Uhlenbeck oscillator in classical mechanics.
- It was demonstrated that the same results can be achieved by applying a generalized Neiderer's transformation.
- Physical applications?
- Supersymmetric extensions?

THANK YOU FOR ATTENTION!