On consistency of the interacting (anti)holomorphic higher-spin sector

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Generating systems

Within unfolded formalism Fronsdal equations on AdS_4 background can be equivalently rewritten as Central on-mass-shell theorem (M. Vasiliev 1989)

$$\mathrm{d}_x + \Omega_{AdS} * \omega + \omega * \Omega_{AdS} = \Upsilon(\Omega_{AdS}, \Omega_{AdS}, C)\,,$$

$$\mathrm{d}_x C + \Omega_{AdS} * C - C * \Omega_{AdS} = 0\,,$$

where $\Omega_{AdS} = \{\omega_{\alpha\beta}, \bar{\omega}_{\dot{\alpha}\dot{\beta}}, e_{\alpha\dot{\beta}}\}$ – is background connection and $\omega(Y|x)$ and $C(Y|x)$ are generating functions for HS potentials and curvatures

$$\omega(Y,x) = \sum_{n,m} \mathrm{d}x^{\mu} \omega_{\mu\,\alpha_1\ldots\alpha_n,\dot{\alpha}_1\ldots\dot{\alpha}_m}(x) y^{\alpha_1}\ldots y^{\alpha_n} \bar{y}^{\dot{\alpha}_1}\ldots\bar{y}^{\dot{\alpha}_m};\;\; m+n = 2(s-1),$$

$$C(Y,x) = \sum_{n,m} C_{\alpha_1\ldots\alpha_n,\dot{\alpha}_1\ldots\dot{\alpha}_m}(x) y^{\alpha_1}\ldots y^{\alpha_n} \bar{y}^{\dot{\alpha}_1}\ldots\bar{y}^{\dot{\alpha}_m};\;\; |m-n| = 2s.$$

This unfolded form suggest to look for nonlinear corrections in the following way

$$\mathrm{d}_x\omega + \omega * \omega = \Upsilon(\omega,\omega,C) + \Upsilon(\omega,\omega,C,C) + \ldots,$$

$$\mathrm{d}_x C + \omega * C - C * \omega = \Upsilon(\omega, C, C) + \ldots.$$

Concept of generating systems

$$\omega(y|x) \Longrightarrow W(Z,Y|x) \quad C(Y|x) \Longrightarrow B(Z,Y|x) \quad \mathrm{d}_Z W = \ldots \quad \mathrm{d}_Z B = \ldots$$

Vasiliev generating system

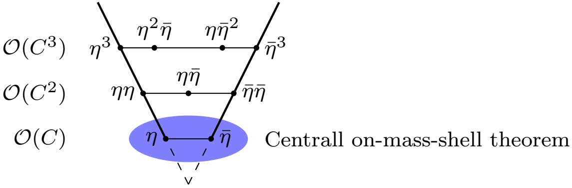
Vasiliev generating system in $d = 4$ [M. Vasiliev 1992]

$$
\begin{aligned}
&\mathrm{d}_x W + W * W = 0\,, \qquad (1)\\
&\mathrm{d}_x S + [W, S]_* = 0\,,\\
&\mathrm{d}_x B + [W, B]_* = 0\,, \qquad (2)\\
&S * S = i(\theta^A \theta_A + \eta B * \gamma + \bar{\eta} B * \bar{\gamma})\,,\\
&[S, B]_* = 0\,.
\end{aligned}
$$

$$(f \star g)(Z, Y) = \frac{1}{(2\pi)^4} \int dU dV f(Z + U, Y + U) g(Z - V, Y + V) e^{iU_A V^A}.$$

$$(1) \Longleftrightarrow \; \mathrm{d}_x \omega + \omega * \omega = \Upsilon^{\eta}(\omega, \omega, C) + \Upsilon^{\bar{\eta}}(\omega, \omega, C) + \dots$$

$$(2) \Longleftrightarrow \mathrm{d}_x C + [\omega, C]_* = \Upsilon^{\eta}(\omega, C, C) + \Upsilon^{\bar{\eta}}(\omega, C, C) + \dots$$



(Anti)holomorphic generating system

Generating system for (anti)holomorphic sector proposed by Vyacheslav Didenko (2022) looks as follows

$$
\begin{aligned}
&\mathrm{d}_x W + W * W = 0\,,\\
&\mathrm{d}_z W + \{W, \Lambda\}_* + \mathrm{d}_x \Lambda = 0\,,\\
&\mathrm{d}_z \Lambda = C * \gamma\,,\\
&\mathrm{d}_x C * \gamma = \mathrm{d}_z \{W, \Lambda\}_*\,,\\
&W \in \mathbf{C}^0\,,\ \ \Lambda[C] := \theta^\alpha z_\alpha \int_0^1 d\tau\tau\, e^{i\tau z_\alpha y^\alpha} C(-\tau z|x) \in \mathbf{C}^1\,,
\end{aligned}
$$

where product is given by [A. Sharapov, E.Skvortsov 2022]

$$(f * g)(z, y) = \int du\, dv\, dP\, dQ\, e^{iu_\alpha v^\alpha - iP_\alpha v^\alpha + iu_\alpha Q^\alpha} f(z+u, y+P) g(z+v, y+Q).$$

and $\mathrm{d}_z := \theta^\alpha \frac{\partial}{\partial z^\alpha}$. Product can be understood as limit of the following β-product [V. Didenko, O. Gelfond, AK, M. Vasiliev 2019] which comes from reordering and stretching

$$
\begin{aligned}
(f *_\beta g)(z, y) = \int du\, dv\, dP\, dQ\, \frac{(1-\beta)^4}{\beta^2(2-\beta)^2} \exp\bigg\{ -\frac{i(1-\beta)^2}{\beta(2-\beta)} u_\alpha v^\alpha + \frac{i(1-\beta)^2}{\beta(2-\beta)} P_\alpha v^\alpha - \\
-\frac{i(1-\beta)^2}{\beta(2-\beta)} u_\alpha Q^\alpha + \frac{i}{\beta(2-\beta)} P_\alpha Q^\alpha \bigg\} f\left(\frac{1-\beta}{1-\beta}(z+u), y+U\right) g\left(\frac{1-\beta}{1-\beta}(z+v), y+V\right),
\end{aligned}
$$

Classes of functions and troubles

Generating function for functional classes $\mathbf{C}^r$ for $r = 0, 1, 2$ is defined as follows

$$\int \mathscr{D}\rho \int_0^1 d\mathcal{T}\,(1-\mathcal{T})^{1-r}\mathcal{T}^{r-1}\,\exp\{i\mathcal{T}z_\alpha(y-B)^\alpha + i(1-\mathcal{T})y^\alpha A_\alpha - i\mathcal{T}B_\alpha A^\alpha\},.$$

Here A and B might depend on various ρs and derivarives over full (anti)holomorphic spinorial arguments of fields ω and C which are denoted as

$$p_\alpha := -i\frac{\partial}{\partial \mathsf{y}_C}\,,\ \ t_\alpha := -i\frac{\partial}{\partial \mathsf{y}_\omega}\,.$$

So the exponent should be understood as follows

$$\exp\{i\mathcal{T}z_\alpha(y - B(t,p_1,p_2,\dots))^\alpha + i(1-\mathcal{T})y^\alpha A(t,p_1,p_2,\dots)_\alpha -$$

$$-i\mathcal{T}B(t,p_1,p_2,\dots)_\alpha A(t,p_1,p_2,\dots)^\alpha\}\omega(\mathsf{y}_\omega|x)C(\mathsf{y}_{C_1}|x)C(\mathsf{y}_{C_2}|x)\dots\Big|_{\mathsf{y}_\omega=0,\ \mathsf{y}_{C_1}=0,\ \mathsf{y}_{C_2}=0,\dots}$$

Troubles with multiplication

$$\mathbf{C}^1 * \mathbf{C}^1 = \infty \times 0\,,\ \ \mathbf{C}^0 * \mathbf{C}^2 = \infty \times 0\,.$$

Hence no Leibniz rule for

$$\mathrm{d}_z\,(W * \Lambda) = \underbrace{\mathrm{d}_z W * \Lambda}_{undefined} - \underbrace{W * \mathrm{d}_z\Lambda}_{undefined}\,.$$

Extracting the vertices and consistency

One-form sector is similiar to Vasiliev theory

$$\mathrm{d}_z W + \{W, \Lambda\}_* + \mathrm{d}_x \Lambda = 0 \quad \Longrightarrow \quad W(z, y|x) = \omega(y|x) + W_{\omega C} + W_{\omega CC} + \dots$$

Plugging to zero-curvature condition

$$\mathrm{d}_x W + W * W = 0 \quad \Longrightarrow \quad \mathrm{d}_x \omega + \omega * \omega = -\mathrm{d}_x W_{\omega C} - \omega W_{\omega C} - W_{\omega C} * W_{\omega C} + \dots$$

Zero-form is highly different. For $W \in \mathbf{C}^0$ the following projection identity holds

$$\mathrm{d}_z \left(W_* \Lambda\right) = -\left(\int e^{iuv}\, W(z, y+u) C(y+v)\right)\Bigg|_{z=-y} * \frac{1}{2}\theta_\alpha \theta^\alpha e^{izy}\,, \tag{1}$$

$$\mathrm{d}_z \left(\Lambda * W_{C^n}\right) = \left(\int e^{iuv}\, C(y+u) W(z, -y-v)\right)\Bigg|_{z=-y} * \frac{1}{2}\theta_\alpha \theta^\alpha e^{izy}\,. \tag{2}$$

Plugging obtained Ws

$$\mathrm{d}_x C * \gamma = \mathrm{d}_z \{W, \Lambda\}_* \quad \Longrightarrow \quad \mathrm{d}_x C * \gamma = C * \omega * \gamma - \omega * C * \gamma + \mathcal{O}(C^2) * \gamma + \dots$$

All verticies are known explicitly to all orderds [V. Didenko, M. Povarnin 2024]
Checking for consistency

$$\mathrm{d}_x^2 C * \gamma = \mathrm{d}_x \mathrm{d}_z \{W, \Lambda\}_* = -\mathrm{d}_z \mathrm{d}_x \{W, \Lambda\} = \dots = \mathrm{d}_z \mathrm{d}_z (W * W)\,.$$

Absence of Leibniz rule and consequences

Consider the linear space of analytic functions in two variables x and z. One can pick up the following basis

$$z^m x^n$$

Then we define "differential"in z as

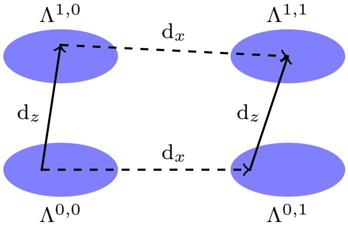
$$\mathrm{d}_z : \Lambda^{0,0} \to \Lambda^{1,0} \quad \mathrm{d}_z(z^m x^n) := \mathrm{d}z f_{m,n}(z) x^n \, ,$$

$$\mathrm{d}_z : \Lambda^{0,1} \to \Lambda^{1,1} \quad \mathrm{d}_z(\mathrm{d}x z^m x^{n-1}) := \mathrm{d}z \wedge \mathrm{d}x f_{m,n}(z) x^{n-1} \, ,$$

while $\mathrm{d}_x := \mathrm{d}x \frac{\partial}{\partial x}$. With such definition one can show that

$$(\mathrm{d}_x \mathrm{d}_z + \mathrm{d}_z \mathrm{d}_x) F(z,x) = 0$$

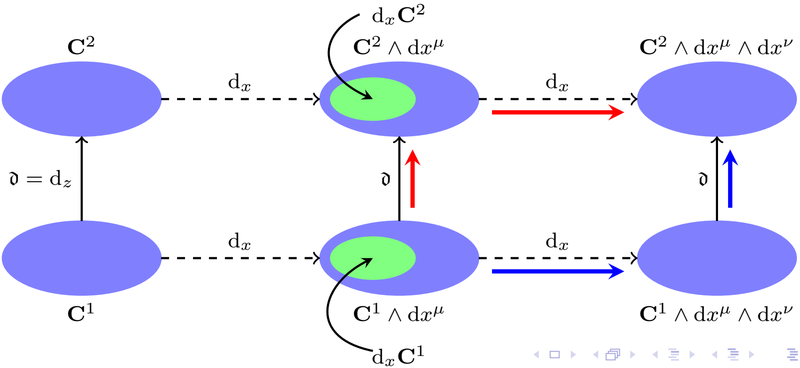
for any analytic in x and z function $F(z,x)$.
Or even more generally



New (anti)holomorphic system

What if we modify the "differential"? $\mathrm{d}_z \to \mathfrak{d}$

$$\mathrm{d}_x W + W * W = 0\,,$$
$$\mathrm{d}_z W + \{W,\Lambda\}_* + \mathrm{d}_x\Lambda = 0\,,$$
$$\mathfrak{d}\Lambda = C * \gamma\,,$$
$$\mathrm{d}_x C * \gamma = \mathfrak{d}\{W,\Lambda\}_*\,,$$
$$W \in \mathbf{C}^0\,,\ \ \Lambda[C] := \theta^\alpha z_\alpha \int_0^1 d\tau\tau\, e^{i\tau z_\alpha y^\alpha} C(-\tau z|x) \in \mathbf{C}^1\,,$$



New (anti)holomorphic system

Map from $\mathbf{C}^1$ to $\mathbf{C}^2$ is defined in the same way $\mathfrak{d}\big|_{\mathbf{C}^1} = \mathrm{d}_z$

$$\mathfrak{d} : \mathbf{C}^1 \to \mathbf{C}^2 \quad \mathfrak{d}\left\{\theta^\alpha z_\alpha \int_0^1 d\tau\tau\, e^{i\tau z_\alpha y^\alpha} C(-\tau z|x)\right\} := C(y|x) * \gamma \,.$$

To define the map between $\mathbf{C}^1 \wedge \mathrm{d}x^\mu$ and $\mathbf{C}^2 \wedge \mathrm{d}x^\mu$ we split $\mathbf{C}^1 \wedge \mathrm{d}x^\mu$ into subspaces (we assume that there are no d_x-chomologies)

$$\mathbf{C}^1 \wedge \mathrm{d}x^\mu = \text{exact forms} \oplus \mathbf{C}^1 \wedge \mathrm{d}x^\mu/\text{exact forms}\,.$$

$$\mathfrak{d}\Big|_{\text{exact forms}} := \mathrm{d}_z\,, \qquad \mathfrak{d}\Big|_{\text{non-exact forms}} := \textit{whatever}$$

Such system generates the same vertices for one-form as original system does. But for zero-forms it can be schematically written as

$$\mathrm{d}_x C * \gamma = \mathfrak{d}\{W, \Lambda\}_* = \mathfrak{d}\big|_{e.}(\text{exact part of } \{W, \Lambda\}_*) + \mathfrak{d}\big|_{n.e.}(\text{non-exact part of } \{W, \Lambda\}_*)$$

$$\mathrm{d}_x C * \gamma = \mathrm{d}_x F(\omega, C, \dots C) + \textit{whatever}$$

First contribution is trivial since it can eliminated by field redefinitions. On the other hand $\mathfrak{d}\mathrm{d}_x\{W, \Lambda\}_* = \mathfrak{d}\mathrm{d}_z(W * W)$.

Shifted homotopies

Operator of shifted homotopy [O. Gelfond, M. Vasiliev 2018] is defined as follows

$$\triangle_q \, J(z,y|\theta) := (z+q)^\alpha \frac{\partial}{\partial\theta^\alpha} \int_0^1 \frac{dt}{t} \, J(tz-(1-t)q, y|t\theta)\,,$$

where q is arbitrary z-independent spinor which also can be an operator.

$$\mathrm{d}_z \, \triangle_q + \triangle_q \, \mathrm{d}_z = 1 - h_q\,.$$

Here h_q is the projector to d_z-cohomologies defined as

$$h_q f(z,y|\theta) := f(-q,y|0)\,.$$

What makes shifted homotopies special are star-exchange relations [V. Didenko, O. Gelfond, AK, M. Vasiliev 2018]. These relations were originally developed for Vasiliev star-product. However for either one of the multiplied functions is z-independent both these products give the same result. For y-independent shift q the following relations holds ($p_f := -i\frac{\partial}{\partial y}$)

$$\triangle_{q-p_f} \, \big(f(y) * \Gamma(z,y)\big) = f(y) * \triangle_q \, \Gamma(z,y)\,,\ \triangle_q \, \big(\Gamma(z,y) * f(y)\big) = \triangle_{q+p_f} \, \Gamma(z,y) * f(y),$$

$$\triangle_{y+q} \, \big(f(y) * \Gamma(z,y)\big) = f(y) * \triangle_{y+q} \, \Gamma(z,y)\,,\ \ \triangle_{-y-p_f} \, \big(f(y) * \Gamma(z,y)\big) = f(y) * \triangle_{-y+p_f} \, \Gamma(z,y)\,,$$

$$\triangle_{y-p_f} \, \big(\Gamma(z,y) * f(y)\big) = \triangle_{y+p_f} \, \Gamma(z,y) * f(y)\,,\ \ \triangle_{q-y} \, \big(\Gamma(z,y) * f(y)\big) = \triangle_{q-y} \, \Gamma(z,y) * f(y)\,.$$

Shifted homotopies

Immidiate application of the shifted homotopy formalism is projection identity

$$\left(\int \frac{\mathrm{d}^2u\,\mathrm{d}^2}{(2\pi)^4} e^{iu_\alpha v^\alpha}\, C(y+u)W(z,-y-v)\right)\Bigg|_{z=-y} = h_{-y-\widehat{p}}\Big\{C(y) * \pi\big(W(z,y)\big)\Big\}$$

$$\left(\int \frac{\mathrm{d}^2u\,\mathrm{d}^2v}{(2\pi)^4} e^{iu_\alpha v^\alpha}\, W(z,y+u)C(y+v)\right)\Bigg|_{z=-y} = h_{y-\widehat{p}}\Big\{W(z,y) * C(y)\Big\}\,.$$

Here $\widehat{p}$ stands for derivative with respect to full spinorial argument of the C-field written explicitly and automorphism π is defined as follows

$$\pi\big[W(z,y)\big] := W(-z,-y)\,.$$

$$\mathrm{d}_x C = h_{-y-\widehat{p}}\Big\{C(y) * \pi\big(W(z,y)\big)\Big\} - h_{y-\widehat{p}}\Big\{W(z,y) * C(y)\Big\}$$

Consistency of the previous equation implies that the following holds due to equation imposed on C and W

$$\mathrm{d}_x^2 C = h_{-y-\widehat{p}}\Big\{C(y)*\pi\big[W(z,y)\big]\Big\}*h_{-y+\widehat{q}}\Big(\pi\big[W(z,y)\big]\Big) - h_{-y-\widehat{p}}\Big\{C(y)*\pi\big[W(z,y)*W(z,y)\big]\Big\}$$
$$- h_{y-\widehat{p}}\Big\{W(z,y) * C(y)\Big\} * h_{-y+\widehat{q}}\Big(\pi\big[W(z,y)\big]\Big) + h_{y-\widehat{p}}\Big\{W(z,y) * W(z,y) * C(y)\Big\} +$$
$$+ h_{y+\widehat{q}}\Big(W(z,y)\Big) * h_{-y-\widehat{p}}\Big\{C(y) * \pi\big[W(z,y)\big]\Big\} - h_{y+\widehat{q}}\Big(W(z,y)\Big) * h_{y-\widehat{p}}\Big\{W(z,y) * C(y)\Big\}\,.$$

Consistency check

$\omega\omega C\ldots C$ ordering is almost kinematic

$$\mathrm{d}_x^2 C\Big|_{\omega\omega C^N} = h_{y-\widehat{p}}\{\omega * W_{\omega C^{N-1}} * C\} - h_{y-\widehat{p}}\{\omega * W_{\omega C^{N-1}} * C\} \equiv 0\,.$$

$\omega C\omega C\ldots C$ is less obvious

$$\begin{aligned}\mathrm{d}_x^2 C\Big|_{\omega C\omega C^N} = &-h_{y-\widehat{p}}\Big\{\omega(y) * C(y)\Big\} * h_{-y+\widehat{q}}\Big(\pi\big[W_{\omega C^N}(z,y)\big]\Big) + \\ &+ h_{y-\widehat{p}}\Big\{W_{\omega C}(z,y) * W_{\omega C^{N-1}}(z,y) * C(y)\Big\} + \\ &+ h_{y+\widehat{q}}\Big(\omega(y)\Big) * h_{-y-\widehat{p}}\Big\{C(y) * \pi\big[W_{\omega C^N}(z,y)\big]\Big\} - \\ &- h_{y+\widehat{q}}\Big(W_{\omega C}(y)\Big) * h_{y-\widehat{p}}\Big\{W_{\omega C^{N-1}}(z,y) * C(y)\Big\}\,.\end{aligned}$$

To proceed we need explicit expression for $W_{\omega C}$ which follows from

$$\mathrm{d}_z W + W * \Lambda + \Lambda * W + \mathrm{d}_x \Lambda = 0\,.$$

$$W_{\omega C} = \omega * C * \triangle_{p+t}\triangle_p\,\gamma\,.$$

And for higher orders the following russian doll-like formula is valid

$$W_{\omega C^N} = -\triangle_0\left(W_{\omega C^{N-1}} * \Lambda\right).$$

Consistency check

$$-\,h_{y+\widehat{q}}\Big(W_{\omega C}(y)\Big)*h_{y-\widehat{p}}\Big\{W_{\omega C^{N-1}}(z,y)*C(y)\Big\}=$$

$$=-\omega*C*h_{y+\widehat{q}}\;\triangle_{p_1+t_1}\triangle_{p_1}\;\gamma*h_{y-p_{N+1}}\Big(W_{\omega C^{N-1}}*C\Big)=$$

$$=-\omega*C*h_{\widehat{q}}\;\triangle_{-y+p_1+t_1}\triangle_{-y+p_1}\;\gamma*h_{y-p_{N+1}}\Big(W_{\omega C^{N-1}}*C\Big)=$$

$$=-\omega*C*h_0\Big(\;\triangle_{-y+p_1+t_1}\triangle_{-y+p_1}\;\gamma*h_{y-p_{N+1}}\big(W_{\omega C^{N-1}}*C\big)\Big)$$

Twist in the arguments is valid due to

$$h_c\;\triangle_b\triangle_a\;\gamma=2\int d^3\tau_+\,\delta(1-\sum_{i=1}^{3}\tau_i)(b-c)_\alpha(a-c)^\alpha\exp\Big\{-i(\tau_1c+\tau_2b+\tau_3a)_\alpha y^\alpha\Big\}\,.$$

Using russian doll-like representation for $W_{\omega C^N}$ and star-exchange relations one obtains

$$\mathrm{d}_xC\Big|_{\omega C\omega C^N}=\omega*C*\Big\{(h_{-y+p_1+t_1}-h_{-y+p_1})\;\triangle_{p_{N+1}}\;\Big(\pi[W_{\omega C^{N-1}}]*\;\triangle_{p_{N+1}}\;\gamma\Big)+$$
$$+\,h_{y+p_{N+1}}\big(\;\triangle_{p_1+t_1}\triangle_{p_1}\;\gamma*W_{\omega C^{N-1}}\big)-$$
$$-\,h_{p_{N+1}}\Big(\;\triangle_{-y+p_1+t_1}\triangle_{-y+p_1}\;\gamma*h_{y+p_{N+1}}(W_{\omega C^{N-1}}\Big)\Big\}*C\,.$$

Consistency check

Next steps in simplification are rather technical but impossible without new sort of identities that generalizes projection identity

$$\mathrm{d}_z\Big(\Gamma(z,y)* \triangle_c \gamma\Big) = h_{y-c}\big(\Gamma(z,y)\big) * \gamma \quad \forall\Gamma(z,y) \in \mathbf{C}^0\,,$$

$$\mathrm{d}_z\big(\triangle_c \gamma * \Gamma(z,y)\big) = \gamma * h_{-y-c}\Gamma(z,y)\,,$$

which after some algebra allows to bring r.h.s. to the form

$$\omega{*}C{*}\Big\{-h_{-y+p_1} \triangle_{-y+p_1+t_1} \big(\pi[W_{\omega C^{N-1}}]* \triangle_{p_{N+1}} \gamma\big)+h_{y+p_{N+1}}(\triangle_{p_1+t_1}\triangle_{p_1} \gamma{*}W_{\omega C^{N-1}})\}{*}C\,.$$

The remaining terms vanish due another new identity

$$h_{y+c}(\triangle_a\triangle_b \gamma * \Gamma(z,y)) = h_{-y+b} \triangle_{-y+a} \big(\pi[\Gamma(z,y)]* \triangle_c \gamma\big) \quad \forall\Gamma(z,y) \in \mathbf{C}^0\,.$$

Conclusion

Results and open questions

- Consistency for orderings $\omega\omega C\dots C$ and $\omega C\omega C\dots C$ is checked explicitly. It rests on new star-exchange-like identities for limiting star-product. Are there more?
- Discovery of identities that generalize projection identity makes Λ less unique objects as it was before and provides freedom for constructing full theory including mixed sector (work in progress)
- Connection with Vasiliev theory through new "differential" $\mathfrak{d}$ (work in progress)