Wilson networks in AdS and global conformal blocks

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AdS_2 space

The AdS_2 gravity can be formulated in terms of $sl(2,\mathbb{R})$ gauge connections $A = A(\rho, z)$ with the zero curvature condition $F = dA + A \wedge A = 0$. The zero curvature condition can be realized dynaimcally via the BF action

$$S_{BF}[A, B] = \frac{1}{2} \int_{\mathcal{M}_2} Tr\, BF$$

Introducing the $sl(2,\mathbb{R})$ commutation relations

$$[J_n, J_m] = (n-m)J_{n+m} \quad \text{with} \quad n, m = -1, 0, 1$$

the solution of the zero-curvature condition $F = 0$ can be cast into the form

$$A = e^{-\rho J_0} J_1 dz e^{\rho J_0} + J_0 d\rho \qquad \textbf{(Banados 1995)}$$

The associated metric of the AdS_2 spacetime is given by

$$ds^2 = e^{2\rho} dz^2 + d\rho^2$$

Wilson lines and intertwiners

- Gravitational Wilson line:

$$W_h[L] = \mathbb{P}e^{-\int_L A}$$

  - L is a path from x_1 to x_2.
  - The index $h \longrightarrow$ the connection A takes values in the $sl(2,\mathbb{R})$ module $\mathcal{R}_h$ of weight $-h$.

  Useful property:

  - Wilson line associated with a flat connection depends only on the x_1, x_2.

  Direct calculation using A shows that

$$W_h[x_1, x_2] = e^{-\rho_2 J_0} e^{z_{12} J_1} e^{\rho_1 J_0} \qquad \text{where} \quad z_{ij} = z_i - z_j$$

- Intertwiners are defined as invariant tensors from $Inv(\mathcal{R}^*_{h_1} \otimes \mathcal{R}_{h_2} \otimes \mathcal{R}_{h_3})$, i.e

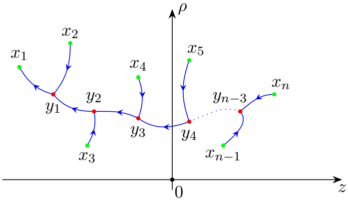
$$I_{h_1 h_2 h_3} : \quad \mathcal{R}_{h_2} \otimes \mathcal{R}_{h_3} \to \mathcal{R}_{h_1}$$

  with the invariance property

$$I_{h_1 h_2 h_3} U_{h_2} U_{h_3} = U_{h_1} I_{h_1 h_2 h_3}$$

  where U_h are $SL(2,\mathbb{R})$ operators of the corresponding representations.

Wilson line networks



Wilson line network operator:

$$\widehat{W}^{h_1\dots h_n}_{\tilde{h}_1\dots\tilde{h}_{n-3}}(\mathbf{x},\mathbf{y}) := \Big(W_{h_1}[y_1,x_1]I_{h_1 h_2 \tilde{h}_1} W_{\tilde{h}_1}[y_2,y_1]I_{\tilde{h}_1 h_3 \tilde{h}_2}\dots W_{\tilde{h}_{n-3}}[y_{n-2},y_{n-3}]I_{\tilde{h}_{n-3}h_{n-1}h_n}\Big)$$

$$\times\Big(W_{h_2}[x_2,y_1]\dots W_{h_{n-1}}[x_{n-1},y_{n-2}]W_{h_n}[x_n,y_{n-2}]\Big)$$

AdS vertex function is matrix element of the Wilson line network operator with the cap states taken as Ishibashi states $|a_k\rangle$ **(Ishibashi 1989)**

$$\mathcal{V}_{h\tilde{h}}(\mathbf{x}) := \langle a_1|\,\widehat{W}^{h_1\dots h_n}_{\tilde{h}_1\dots\tilde{h}_{n-3}}(\mathbf{x},\mathbf{y})\,|a_2\rangle\otimes|a_3\rangle\otimes\dots\otimes|a_n\rangle$$

- Ishibashi states $\longrightarrow$ AdS_2 spacetime invariance of the AdS vertex function.
- The AdS vertex function is independent of positions of the vertices y_i **(Bhatta, et al. 2016; Besken, et al. 2016; Alkalaev, et al. 2020)**.

AdS vertex function parametrization, global conformal blocks and extrapolate dictionary relation

- The AdS vertex functions can be parameterized as

$$\mathcal{V}_{h\tilde{h}}(\mathbf{x}) = \mathcal{V}_{h\tilde{h}}(c_{12}, \ldots, c_{n-1\,n}, c_{13}, \ldots, c_{n-2\,n})$$

where we introduced the AdS invariant variables:

$$c_{ij} = e^{\rho_i - \rho_j} + e^{\rho_i - \rho_j} + (z_i - z_j)^2 e^{\rho_i + \rho_j} - 2$$

- The conformal n-point correlators can be expanded into the conformal blocks as

$$\langle \hat{\mathcal{O}}_1(z_1) \cdots \hat{\mathcal{O}}_n(z_n) \rangle = \sum_{\tilde{h}} C_{12\tilde{h}_1} \ldots C_{\tilde{h}_{n-2}\,n-1\,n}\, \mathcal{F}_{h\tilde{h}}(\mathbf{z})$$

where C_{ijk} are model-dependent structure constant and $\mathcal{F}_{h\tilde{h}}(\mathbf{z})$ are conformal blocks. The $c \to \infty$ asymptotic of the block depends on the asymptotics of $h, \tilde{h}$:

$$h, \tilde{h} = \mathcal{O}(c^1) : \text{heavy operators}$$
$$h, \tilde{h} = \mathcal{O}(c^0) : \text{light operators}$$

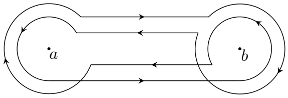
Global conformal block – all operators are light.

- The extrapolate dictionary relation (Alkalaev, Kanoda, Khiteev 2024)

$$\lim_{\rho \to \infty} e^{\rho \sum_{i=1}^n h_i}\, \mathcal{V}_{h\tilde{h}}(\mathbf{x})\big|_{\rho_1 = \rho_2 = \ldots = \rho_n = \rho} = C_{h\tilde{h}}\, \mathcal{F}_{h\tilde{h}}(\mathbf{z})$$

where $\mathcal{F}_{h\tilde{h}}(\mathbf{z})$ is the global conformal block in the CFT_1.

Matrix elements of the Wilson lines and n-point AdS vertex function



Proposition 1: The holographic reconstruction of the Wilson matrix elements

$$\langle a|W_h[0,x]|h,m\rangle = K_h \oint_{P[w,\bar{w}]} du \;\; K(x,u|h)\,\langle a|W_h[0,(u,\rho)]|h,m\rangle_\partial$$

$$\langle h,m|W_h[x,0]|a\rangle = K_h \oint_{P[w,\bar{w}]} dv \;\; K(x,v|h)\,\langle h,m|W_h[(v,\rho),0]|a\rangle_\partial$$

where

- $\langle a|W_h[0,y_1]|h,m\rangle_\partial$ – boundary asymptotics of Wilson matrix elements,
- K(x,u|h) – smearing function,
- $P[w,\bar{w}]$ – Pochhammer contour around points $w = z + ie^{-\rho}$ and $\bar{w} = z - ie^{-\rho}$.

Proposition 2: The holographic reconstruction formula for the n-point AdS vertex functions is given by

$$\mathcal{V}_{h\tilde{h}}(\mathbf{x}) = C_{h\tilde{h}} \prod_{k=1}^{n} K_{h_k} \oint_{P[w_k,\bar{w}_k]} du_k \;\; K(x_k,u_k|h_k)\mathcal{F}_{h\tilde{h}}(\mathbf{u})$$

where $\mathcal{F}_{h\tilde{h}}(\mathbf{u})$ is n-point global conformal block.

2-point AdS vertex function

The 2-point AdS vertex function:

$$\mathcal{V}_{h_1 h_2}(x_1, x_2) = \frac{\delta_{h_1 h_2}}{(-2h_1+1)^{\frac{1}{2}}} \left(\frac{\xi(x_1, x_2)}{2} \right)^{h_1} {}_2F_1\left(\frac{h_1}{2}, \frac{h_1}{2} + \frac{1}{2}; h_1 + \frac{1}{2} | \xi(x_1, x_2)^2 \right)$$

$$= \delta_{h_1 h_2} C_{h_1 h_2} G_{bb}(x_1, x_2 | h_1)$$

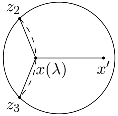
where where the AdS invariant distance is defined as

$$\xi(x, x') := \frac{2e^{-\rho-\rho'}}{e^{-2\rho} + e^{-2\rho'} + (z - z')^2}$$

and $G_{bb}(x_1, x_2 | h_1)$ is the bulk-to-bulk propagator of free scalar fields in AdS_2

- Similar results were obtained earlier (Castro, et al. 2018), but the construction considered there is defined only in the case $n = 2$.

3-point AdS vertex functions



Proposition 3: In the case of two points on the boundary the AdS vertex function is proportional to the geodesic Witten diagram

$$\int_{\gamma_{23}} d\lambda G_{bb}(x(\lambda), x_1|h_1) G_{b\partial}(x(\lambda), z_2|h_2) G_{b\partial}(x(\lambda), z_3|h_3) \propto \mathcal{V}_{h_1 h_2 h_3}(x_1, z_2, z_3)$$

where $G_{bb}(x(\lambda), x_1|h_1)$ is bulk-to-bulk propagator and $G_{b\partial}(x(\lambda), z_2|h_2)$ is bulk-to-boundary propagator.

Proposition 4: In case of one point on the boundary the 3-point Witten diagram can be expressed as a linear combination of the 3-point AdS vertex functions.

$$\int_{\mathbf{AdS}_2} d^2x\, G(x, x_1|h_1) G(x, x_2|h_2) G(x, x_3|h_3) \sim \mathcal{V}_{h_1 h_2 h_3}(x_1, x_2, x_3)$$

$$+ \sum_{n=0}^{\infty} \alpha_{h_1 h_2 h_3}\, \mathcal{V}_{h_2+h_3+2n\ \ h_2 h_3}(x_1, x_2, x_3) + \sum_{n=0}^{\infty} \beta_{h_1 h_2 h_3}\, \mathcal{V}_{h_1\ \ h_1+h_3+2n\ \ h_3}(x_1, x_2, x_3)$$

$$+ \sum_{n=0}^{\infty} \gamma_{h_1 h_2 h_3}\, \mathcal{V}_{h_1 h_2\ \ h_1+h_2+2n}(x_1, x_2, x_3)\,.$$

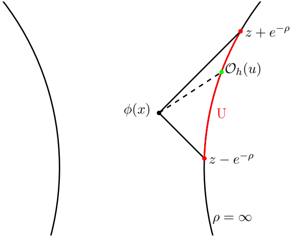
Conclusion and outlooks

- Formulated the one-to-one holographic correspondence between the global conformal block and the AdS vertex function.
- Ishibashi states as cap states $\longrightarrow$ AdS_2 spacetime invariance of the AdS vertex function.
- The 3-point Witten diagram expressed as a linear combination of the 3-point AdS vertex functions.

Further developments:

- n-point Witten diagram = linear combination of the n-point AdS vertex function?
- Correspondence between scalar field theory in AdS_2 and theory of the $sl(2, \mathbb{R})$ flat connections.

HKLL formalism (Hamilton, et al. 2006)



HKLL representation of a scalar field in AdS_2:

$$\phi(x) = \int_U du\, K(x, u|h)\mathcal{O}_h(u)$$
$$K(x, u|h) = \frac{e^{-(1-h)\rho}}{(e^{-2\rho} + (z-u)^2)^{1-h}}$$

- $\mathcal{O}_h(u)$ is a primary operator of conformal weight h.
- Mass of the scalar field related with conformal weight as $m^2 = h(1-h)$.
- $K(x, u|h)$ is smearing function.
- Integration contour U lies on the conformal boundary of AdS_2.
- AdS_2 spacetime has two conformal boundaries but we consider only one.