Universal quantum and Macdonald dimensions

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Plan of the talk

1. Vogel's Universality
2. Refinement
3. Refinement of symmetric functions
4. Root systems
5. Orthogonal Macdonald polynomials associated with root systems
7. Quantum dimensions and their universalization
7. Macdonald dimensions and their universalization
8. Conclusion

Vogel's Universality

Vogel's parameters for simple Lie algebras:

Root system	Lie algebra	α	β	γ	$t = \alpha + \beta + \gamma$
A_n	sl_{n+1}	-2	2	$n+1$	$n+1$
B_n	so_{2n+1}	-2	4	$2n-3$	$2n-1$
C_n	sp_{2n}	-2	1	$n+2$	$n+1$
D_n	so_{2n}	-2	4	$2n-4$	$2n-2$
G_2	g_2	-2	$\frac{10}{3}$	$\frac{8}{3}$	4
F_4	f_4	-2	5	6	9
E_6	e_6	-2	6	8	12
E_7	e_7	-2	8	12	18
E_8	e_8	-2	12	20	30

We call a quantity **universal** if it can be expressed as an analytic function of Vogel's parameters.
Dimensions of adjoint representation in Vogel's parameters [Vogel '95]:

$$\dim \mathrm{Adj} = \frac{(\alpha - 2t)(\beta - 2t)(\gamma - 2t)}{\alpha\beta\gamma} \qquad (1)$$

Examples of universal quantities

- Representation theory:
  - Dimensions of adjoint and its descendant representations [Vogel '95,'99; Landsberg, Manivel '06; Avetisyan, Isaev, Krivonos, Mkrtchyan '23; Isaev, Krivonos '24]
  - Generating function for the eigenvalues of higher Casimir operators on the adjoint representation [Sergeev, Veselov, Mkrtchyan '12]
- In Chern-Simone theory:
  - Partition function [Mkrtchyan, Veselov '12; Mkrtchyan '13]
  - Expectation value of unknotted Wilson loop in S^3 in the adjoint representation (quantum dimensions) [Westbury 06'; Mkrtchyan, Veselov '12]
  - Refined partition function [Krefl, Schwarz '13, Avetysyan, Mkrtchyan '22]
- In knot theory:
  - Universal knot polynomial of (2,3)-strand torus knots in adjoint representation [Mironov, Mkrtchyan, Morozov '16]
  - Universal Racah matrices [Mironov, Morozov '16]

Refinement of Chern-Simons theory

Refinement involves the insertion of additional parameters into functions.

The partition function of CS theory on S^3 sphere was given in Witten's seminal paper as the S_{00} element of the S matrix of modular transformations. For an arbitrary gauge group, it is the following [Aganagic, Shakirov '12; Mkrtchyan, Veselov '12; Mkrtchyan '13]

$$Z(\kappa) = \mathrm{Vol}(Q^\vee)^{-1}(\kappa + t)^{-r/2} \prod_{\vec{\alpha}>0} 2\sin\left(\pi \frac{(\vec{\alpha}, \rho)}{\kappa + t}\right) \tag{2}$$

Avetysyan, Mkrtchyan '22 suggested the following expression for S_{00} for the refined CS theory:

$$Z(\kappa, y) = \mathrm{Vol}(Q^\vee)^{-1}\delta^{-r/2} \prod_{m=0}^{y-1} \prod_{\vec{\alpha}>0} 2\sin\pi \frac{y(\vec{\alpha}, \rho) - m(\vec{\alpha}, \vec{\alpha})/2}{\kappa + yt} \tag{3}$$

$\delta = \kappa + yt$, y is the refinement parameter.

ADE-universalization in refined case

- CS partition function was universalized [Mkrtchyan, Veselov '12; Mkrtchyan '13].
- One can extend the universal CS partition function to universal refined CS partition function in case of simply-laced (ADE) root systems:

$$(\alpha, \beta, \gamma, \kappa) \;\to\; (\alpha, y\beta, y\gamma, t = yh^\vee, \kappa) \tag{4}$$

in particular [Krefl, Schwarz '13]:

$$\text{refined } SU(N)\ (A_{N-1})\ \text{CS}: \qquad (-2, 2y, yN, yN, \kappa), \tag{5}$$

$$\text{refined } SO(2N)\ (D_N)\ \text{CS}: \qquad (-2, 4y, 4yN - 4y, 2yN - 2y, \kappa). \tag{6}$$

So there exists ADE-universalization of CS refined CS partition function, that is why it was natural to expect ADE-universalization of other refined quantities, and we got it for Macdonald dimensions in simply-laced case.

ADE-universalization

Different factorization points:

	q^{ρ}	q^{r}
$\chi_\lambda(x)$	$\boxed{D_\lambda}$	$\tilde{D}_\lambda$

	q^{ρ_k}	q^{r_k}
$P_\lambda(x \mid t_{\vec{\alpha}} \mid q, t)$		$\boxed{MD_\lambda}$

$$\rho = r \quad \text{for root systems} \quad A_n, \; D_n, \; E_n \tag{7}$$

ADE-universalization of Macdonald dimensions

$$\boxed{MD_{\text{Adj}} = -\frac{\{t^{\alpha/2+\beta+\gamma}\}\{t^{\alpha+\beta/2+\gamma}\}\{t^{\alpha+\beta+\gamma/2}\}\{t^{\alpha+\beta+\gamma}\}\{q\,t^{\alpha+\beta+\gamma}\}}{\{t^{\alpha/2}\}\{t^{\beta/2}\}\{t^{\gamma/2}\}\{q\,t^{\alpha+\beta+\gamma-1}\}}} \tag{8}$$

Refinement in knot theory

Jones polynomial of knots can be calculated with Euler characteristic of the Khovanov homology of a knot [M. Khovanov '00]

$$J(K,q) = \sum_{i,j} (-1)^i q^j \dim H^{i,j}(K). \tag{9}$$

This expression can be refined in the following way

$$Kh(K,q,t) = \sum_{i,j} t^i q^j \dim H^{i,j}(K), \tag{10}$$

which gives Khovanov polynomial invariant.

Symmetric functions (SF)

$$f_\lambda(x_1, x_2, \dots, x_n): \quad S_n f_\lambda = f_\lambda. \tag{11}$$

S_n — symmetric group, λ — partitions:

$$\lambda = [\lambda_1, \lambda_2, \dots, \lambda_n], \quad \lambda_1 \geq \lambda_2 \geq \dots \geq \lambda_n \geq 0, \quad |\lambda| = \sum \lambda_i. \tag{12}$$

Power sum SF

$$p_k = \sum_{i=1}^{n} x_i^k, \quad p_\lambda = \prod_{i=1}^{l(\lambda)} p_{\lambda_i} \tag{13}$$

Schur SF:

$$\langle p_\lambda(x), p_\mu(x) \rangle = \delta_{\lambda,\mu} z_\lambda \quad \Longrightarrow \quad \langle S_\lambda(x), S_\mu(x) \rangle = \delta_{\lambda,\mu} \tag{14}$$

and the "level" condition:

$$S_\lambda \sim p_\mu \quad \text{if} \quad |\lambda| = |\mu|. \tag{15}$$

For example:

$$S_{[1]} = p_{[1]}, \; S_{[2]} = \frac{1}{2}\left(p_{[2]} + p_{[1,1]}\right), \; S_{[1,1]} = \frac{1}{2}\left(p_{[1,1]} - p_{[2]}\right).$$

Refinement in symmetric functions

$$\langle p_\lambda(x), p_\mu(x)\rangle = \delta_{\lambda,\mu} z_\lambda \tag{16}$$

Refinement in case of different symmetric functions: Schur polynomials, Macdonald polynomials and Hall-Littlewood polynomials

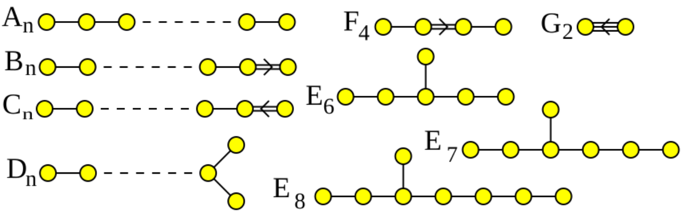
	Schurs	Hall-Littlewoods	Macdonalds
$\langle p_\lambda(x), p_\lambda(x)\rangle$	z_λ	$z_\lambda(t) = z_\lambda \prod_{i=1}^{l(\lambda)} \frac{1}{1-t^{\lambda_i}}$	$z_\lambda(q,t) = z_\lambda \prod_{i=1}^{l(\lambda)} \frac{1-q^{\lambda_i}}{1-t^{\lambda_i}}$
$\sum_\lambda f_\lambda(x) f_\lambda(y)$	$\prod_{i,j} \frac{1}{1-x_i y_j}$	$\prod_{i,j} \frac{1-t x_i y_j}{1-x_i y_j}$	$\prod_{i,j} \frac{(t x_i y_j;q)_\infty}{(x_i y_j;q)_\infty}$

Root systems

R — root system on Euclidean space V:

- $R = R_+ \cup (-R_+)$,
- $\vec{\alpha} \in R$ — roots, $\vec{\alpha}^\vee = \frac{2\vec{\alpha}}{(\vec{\alpha},\vec{\alpha})}$ — coroots,
- $\vec{\alpha} \in R_+$ — positive roots,
- $\vec{\alpha}_i$ — basis of V — simple roots,
- ω_i — fundamental weights: $(\omega_i, \vec{\alpha}_j^\vee) = \delta_{ij}$
- dominant weights: $(\lambda \in V \,|\, (\lambda, \vec{\alpha}^\vee) \in \mathbb{N})$
- W_R — Weyl group — group of reflections

Dynkin diagrams:



A_n, D_n, E_n — **simply-laced** root systems.

Orthogonal Macdonald polynomials associated with root systems

Macdonald polynomials $P_\lambda^{(R,S)}(x \,|\, t_{\vec{\alpha}} \,|\, q, t)$ I.G. Macdonald '87,'00.

- (R, S) — admissible pair, $W_R = W_S$
- $x_1, x_2, \ldots, x_n$
- W_R
- λ — dominant weights
- $\langle P_\lambda, P_\mu \rangle = 0$, $\lambda \neq \mu$

For a reduced root system R there exist a unique family of polynomials P_λ:

$$P_\lambda = m_\lambda + \sum_{\mu<\lambda} c_{\lambda\mu} m_\mu, \tag{17}$$

$$\langle P_\lambda, m_\mu \rangle = 0 \text{ if } \mu < \lambda, \tag{18}$$

which satisfy the orthogonality condition:

$$\langle P_\lambda, P_\mu \rangle = 0, \qquad \lambda \neq \mu. \tag{19}$$

Macdonald polynomials: essential definitions

- **Scalar product**

$$\langle f(x), g(x)\rangle = |W|^{-1}[f(x)g(x^{-1})\Delta]_0, \tag{20}$$

$$\Delta := \prod_{\vec{\alpha}\in R} \frac{\left(t_{2\vec{\alpha}}^{1/2} e^{\vec{\alpha}}; q_{\vec{\alpha}}\right)_\infty}{\left(t_{\vec{\alpha}} t_{2\vec{\alpha}}^{1/2} e^{\alpha}; q_{\vec{\alpha}}\right)_\infty}, \tag{21}$$

the operator $[\dots]_0$ means picking up the constant term of polynomial, $(a;q)_\infty = \prod_{j=0}^{\infty}(1-aq^j)$.

- **Basis**

$$m_\lambda = \sum_{\omega\in W} e^{\omega\lambda}, \tag{22}$$

W — Weyl group

- **Dominance order**

$$\lambda \geq \mu \;\;\longleftrightarrow \lambda - \mu \in Q^+, \tag{23}$$

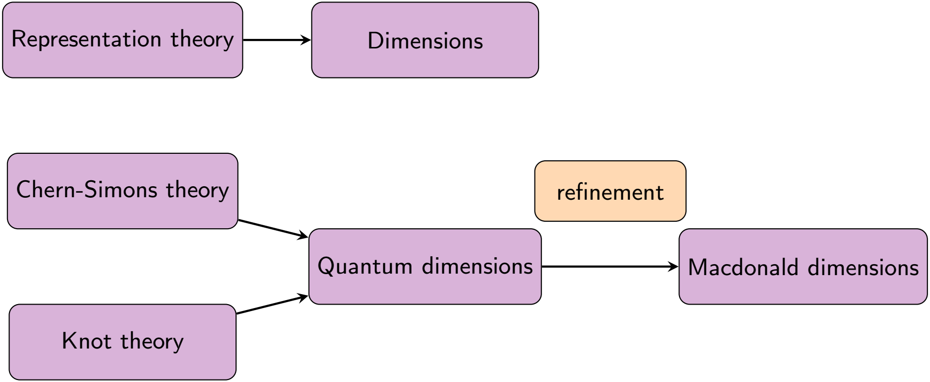
$Q^+ = \mathbb{N}R_+$ — positive cone of positive roots.

Parameters of Macdonald polynomials $P_\lambda^{(R,R)}(x\,|\,t_{\vec\alpha}\,|\,q,t) = P_\lambda^R(x\,|\,t_{\vec\alpha}\,|\,q,t)$

$$q_{\vec\alpha} = q, \quad t_{\vec\alpha} = q^{k_{\vec\alpha}}, \quad k_{\vec\alpha} = k_{\vec\beta} \ \text{ if } \ |\vec\alpha| = |\vec\beta| \tag{24}$$

	algebra	χ_λ	$P_\lambda(x\,|\,t_{\vec\alpha}|q,t)$
A_n	sl_{n+1}	$S_\lambda(x)$	$P_\lambda^{A_n}(x\,|\,q,t)$
B_n	so_{2n+1}	$So_\lambda(x)$	$P_\lambda^{B_n}(x\,|\,t_s\,|\,q,t)$
C_n	sp_{2n+1}	$Sp_\lambda(x)$	$P_\lambda^{B_n}(x\,|\,t_l\,|\,q,t)$
D_n	so_{2n}	$So_\lambda(x)$	$P_\lambda^{D_n}(x\,|\,q,t)$
E_n	e_n		$P_\lambda^{E_n}(x\,|\,q,t)$
F_4	f_4		$P_\lambda^{F_4}(x\,|\,t_s\,|\,q,t)$
G_2	g_2		$P_\lambda^{G_2}(x\,|\,t_3\,|\,q,t)$

General logic



Quantum dimensions

Quantum dimensions emerge in Chern-Simons theory and knot theory. They are:

- expectation values of unknotted Wilson loop in S^3;
- invariants of unknots in knot theory;
- characters of the representations of Lie algebras evaluated at Weyl vector ρ:

$$D_\lambda = \chi_\lambda(q^\rho) = \prod_{\vec{\alpha}>0} \frac{[(\vec{\alpha}, \lambda+\rho)]_q}{[(\vec{\alpha}, \rho)]_q}, \quad [n]_q = \frac{q^n - q^{-n}}{q - q^{-1}} \tag{25}$$

Universal quantum dimensions

Different evaluation points:

$$\rho = \frac{1}{2}\sum_{\vec{\alpha}>0}\vec{\alpha}, \qquad r = \rho^{\vee} = \frac{1}{2}\sum_{\vec{\alpha}>0}\vec{\alpha}^{\vee} = \frac{1}{2}\sum_{\vec{\alpha}>0}\frac{2\vec{\alpha}}{(\vec{\alpha},\vec{\alpha})} \neq \frac{1}{2}\frac{2\rho}{(\rho,\rho)} \tag{26}$$

$$\boxed{D_\lambda = \chi_\lambda(q^\rho)}, \qquad \tilde{D}_\lambda = \chi_\lambda(q^r) \tag{27}$$

$\rho = r$ and $D_\lambda = \tilde{D}_\lambda$ for simply-laced algebras (ADE).

Quantum dimensions of adjoint representation in Vogel's parameters (universal quantum dimensions) [Mkrtchyan, Veselov '12]

$$D_{\text{Adj}} = \frac{[(\alpha-2t)/2]_q}{[\alpha/2]_q}\frac{[(\beta-2t)/2]_q}{[\beta/2]_q}\frac{[(\gamma-2t)/2]_q}{[\gamma/2]_q}, \quad \alpha+\beta+\gamma = t \tag{28}$$

Macdonald dimensions

k—Weyl vectors:

$$\rho_k = \frac{1}{2}\sum_{\vec{\alpha}>0} k_{\vec{\alpha}}\,\vec{\alpha}, \qquad r_k = \frac{1}{2}\sum_{\vec{\alpha}>0} k_{\vec{\alpha}}\,\vec{\alpha}^{\vee}, \qquad k_{\vec{\alpha}} = k_{\vec{\beta}} \ \text{ if } \ |\vec{\alpha}| = |\vec{\beta}| \tag{29}$$

$$\tag{30}$$

Factorization of Macdonald polynomials at the point q^{r_k}:

$$P^R_\lambda(x = q^{r_k} \mid t_{\vec{\alpha}} \mid q, t) = \prod_{\vec{\alpha}>0} \prod_{j=0}^{(\vec{\alpha}^{\vee},\lambda)-1} \frac{\{q^j\, t_{\vec{\alpha}}\, q^{(\rho_k,\vec{\alpha}^{\vee})}\}}{\{q^j\, q^{(\rho_k,\vec{\alpha}^{\vee})}\}}, \quad \{n\} = n - n^{-1} \tag{31}$$

Macdonald dimensions — Macdonald polynomials evaluated at $x = q^{r_k}$

$$MD^R_\lambda = P^R_\lambda(q^{r_k} \mid t_{\vec{\alpha}} \mid q, t) \tag{32}$$

Macdonald polynomials at the point $x = q^{\rho_k}$ do not factorize.

Universal Macdonald dimensions for simply-laced root systems

We present universalized Macdonald dimensions for simply-laced (ADE) root systems:

$$MD_{\mathrm{Adj}} = -\frac{\{t^{\alpha+\beta/2+\gamma}\}\{t^{\alpha+\beta+\gamma/2}\}\{t^{\alpha+\beta+\gamma}\}\{q\,t^{\alpha+\beta+\gamma}\}}{\{t^{\alpha/2}\}\{t^{\beta/2}\}\{t^{\gamma/2}\}\{q\,t^{\alpha+\beta+\gamma-1}\}} \tag{33}$$

and in more symmetric form:

$$MD_{\mathrm{Adj}} = -\frac{\{t^{\alpha/2+\beta+\gamma}\}\{t^{\alpha+\beta/2+\gamma}\}\{t^{\alpha+\beta+\gamma/2}\}\{t^{\alpha+\beta+\gamma}\}\{q\,t^{\alpha+\beta+\gamma}\}}{\{t^{\alpha/2}\}\{t^{\beta/2}\}\{t^{\gamma/2}\}\{q\,t^{\alpha+\beta+\gamma-1}\}} \tag{34}$$

Different factorization points:

	q^{ρ}	q^{r}
$\chi_\lambda(x)$	$\boxed{D_\lambda}$	$\tilde{D}_\lambda$

	q^{ρ_k}	q^{r_k}
$P_\lambda(x \mid t_{\vec{\alpha}} \mid q, t)$		$\boxed{MD_\lambda}$

Conclusion

- In case of Macdonald dimensions we get only ADE-universalization because in this case two Weyl vectors coincide
$$\rho_k = r_k. \tag{35}$$
- Are there other systems of polynomials that factorize at q^{ρ_k}?
- Are there any other universal quantities in refined case?

Thank you for your attention!