Higher-spin fields in spinor space

(work in progress)

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main statements

- action principle for classical $4d$ integer-spin fields is formulated in space formed by $sl(2,\mathbb{C})$-spinors and a Lorentz-invariant proper-time coordinate
- e.o.m. get very simple and put no constraints on spinor-dependence of fields
- relation to the space-time picture is provided by unfolded equations

outline

- $4d$ higher-spin gravity and spinors
- unfolded equations and unfolding maps
- scalar field
- spin-1 field
- integer-spin fields

$4d$ higher-spin gravity and spinors

- higher-spin gravity - a theory of interacting massless fields of all spins (including graviton) with ∞-dim gauge symmetry.
- quantum gravity requires new geometry: worldsheet in string theory, twistors for self-dual theories, spinfoam in loop quantum gravity.
- in $4d$ higher-spin gravity, Weyl spinors are natural auxiliary variables - it looks reasonable to try to implement the whole dynamics in terms of them.
- higher spins in spinor variables: [Ponomarev 2104.02770, NM 2301.02207].

$4d$ higher-spin gravity and spinors

- Higher-spin gauge algebra [Fradkin, Vasiliev, 1987] - infinite-dimensional associative algebra of star-product (Weyl algebra)

$$f(Y) * g(Y) = f(Y) \exp\{i \overleftarrow{\partial}_A \epsilon^{AB} \overrightarrow{\partial}_B\} g(Y), \tag{1}$$

where $Y^A = \{y^\alpha, \bar{y}^{\dot{\alpha}}\}$ is a pair of Grassmann-even Weyl spinors. The only independent Lorentz-invariant combinations built out of Y are

$$\hat{N} := y^\alpha \partial_\alpha, \quad \hat{\bar{N}} := \bar{y}^{\dot{\alpha}} \bar{\partial}_{\dot{\alpha}}. \tag{2}$$

- Spinor bilinears $T^{AB} = Y^A Y^B$ form $sp(4, \mathbb{R})$ subalgebra

$$[T^{AA}, T^{BB}]_* = -2i\epsilon^{AB} T^{AB}, \tag{3}$$

isomorphic to $so(3,2)$ symmetry algebra of AdS_4, which is a vacuum solution of Vasiliev theory.

$4d$ higher-spin gravity and spinors

- In terms of $sl(2,\mathbb{C})$, symmetric traceless Lorentz tensors correspond to symmetric spinor-tensors

$$\{T^{a_1\dots a_n} : \eta_{a_1 a_2} T^{a_1\dots a_n} = 0\} \Longleftrightarrow T^{\alpha_1\dots\alpha_n,\dot{\alpha}_1\dots\dot{\alpha}_n}. \tag{4}$$

- Antisymmetric Lorentz tensors correspond to (2,0)⊕(0,2) spinor-tensors

$$\{T^{a,b} = -T^{b,a}\} \Longleftrightarrow T^{\alpha_1\alpha_2} \oplus \bar{T}^{\dot{\alpha}_1\dot{\alpha}_2}. \tag{5}$$

- For derivatives this yields

$$\nabla^{\alpha_1\dot{\alpha}_1}\nabla^{\alpha_2\dot{\alpha}_2} \Longleftrightarrow \nabla^a\nabla^b - \frac{1}{4}\eta^{ab}\Box. \tag{6}$$

- Practical profit: working in terms of spinors automatically projects out traces and boxes of tensor fields, restricting to irreducible on-shell components.

unfolded equations and unfolding maps

- to formulate higher-spin gravity in a manifestly diffeomorphism- and gauge-invariant way [Vasiliev'89-94], a special first-order formalism was developed named unfolded dynamics approach [Vasiliev, hep-th/0504090].
- Master-fields of Vasiliev theory, living in the fiber bundle of $\mathbb{C}^2$ over AdS_4, are 1-form $\omega(Y|x)$ ("potentials") and 0-form $C(Y|x)$ ("field strengths").
- solving generating Vasiliev equations leads to the unfolded system of the form

$$\mathrm{d}\omega(x|Y) + \omega * \omega + V^1_\omega(\omega,\omega,C) + V^2_\omega(\omega,\omega,C,C) + ... = 0 \quad (7)$$

$$\mathrm{d}C(x|Y) + \omega * C - C * \hat{\pi}\omega + V^2_C(\omega,C,C) + V^3_\omega(\omega,C,C,C) + ... = 0 \quad (8)$$

- These equations can be interpreted as (locally) restoring the space-time dependence of higher-spin fields from the spinorial one.
- From the space-time point of view, spinors Y encode expansions of higher-spin fields in on-shell derivatives.
- Two boundary problems $C(x|0) \to C(x|Y)$, $C(0|Y) \to C(x|Y)$ generate unfolding maps $C(x) \to C(Y)$, $C(Y) \to C(x)$.

unfolded equations and unfolding maps

- Problem: to formulate the theory of relativistic fields solely in terms of spinors, with no reference to the space-time.
- Motivation: spinor space may provide a fundamental geometry for higher-spin gravity.
- Idea: one needs an unfolding map defining fields living in $\mathbb{R}^{1,3} \times (\mathbb{C}^2 \times \mathbb{R})$ and unfolded-like equations $\nabla_{\alpha\dot{\alpha}}\Phi = P_{\alpha\dot{\alpha}}(Y,\tau)\Phi$ defining Poincaré reps – this allows one to connect space-time fields to spinor-space fields living in $\mathbb{C}^2 \times \mathbb{R}$.
- Technicalities: to construct an unfolded formulation of a theory, one should first define an unfolding map $x \to x|Y$, and then deduce unfolded equations as identities, satisfied by the unfolded fields [NM, 2402.14164].
- Antipodality of spinor and space-time realizations of Poincaré UIR:
  - in space-time: universal $\hat{P}_a = -i\frac{\partial}{\partial x^a}$ for all masses and spins, but $\hat{M}_{a,b} = -ix_{[a}\partial_{b]} + (\hat{S}_{a,b})^I{}_J$, thus fields are $\phi^I(x)$.
  - in spinor space: universal $\hat{M}_{\alpha\beta} = y_{(\alpha}\partial_{\beta)}$ for all masses and spins, but $\hat{P}_{\alpha\dot{\alpha}} = a^{m,s}_{\hat{N},\hat{\bar{N}}}\partial_\alpha\bar{\partial}_{\dot{\alpha}} + b^{m,s}_{\hat{N},\hat{\bar{N}}}y_\alpha\bar{y}_{\dot{\alpha}} + (c^{m,s}_{\hat{N},\hat{\bar{N}}}\bar{y}_{\dot{\alpha}}\partial_\alpha + h.c.)$, thus fields are $\varphi(Y,\tau)$.

scalar field

- Consider an unfolding map for an off-shell scalar field [NM, 2208.04306]

$$\phi(x) \to \Phi(x|Y,\tau) := \exp(y^{\alpha}\bar{y}^{\dot{\alpha}}\nabla_{\alpha\dot{\alpha}} + \tau\Box)\phi(x) = \int \frac{d^4z}{(4\pi\tau)^2} e^{-\frac{(x+y\bar{y}-z)^2}{4\tau}}\phi(z) \tag{9}$$

- This corresponds to the unfolded-like equation

$$\nabla_{\alpha\dot{\alpha}}\Phi(x|Y,\tau) = \frac{1}{\hat{N}+1}(\partial_{\alpha}\bar{\partial}_{\dot{\alpha}} - y_{\alpha}\bar{y}_{\dot{\alpha}}\frac{\partial}{\partial\tau})\Phi(x|Y,\tau), \tag{10}$$

which generates a representation of Poincaré algebra as

$$P_{\alpha\dot{\alpha}} = \frac{1}{\hat{N}+1}(\partial_{\alpha}\bar{\partial}_{\dot{\alpha}} - y_{\alpha}\bar{y}_{\dot{\alpha}}\frac{\partial}{\partial\tau}), \quad M_{(\alpha\beta)} = y_{(\alpha}\partial_{\beta)}, \quad \bar{M}_{(\dot{\alpha}\dot{\beta})} = \bar{y}_{(\dot{\alpha}}\bar{\partial}_{\dot{\beta})} \tag{11}$$

- P^2-casimir is Y-independent

$$P^2 := \frac{1}{2}P_{\alpha\dot{\alpha}}P^{\alpha\dot{\alpha}} = \frac{\partial}{\partial\tau}. \tag{12}$$

- $x=0$-boundary problem for (10) is solved as

$$\varphi(y\bar{y},\tau) \to \Phi(x|Y,\tau) := \exp(x^{\alpha\dot{\alpha}}P_{\alpha\dot{\alpha}})\varphi(y\bar{y},\tau). \tag{13}$$

scalar field

- An action principle in spinor space can be deduced from the space-time one by means of a $(Y,\tau)\to x$ unfolding map

$$\phi(x)=\Phi(x|Y=0,\tau=0)=\exp(x^{\alpha\dot\alpha}P_{\alpha\dot\alpha})\varphi(y\bar y,\tau)|_{Y,\tau=0}, \tag{14}$$

$$S=-\frac{1}{2}\int d^4x\phi(x)(\Box-m^2)\phi(x)=-\frac{1}{2}\varphi\delta^4(\overleftarrow{P}+\overrightarrow{P})(\frac{\partial}{\partial\tau}-m^2)\varphi|_{Y,\tau=0} \tag{15}$$

- Making use of the delta-function representation

$$\delta(z)=\lim_{\epsilon\to0}\frac{1}{\sqrt{4\pi\epsilon}}\exp\{-\frac{z^2}{4\epsilon}\}, \tag{16}$$

the action can be rewritten as

$$S[\varphi]=\lim_{\epsilon\to0}\frac{(-1)}{32\pi^2\epsilon^2}(e^{(-4\epsilon)^{-1}\frac{\partial}{\partial\tau}}\varphi)\exp\{-\frac{\overleftarrow{P}\cdot\overrightarrow{P}}{2\epsilon}\}(\frac{\partial}{\partial\tau}-m^2)(e^{(-4\epsilon)^{-1}\frac{\partial}{\partial\tau}}\varphi)|_{Y,\tau=0}. \tag{17}$$

- Compare with the supertrace on the Weyl algebra

$$\mathrm{str}(f*g)=f(Y)\exp(i\overleftarrow{\partial}_A\overrightarrow{\partial}^A)g(Y)|_{Y=0}. \tag{18}$$

- Poincaré-invariance of S is ensured by $\delta^4(\overleftarrow{P}+\overrightarrow{P})$ (for translations) and Lorentz-covariance plus Y-independence (for rotations).

scalar field

- e.o.m. following from S are

$$(\frac{\partial}{\partial\tau} - m^2)\varphi(y\bar{y},\tau) = 0 \qquad (19)$$

and put no restrictions on Y-dependence of φ.

- General solution is

$$\varphi = \widetilde{\varphi}(y\bar{y})e^{\tau m^2},\ \forall\widetilde{\varphi}. \qquad (20)$$

$$P_{\alpha\dot{\alpha}} = \frac{1}{\hat{N}+1}(\partial_\alpha\bar{\partial}_{\dot{\alpha}} - y_\alpha\bar{y}_{\dot{\alpha}}m^2). \qquad (21)$$

This corresponds to τ being a Lorentz-invariant evolution parameter, while spinors covariantly parameterize $3d$ Cauchy hypersurface, which is light-like (even for massive fields). Massless fields are "static".

- This separation of τ and Y variables is a distinguished feature of the proposed construction. In fact, the freedom is much larger: P^2 can be defined arbitrarily, with the only condition $[P^2, Y^A] = [P^2, \partial_A] = 0$. Analysis of unfolded off-shell supersymmetric systems [NM 2201.01674] hints towards a 2^{nd}-order realization of P^2. In general case, $\tau = 0$ projection gets replaced with projection onto $\ker P^2$.

spin-1 field

- Proca equations for a spin-1 field [Proca, 1936]:

$$(\Box - m^2)A_n(x) = 0, \tag{22}$$

$$\partial^n A_n(x) = 0 \tag{23}$$

- Massless limit corresponds to the Maxwell equations in the Lorenz gauge, with a residual gauge symmetry

$$A_n(x) \to A_n(x) + \partial_n f(x), \quad \Box f(x) = 0. \tag{24}$$

spin-1 field

- Consider the following unfolded map for a vector field [NM 2402.14164]

$$A_{\alpha\dot\alpha}(x) \to A(x|Y,\tau) := e^{y^\beta \bar y^{\dot\beta}\nabla_{\beta\dot\beta}+\tau\Box}\{A_{\alpha\dot\alpha}(x)y^\alpha\bar y^{\dot\alpha}+\frac{1}{2}F_{\alpha\alpha}(x)y^\alpha y^\alpha+\frac{1}{2}\bar F_{\dot\alpha\dot\alpha}(x)\bar y^{\dot\alpha}\bar y^{\dot\alpha}\}$$

  where (anti-)self-dual components of the Maxwell tensor are

$$F_{\alpha\alpha}(x) := \nabla_{\alpha\dot\beta}A_\alpha{}^{\dot\beta}(x), \quad \bar F_{\dot\alpha\dot\alpha}(x) := \nabla_{\beta\dot\alpha}A^\beta{}_{\dot\alpha}(x). \tag{25}$$

- This map corresponds to an unfolded-type equation

$$\nabla_{\alpha\dot\alpha}A(x|Y,\tau) = P_{\alpha\dot\alpha}A(x|Y,\tau), \tag{26}$$

  which in addition imposes a transversality constraint on $A_{\alpha\dot\alpha}(x)$

$$\nabla^{\alpha\dot\alpha}A_{\alpha\dot\alpha}(x) = 0. \tag{27}$$

- Poincaré-translations in the spinor space are realized as

$$\begin{aligned} P_{\alpha\dot\alpha} \quad = \quad & \frac{1}{(\hat N+1)(\hat{\bar N}+1)}\{\frac{\hat N+\hat{\bar N}}{2}\partial_\alpha\bar\partial_{\dot\alpha}+\frac{\hat N+\hat{\bar N}+4}{2}y_\alpha\bar y_{\dot\alpha}\frac{\partial}{\partial\tau}- \\ & -\bar y_{\dot\alpha}\partial_\alpha(\Pi^+-2\frac{\partial}{\partial\tau}\Pi^0)-y_\alpha\bar\partial_{\dot\alpha}(\Pi^--2\frac{\partial}{\partial\tau}\Pi^0)\} \end{aligned} \tag{28}$$

  on the spin-1 module of the form ("external" indices are not really external)

$$A(Y,\tau) = Y^M Y^N A_{MN}(y\bar y,\tau). \tag{29}$$

spin-1 field

- Analogously to the scalar field case, one defines a spinor-space action

$$S[A(Y,\tau)] = -\frac{1}{4}A\{(\overleftarrow{\partial}_M \overrightarrow{\partial}^M)^2 \delta^4(\overleftarrow{P} + \overrightarrow{P})\}(\frac{\partial}{\partial\tau} - m^2)A|_{Y,\tau=0}, \qquad (30)$$

where $\overleftarrow{\partial}_M \overrightarrow{\partial}^M$ acts after evaluation of the delta-function ($[\partial_M, P_{\alpha\dot{\alpha}}] \neq 0$).

- e.o.m. and their general solutions are

$$(\frac{\partial}{\partial\tau} - m^2)Y^M Y^N A_{MN}(y\bar{y},\tau) = 0 \implies A = A_{MN}(y\bar{y})Y^M Y^N e^{m^2\tau}, \qquad (31)$$

and again there are no restrictions on Y-dependence.

- In the space-time picture, instead of Proca action, this corresponds to

$$S[A] \quad = \quad \int d^4x\{c_1 \cdot A^{\|}_a(\Box - m^2)A^{\|a} + c_2 \cdot F_{a,b}(\Box - m^2)F^{a,b}\}. \qquad (32)$$

But in the spinor-space picture, F and $\bar{F}$ are independent elementary fields and the action is of first order in the evolution parameter τ, while the very A_a is in fact absent.

spin-1 field

- In the massless case $\frac{\partial}{\partial\tau}A = 0$, on the spinor-space field

$$A(Y) = \underbrace{A_{\alpha\dot\alpha}(y\bar y)y^\alpha\bar y^{\dot\alpha}}_{A^{\parallel}} + \underbrace{\frac{1}{2}F_{\alpha\alpha}(y\bar y)y^\alpha y^\alpha}_{F} + \underbrace{\frac{1}{2}\bar F_{\dot\alpha\dot\alpha}(y\bar y)\bar y^{\dot\alpha}\bar y^{\dot\alpha}}_{\bar F}, \qquad (33)$$

Poincaré translations are realized as

$$P_{\alpha\dot\alpha}F = \frac{\hat N + \hat{\bar N}}{2(\hat N + 1)(\hat{\bar N} + 1)}\partial_\alpha\bar\partial_{\dot\alpha}F, \quad P_{\alpha\dot\alpha}\bar F = \frac{\hat N + \hat{\bar N}}{2(\hat N + 1)(\hat{\bar N} + 1)}\partial_\alpha\bar\partial_{\dot\alpha}\bar F,$$

$$P_{\alpha\dot\alpha}A^{\parallel} = \frac{1}{(\hat N + 1)(\hat{\bar N} + 1)}\{\frac{\hat N + \hat{\bar N}}{2}\partial_\alpha\bar\partial_{\dot\alpha}A^{\parallel} - \bar y_{\dot\alpha}\partial_\alpha F - y_\alpha\bar\partial_{\dot\alpha}\bar F\}. \qquad (34)$$

- F and $\bar F$ correspond to physical ± 1-helicities.
- zero-helicity $A^{\parallel}$ corresponds to the residual pure-gauge contributions in the Lorenz gauge

$$A^{\parallel} = y^\alpha\bar y^{\dot\alpha}P_{\alpha\dot\alpha}\varphi(y\bar y) = \hat N\varphi(y\bar y), \quad \varphi = \hat N^{-1}A^{\parallel}. \qquad (35)$$

- The module is manifestly indecomposable, which reflects the non-existence of a Lorentz-invariant complete gauge.

integer-spin fields

- Fierz equations for a spin-s field [Fierz, 1939]:

$$(\Box - m^2)A_{n_1 n_2 \dots n_s}(x) = 0, \quad (36)$$
$$\partial^{n_1} A_{n_1 n_2 \dots n_s}(x) = 0, \quad (37)$$
$$\eta^{n_1 n_2} A_{n_1 n_2 \dots n_s}(x) = 0. \quad (38)$$

- Massless limit corresponds to the Fronsdal equations in TT-gauge, with the residual gauge symmetry

$$A_{n_1 n_2 \dots n_s} \to A_{n_1 n_2 \dots n_s} + \partial_{n_s} f_{n_1 \dots n_{s-1}} \quad (39)$$

for a gauge function $f_{n_1 \dots n_{s-1}}(x)$ subjected to the same equations.

- Lagrangian requires a bunch of auxiliary symmetric traceless fields of ranks $s-2, s-3, \dots 0$ [Singh, Hagen, 1974]. For massless fields, only $s-2$ is necessary [Fronsdal, 1978].

integer-spin fields

- Straightforward generalization of the $s=1$ case to an arbitrary integer s gives
$$\Phi_s(Y,\tau)=(Y^A)^{2s}\Phi_{A(2s)}(y\bar{y},\tau), \tag{40}$$
produced from a transverse space-time field $\phi_{\alpha(s),\dot{\alpha}(s)}(x)$ as
$$\Phi_s(x|Y,\tau)=e^{y^\beta\bar{y}^{\dot{\beta}}\nabla_{\beta\dot{\beta}}+\tau\Box}\frac{1}{2}(e^{y^\gamma\bar{\partial}^{\dot{\gamma}}\nabla_{\gamma\dot{\gamma}}}+e^{\partial^\gamma\bar{y}^{\dot{\gamma}}\nabla_{\gamma\dot{\gamma}}})\phi_{\alpha(s),\dot{\alpha}(s)}(x)(y^\alpha\bar{y}^{\dot{\alpha}})^s. \tag{41}$$
- Poincaré algebra is realized via
$$P_{\alpha\dot{\alpha}}=a_{\hat{N},\hat{\bar{N}}}\partial_\alpha\bar{\partial}_{\dot{\alpha}}+b_{\hat{N},\hat{\bar{N}}}y_\alpha\bar{y}_{\dot{\alpha}}+c_{\hat{N},\hat{\bar{N}}}\bar{y}_{\dot{\alpha}}\partial_\alpha+\bar{c}_{\hat{N},\hat{\bar{N}}}y_\alpha\bar{\partial}_{\dot{\alpha}} \tag{42}$$
for certain Lorentz-invariant operators a, b, c and $\bar{c}$.
- Poincaré-invariant action is
$$S[\Phi_s]=-\frac{1}{2(2s!)^2}\Phi_s\{(\overleftarrow{\partial}_M\overrightarrow{\partial}^M)^{2s}\delta^4(\overleftarrow{P}+\overrightarrow{P})\}(\frac{\partial}{\partial\tau}-m^2)\Phi_s|_{Y,\tau=0}, \tag{43}$$
which leads to e.o.m. that impose no restrictions on Y-dependence
$$(\frac{\partial}{\partial\tau}-m^2)\Phi_s(Y,\tau)=0,\quad \Phi_s(Y,\tau)=(Y^A)^{2s}\Phi_{A(2s)}(y\bar{y})e^{m^2\tau} \tag{44}$$
- In the massless case, a set of all intermediate helicities forms a maximal gauge submodule, so the physical are only $\pm s$, i.e.
$$\Phi_s^{phys}(Y,\tau)=(y^\alpha)^{2s}\Phi_{\alpha(2s)}(y\bar{y})+(\bar{y}^{\dot{\alpha}})^{2s}\bar{\Phi}_{\dot{\alpha}(2s)}(y\bar{y}). \tag{45}$$

conclusions

- an action principle for the free classical integer-spin Poincaré fields in the spinor space (Y^A, τ) is constructed.
- e.o.m. only fix τ-dependence, while Y-dependence is constrained by the spin value.
- for massless fields, space-time gauge symmetry gets transformed into a manifestly indecomposable structure of spinor modules.
- relation to the space-time picture is provided by corresponding unfolded-like equations.
- further directions:
  - conservation laws and charges
  - inherent spinor-space symmetries
  - black-hole-like solutions
  - interactions and locality
  - quantization