De Sitter entropy: on-shell versus off-shell

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Result

$$S_V^{\text{on shell}} = \int \frac{\delta Q}{T_H} = \int_0^{T_H} \frac{dT_H}{T_H} \int_{r_{ds}=\frac{1}{2\pi T_H}} d^{d-1}x\sqrt{g}\,\frac{\partial \rho}{\partial T_H}, \quad \rho = \frac{\Lambda(T_H)}{8\pi} \tag{1}$$

and

$$S_A^{\text{off shell}} = -(\alpha\partial_\alpha - 1)W[\alpha]\Big|_{\alpha=1} \sim A, \quad \text{where} \quad \alpha = \frac{\beta}{\beta_H} \tag{2}$$

- We show that bulk entropy that computed on-shell precisely follows the area law in any dimension of space and in any theory of $f(R)$ gravity and coincide with Wald entropy:

$$S_V^{\text{on shell}}(\text{gravity}) = S_A^{\text{off shell}}(\text{gravity}) = \frac{A}{4} f'(R) \tag{3}$$

- Generalize this statment for entanglenment entropy of minimally coupled scalar field:

$$S_V^{\text{on shell}}(\text{matter}) = S_A^{\text{off shell}}(\text{matter}) \tag{4}$$

- Reason are not yet known
- Does it generalize to over space with Killing horizon with non vanishing cosmological constant (BTZ or Narai) ?

Introduction

It is known that, in general, quantum corrections to the entropy are given by:

$$S_A = \frac{A}{4G\hbar} + \alpha_0 \log\left(\frac{A}{4G\hbar}\right) + \sum_n a_n \left(\frac{A}{4G\hbar}\right)^{-n}. \tag{5}$$

- The first term is the Gibbons-Hawking area law, computed in the semiclassical approximation.
- The leading-order quantum correction is logarithmic and is the most interesting.
- For macroscopic black holes, quantum corrections are negligibly small.
- At the late stages of black hole evaporation, quantum corrections become very important.

The effective temperature of a black hole is given by:

$$\frac{1}{T_H} = \frac{\partial S_A}{\partial M} = 8\pi M + 2\alpha_0 \frac{1}{M}. \tag{6}$$

- For large black holes, the Hawking temperature is inversely related to their mass: $T_H \sim \frac{1}{M}$.
- For small mass, the temperature is linearly related: $T_H \sim M$.

This means that the temperature does not diverge at the final stage, resulting in an increased evaporation time. Moreover, the final stage of black hole evaporation significantly depends on the sign of α_0.

Geometry of the de Sitter Space-Time

The de Sitter space is a vacuum solution to Einstein's equation:

$$R_{\mu\nu} - \frac{1}{2} R g_{\mu\nu} + \Lambda g_{\mu\nu} = 0. \tag{7}$$

The d-dimensional de Sitter space can be visualized as a one-sheeted hyperboloid embedded in a $d+1$-dimensional ambient Minkowski space, described by:

$$dS_d = \{X \in \mathbb{R}^{d+1}, \quad X^\alpha X_\alpha = -X_0^2 + \sum_i X_i^2 = H^{-2}\} \tag{8}$$

The static coordinates of de Sitter space are given by:

$$X = \begin{cases} X^0 = H^{-1}\sqrt{1 - r^2H^2}\sinh(tH) \\ X^i = r z_i, \qquad i = 1, \ldots, d-1 \\ X^d = \pm H^{-1}\sqrt{1 - r^2H^2}\cosh(tH) \end{cases}, \qquad t \in (-\infty, \infty), \; r \in (0, H^{-1}), \tag{9}$$

where z_i are the coordinates on the $(d-2)$-dimensional sphere, and the $\pm$ in X^d defines the right or left de Sitter wedges with the metric:

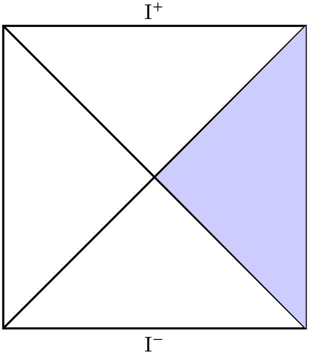
$$ds^2 = -(1 - r^2H^2)dt^2 + \frac{dr^2}{1 - r^2H^2} + r^2 d\Omega_{d-2}^2. \tag{10}$$

The static coordinates are bounded by a Killing horizon:

$$r_{\text{horizon}} = \frac{1}{H}, \tag{11}$$

where the metric degenerates.

Penrose Diagram



f(R) Gravity

De Sitter space serves as a solution in modified gravity theory:

$$W = \frac{1}{16\pi} \int d^d x \sqrt{g} \left(f(R) - 2\Lambda \right) + W_{\text{matter}}. \tag{12}$$

The variation of the action with respect to the metric yields:

$$f'(R) R_{\mu\nu} - \frac{1}{2} f(R) g_{\mu\nu} + \left(g_{\mu\nu} \Box - \nabla_\mu \nabla_\nu \right) f'(R) + \Lambda g_{\mu\nu} = 8\pi T^{\text{matter}}_{\mu\nu}. \tag{13}$$

Thus, the de Sitter space, with Hubble constant equal to H, is a solution to the field equations in the absence of matter if the cosmological constant is given by:

$$\Lambda = \left(\frac{1}{2} f(R) - \frac{1}{d} f'(R) R \right) \Bigg|_{R=(d-1)dH^2}. \tag{14}$$

Thermodynamics of de Sitter Space

An observer in the static patch of de Sitter space sees isotropic radiation with Gibbons-Hawking temperature:

$$T_H = \frac{H}{2\pi}. \tag{15}$$

In the semiclassical approximation, the entropy of de Sitter space obeys the area law:

$$S = \frac{A}{4}. \tag{16}$$

- The Gibbons-Hawking temperature is $10^{-30}K$, which is much lower than the temperature of the cosmic microwave background, $T = 2.73K$.
- This temperature implies the existence of entropy in de Sitter space, which is given by $2.6 \cdot 10^{122}$, vastly exceeding the entropy of all the matter and energy in our Universe, which is on the order of 10^{104}.

The de Sitter Temperature: $T_H = \frac{H}{2\pi}$

1. Imaginary Time Periodicity Trick

$$ds^2 = d\epsilon^2 + \epsilon^2 d(H\tau)^2 + H^{-2} d\Omega^2. \tag{17}$$

2. Tunneling Exponent

$$\Gamma \sim e^{-\frac{2\pi}{H}E}. \tag{18}$$

3. KMS Condition

$$G_{BD}(Z_{1,2}) = \frac{1}{Z}\mathrm{Tr}\left[e^{-\beta_H H}\phi(x_1)\phi(x_2)\right]. \tag{19}$$

4. Reduced Density Matrix

$$|BD\rangle = \sum_{n_\omega} e^{-\pi/H\omega n_\omega}|n_\omega, L\rangle \otimes |n_\omega, R\rangle \quad \to \quad \hat{\rho}_R = \mathrm{Tr}_L|BD\rangle\langle BD| = e^{-\beta_H H_R}. \tag{20}$$

5. Unruh-DeWitt Detector

$$\dot{P}(E_i \to E_j) = g^2|m_{ij}|^2 \int dte^{-(E_j-E_i)t} G_{BD}(Z = \cosh(t)). \tag{21}$$

The probabilities satisfy the relation:

$$\dot{P}(E_i \to E_j) = \dot{P}(E_j \to E_i)e^{-\beta(E_j-E_i)}. \tag{22}$$

Entropy (Replica Trick)

Let us consider a general curved spacetime M with a Killing horizon, and impose periodic boundary conditions $\tau \sim \tau + \beta$, with a non-fixed inverse temperature $\beta = 2\pi\alpha/H$. In the limit $\alpha \to 1$, the Riemannian tensor contains delta-like surface contributions near the horizon:

$$\lim_{\alpha\to 1} R^{M_\alpha}_{\mu\nu\rho\sigma} = R^{M}_{\mu\nu\rho\sigma} + 2\pi(1-\alpha)\left(n^1_\mu n^2_\rho - n^2_\mu n^1_\rho\right)\left(n^1_\nu n^2_\sigma - n^2_\nu n^1_\sigma\right)\delta\left(\Sigma\right) + O\left((1-\alpha)^2\right), \quad (23)$$

where $n^{1,2}$ are two orthonormal vectors orthogonal to the horizon surface Σ. The classical action in the limit $\alpha \to 1$ is given by:

$$\begin{aligned} &W[\alpha] = \\ &= -2\pi\alpha \frac{1}{16\pi}\int d^{d-1}x\sqrt{g}\left(f\left(R^M\right) - 2\Lambda\right) - \\ &-\alpha(1-\alpha)\frac{1}{4}\int d^{d-1}x\sqrt{g}\delta\left(\Sigma\right) f'\left(R^M\right) + O\left((1-\alpha)^2\right). \end{aligned} \quad (24)$$

Applying the replica formula, we find that the only surface term contributes to the entropy:

$$S_A^{\text{off shell}} = -(\alpha\partial_\alpha - 1)W[\alpha]\Big|_{\alpha=1} = \frac{A}{4}f'(R). \quad (25)$$

Entropy (Thermodynamic Potential $F = E - T_H S$)

The partition function for thermal gravity can be evaluated in the semiclassical limit by the on-shell gravity action in Euclidean signature:

$$Z = e^{-\beta_H F} = \int Dg_{ab} e^{-W_E(g)} \approx e^{-W_E(g_{dS})}. \tag{26}$$

Using the thermodynamic potential $F = E_H - TS$, where E is the energy, which is set to zero for empty de Sitter space:

$$E = 0, \tag{27}$$

we can demonstrate that the entropy for the general theory of $f(R)$ gravity is given by:

$$S = -\beta_H F = -W_E(g_{dS}) = \frac{1}{16\pi} \int d^d x \sqrt{g} \, (f(R) - 2\Lambda) = \frac{A}{4} f'(R). \tag{28}$$

Entropy (First Law)

Let us consider how the area A of the cosmological horizon changes when an infinitesimal amount of energy is added, $\delta E = \delta M$, where M is the mass of the black hole. The metric of a Schwarzschild-de Sitter black hole in $d = 4$ dimensions is given by:

$$ds^2 = \left(1 - 2M/r - r^2H^2\right) dt^2 - \frac{dr^2}{(1 - 2M/r - r^2H^2)} - r^2 d\Omega^2. \tag{29}$$

If we add a small amount of energy δM to the empty de Sitter space, then the area of the cosmological horizon decreases as:

$$\delta A_H = -\frac{8\pi}{H}\delta M, \tag{30}$$

while we ignore the area of the black hole horizon since it is of second order in the variation of energy: $\delta A_{BH} \sim \delta M^2$. Assuming that the entropy is $1/4$ of the event horizon, we can show that the first law of thermodynamics holds for the cosmological horizon:

$$\delta(-E) = T_H dS. \tag{31}$$

Local Thermodynamics

The first law of thermodynamics with non-zero pressure is given by:

$$T_H dS_V^{\text{on shell}} = \delta Q = dE + pdV. \tag{32}$$

Let us assume that:

- The temperature is given by the Gibbons-Hawking relation: $T_H = \frac{H}{2\pi}$.
- The energy density is given by the cosmological term:

$$\rho(2\pi T_H) = \frac{\Lambda(2\pi T_H)}{8\pi}, \quad \Lambda = \left(\frac{1}{2}f(R) - \frac{1}{d}f'(R)R\right) + 8\pi T_0^0 \tag{33}$$

- The pressure is given by $p = -\rho$.

Thus, the total entropy of the de Sitter vacuum state is given by:

$$S_V^{\text{on shell}} = \int \frac{\delta Q}{T_H} = \int_0^{T_H} \frac{dT_H}{T_H} \int_{r_{ds} = \frac{1}{2\pi T_H}} d^{d-1}x\sqrt{g}\, \frac{\partial \rho}{\partial T_H}. \tag{34}$$

For $f(R)$ gravity, we can obtain the following value for the entropy:

$$\rho = \frac{1}{8\pi d}\left(f'\left[-(d-1)d(2\pi T_H)^2\right]\left(-(d-1)d(2\pi T_H)^2\right) - \frac{d}{2}f\left[-(d-1)d(2\pi T_H)^2\right]\right), \tag{35}$$

which leads to the conclusion that the entropy of the de Sitter vacuum state is given by:

$$S_V^{\text{on shell}} = \frac{A}{4}f'(R), \tag{36}$$

which exactly coincides with the Wald entropy of the Killing horizon.

Quantum Correction

Let us consider quantum corrections to the Einstein equation due to matter:

$$R_{\mu\nu} - \frac{1}{2}Rg_{\mu\nu} + \Lambda g_{\mu\nu} = 8\pi\langle T_{\mu\nu}^{\text{matter}}\rangle \tag{37}$$

For $d = 4$ dimensional spacetime, the conformally coupled matter stress-energy tensor is given by:

$$\langle T_{\mu\nu}^{\text{matter}}\rangle = \frac{q(s)H^4}{960\pi^2} g_{\mu\nu} \tag{38}$$

where $q(0) = 1$, $q(1/2) = 11/2$, and $q(1) = 62$ are the coefficients for different spins s.

Hence, the self-consistent equation for the corrected Hubble constant takes the form:

$$6H^2 = 8\pi\left(\Lambda + \frac{q(s)H^4}{960\pi^2}\right). \tag{39}$$

As a result, the vacuum energy is given by:

$$\rho = \frac{\Lambda}{8\pi} = \frac{6H^2}{8\pi} - \frac{q(s)H^4}{960\pi^2}. \tag{40}$$

Consequently, the temperature is given by the quantum-corrected Hubble constant:

$$T_H = \frac{H}{2\pi}. \tag{41}$$

Quantum Correction

Using local thermodynamic expressions:

$$S_V^{\text{on shell}} = \int \frac{\delta Q}{T_H} = \int_0^{T_H} \frac{dT_H}{T_H} \int_{r_{ds}=\frac{1}{2\pi T}} d^3x\sqrt{g}\, \frac{\partial \rho}{\partial T_H}. \tag{42}$$

The corrected entropy of the de Sitter vacuum state is given by:

$$S_V^{\text{on shell}} = S_0 + \frac{1}{180} q(s) \log\left(4\pi S_0\right). \tag{43}$$

This formula is exact since it is expressed in terms of the corrected Hubble constant.
The same logarithmic terms were obtained in the context of black holes, and it seems to be universal:

$$S_V^{\text{on shell}} = S_0 + \frac{b}{2} \log\left(S_0\right) + \ldots, \tag{44}$$

where b is the integrated conformal anomaly:

$$b = \int d^4x\sqrt{g}\langle T_\mu^\mu \rangle, \tag{45}$$

which for de Sitter space is given by:

$$b = \frac{q(s)}{90}. \tag{46}$$

$S_V^{\text{on shell}} = \beta_H E - \beta_H F$

The entropy we define as:

$$S_V^{\text{on shell}} = \int_0^T \frac{dT_H}{T_H} \int d^3x\sqrt{g}\, \frac{\partial \rho_{matter}}{\partial T_H}. \tag{47}$$

and energy of the system as:

$$\beta_H E = \int d^4x\sqrt{g}\rho_{matter} = \int d^4x\sqrt{g}\rho_{matter}. \tag{48}$$

To find free energy times invese temterature let us use the following relation:

$$-\beta_H \frac{d}{d\beta_H}\log Z = T_H\frac{d}{dT_H}\log Z = H\frac{d}{dH}\log Z = 2\int d^4x g^{\mu\nu}\frac{\delta}{\delta g_{\mu\nu}}\log Z = -\int d^4x\sqrt{g}\langle T^\mu_\mu\rangle^{matter} \tag{49}$$

hence:

$$\beta_H F = -\log Z = -4\int \frac{dT_H}{T_H}\int d^4x\sqrt{g}\rho_{matter} \tag{50}$$

$S_V^{\text{on shell}} = \beta_H E - \beta_H F$

Let:

$$\rho_{matter} = -\langle T_0^0 \rangle^{matter} = -T^4 f\left(\frac{m}{T}\right), \tag{51}$$

Then we can rewrite the enrtopy as follows:

$$S_V^{\text{on shell}} = \int_0^T \frac{dT}{T} \frac{4}{3}\pi \left(\frac{1}{2\pi T}\right)^3 \frac{\partial}{\partial T}\left[T^4 f\left(\frac{m}{T}\right)\right] = \\ = \frac{1}{6\pi^2} f\left(\frac{m}{T}\right) + \frac{2}{3\pi^2}\int_0^T \frac{dT}{T} f\left(\frac{m}{T}\right) \tag{52}$$

where the first term is indeed energy of the system:

$$\beta E = \int d^4x \sqrt{g} \rho_{matter} = \frac{1}{6\pi^2} f\left(\frac{m}{T}\right) \tag{53}$$

and secod is free energy:

$$\beta F = -4\int \frac{dT}{T}\frac{1}{T}\frac{4}{3}\pi\left(\frac{1}{2\pi T}\right)^3 T^4 f\left(\frac{m}{T}\right) = -\frac{2}{3\pi^2}\int \frac{dT}{T} f\left(\frac{m}{T}\right) \tag{54}$$

Off shell method

To compute etropy off shell we considere the Euclidean static de Sitter space-time with not fixed period β in time. Then expanding renormalized effenctive action W_β^{ren} in terms of $\beta - \beta_H$ we will complute the renormalized entanglenment etropy at Gibbons-Hawkong temperature:

$$S_A^{\text{off shell}} = (\alpha\partial_\alpha - 1)\, W_\alpha^{ren}\Big|_{\alpha=1}, \qquad \alpha = \frac{\beta}{2\pi/H} \tag{55}$$

Since β is an arbirtuary the Euclidean manifold has conical singularities at the horizon surface Σ.

$$W^{ren} = W_{gr}^{bar} + W_{matter}^{div} + O((1-\alpha)^2). \tag{56}$$

$$W_{gr} = \tag{57}$$
$$= \alpha \int_M d^4x\sqrt{g}\left(-\frac{1}{16\pi G_B}\left(R^M + 2\Lambda_B\right) + c_1^B R^M R^M + c_2^B R^M_{\mu\nu} R^{M,\mu\nu} + c_3^B R^M_{\mu\nu\rho\sigma} R^{M,\mu\nu\rho\sigma}\right) +$$
$$+4\pi(1-\alpha)\int d\Sigma\left(-\frac{1}{16\pi G_B} + 2c_1^B R^M + c_2^B R^M_{\mu\nu} n_i^\mu n_i^\nu + 2c_3^B R^M_{\mu\nu\rho\sigma} n_i^\mu n_i^\rho n_j^\nu n_j^\sigma\right) +$$
$$+O\left((1-\alpha)^2\right).$$

The path integral over the scalar field is given by:

$$Z_{matter} = \int D\phi \ e^{-\frac{1}{2}\int_{M_\alpha} d^4x\sqrt{g}\ \phi\left(-\Box + m^2\right)\phi} \tag{58}$$

The heat kernel expansions of effective action of matter is given by:

$$\log Z_{matter} = \frac{1}{2}\int \frac{ds}{s} e^{-sm^2} \int_{M_\alpha} d^4x\sqrt{g}\ \bar{K}_{M_\alpha}(s,x,x) \tag{59}$$

where

$$\bar{K}_{M_\alpha}(s,x,x) = \frac{1}{(4\pi s)^{\frac{d}{2}}}\sum_n \bar{a}_n(x) s^n \tag{60}$$

and

$$\bar{a}_n(x) = a_n^{st}(x) + a_n^\alpha(x)(1-\alpha)\delta(\Sigma) \tag{61}$$

$$S_A^{\text{off shell}} = (\beta\partial_\beta - 1)\frac{1}{2}\left(\zeta'(0,\beta) + \log\left(\mu^2\right)\zeta(0,\beta)\right), \tag{62}$$

where:

$$\zeta(0,\beta) \approx \tag{63}$$

$$\approx \left[\frac{\nu^4}{12} + \frac{\nu^2}{24} - \frac{17}{2880}\right] + \left(\frac{\beta}{\beta_H} - 1\right)\left[\nu^4 + \frac{4}{24}\nu^2 + \frac{3}{64}\right],$$

$$\zeta'(0,\beta) \approx \mathit{const} + \tag{64}$$

$$\approx + \left(\frac{\beta}{\beta_H} - 1\right)\left[-\frac{41}{144}\nu^2 - \frac{1}{8}\nu^4 - \frac{973}{5760} + \frac{1}{192}\left(16\nu^4 + 40\nu^2 + 9\right)\left(\psi\left(\frac{3}{2} + i\nu\right) + \psi\left(\frac{3}{2} - i\nu\right)\right)\right]$$

The bulk entropy:

$$S_V^{\text{on shell}} = \int_0^T \frac{dT}{T}\int d^3x\sqrt{g}\ \frac{\partial\rho_{matter}}{\partial T}. \tag{65}$$

$$\rho_{matter} = \tag{66}$$

$$= -\frac{H^4}{64\pi^2}\left[\left(\frac{1}{4} + \mu^2\right)\left(\frac{9}{4} + \mu^2 - 12\xi\right)\left[\psi\left(\frac{3}{2} + i\mu\right) + \psi\left(\frac{3}{2} - i\mu\right) - \log\left(\frac{m^2}{H^2}\right)\right] + \right.$$

$$\left. + \mu^2\left(\frac{4}{3} - 12\xi\right) + \xi(72\xi - 19) + \frac{16}{15}\right],$$

$$\log Z^{matter} = -\int d^3x\sqrt{g}g^{00}\sum\!\!\!\!\!\!\int_i \phi_i(x)\phi_i^*(x)\log\left(1-e^{-\beta\omega_i}\right) + \tag{67}$$

$$+\int d^3x\sqrt{g}g^{00}\sum\!\!\!\!\!\!\int_i \int_\infty^{m^2} dm^2\partial_{m^2}\left[\phi_i(x)\phi_i^*(x)\right]\log\left(1-e^{-\beta\omega_i}\right) -$$

$$-\beta\int d^3x\sqrt{g}\int_\infty^{m^2} dm^2\sum\!\!\!\!\!\!\int_i \frac{1}{2\omega_i}\left[\triangle_3\partial_{m^2}\phi_i(x)\phi_i^*(x) - \partial_{m^2}\phi_i(x)\triangle_3\phi_i^*(x)\right]n(\beta\omega_i).$$

The first term $\log Z_1^E$ is equal to the standard definition of the partition function: For space time without Killing horizon

$$\log Z^{matter} = -\int d^3x\sqrt{g}g^{00}\sum\!\!\!\!\!\!\int_i \phi_i(x)\phi_i^*(x)\log\left(1-e^{-\beta\omega_i}\right) \tag{68}$$

For space time with Killing horizon:

$$\log Z^{matter} = \tag{69}$$

$$= -\beta\int d^3x\sqrt{g}\int_\infty^{m^2} dm^2\sum\!\!\!\!\!\!\int_i \frac{1}{2\omega_i}\left[\triangle_3\partial_{m^2}\phi_i(x)\phi_i^*(x) - \partial_{m^2}\phi_i(x)\triangle_3\phi_i^*(x)\right]n(\beta\omega_i) \sim$$

$$\sim A$$