The Variety of Higher-Spin Gauge Theories and BRST Formalism

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Problems of the Modern Mathematical Physics

JINR, Dubna, February 14, 2025

Fronsdal Fields

All $m = 0$ HS fields are gauge fields C.Fronsdal 1978

$\varphi_{a_1 \ldots a_s}(x)$ is a rank s symmetric tensor obeying $\varphi^{c}{}_{c}{}^{b}{}_{b a_5 \ldots a_s} = 0$

$$\delta \varphi_{a_1 \ldots a_s}(x) = \partial_{(a_1} \varepsilon_{a_2 \ldots a_s)}(x)\,, \qquad \varepsilon^{b}{}_{b a_3 \ldots a_{s-1}} = 0$$

$$S = \frac{1}{2} \int_{M^d} \left(\varphi^{a_1 \ldots a_s} \Box \varphi_{a_1 \ldots a_s}(\varphi) + \ldots \right)$$

HS gauge theory: theory of maximal HS symmetries that cannot result from spontaneous breakdown of a larger symmetry:

HS symmetries are manifest at ultrahigh energies above any scale including Planck scale

- HS gauge theory should capture effects of Quantum Gravity: restrictive HS symmetry versus unavailable experimental tests
- Lower-spin theories as low-energy limits of HS theory: lower-spin symmetries: subalgebras of HS symmetry
- String Theory as spontaneously broken HS theory?! $(s > 2, m > 0)$

No-go and the Role of $(A)dS$

No HS symmetries in Minkowski space

Weinberg, Coleman-Mandula, Aragone and Deser

Green light: AdS background with $\Lambda \neq 0$ Fradkin, MV, 1987

In agreement with no-go statements the limit $\Lambda \to 0$ is singular

HS Symmetries Versus Riemann Geometry

HS symmetries do not commute with space-time symmetries

$$[T^{a}, T^{HS}] = T^{HS}, \qquad [T^{ab}, T^{HS}] = T^{HS}$$

HS transformations map gravitational fields (metric) to HS fields

Consequence:

Riemann geometry is not appropriate for HS theory:

concept of local event may become illusive!

Related feature: HS interactions contain higher derivatives

Bengtsson, Bengtsson, Brink (1983), Berends, Burgers and H. Van Dam (1984), (1985), Fradkin, MV; Metsaev,...

How non-local HS gauge theory is?!

Types of Unitary HS Gauge Theories

- Space-time dimension d
- Inner (YM) symmetries and SUSY
- Description type: tensor type in any d or spinor type in $d = 3, 4, 5$
- Coxeter type HS symmetry C with $C = Z_2$ for usual HS theories
- Multiparticle extensions

Reductions

- Chern-Simons type with no matter fields:

  Maurier-Cartan equations for HS connection

$$\mathrm{d}\omega + \omega * \omega = 0$$

  No local propagating degrees of freedom

  Nontriviality due to boundary conditions

- Self-dual HS theories in $d = 4$ $\sim$ chiral to make them looking new

  Much simpler than the full HS theory:

  no current interactions: $T_{nm} = 0$

  non-unitary: complex fields in Minkowski signature

- Free Theory

Variety of Formalisms

- **Covariant tensor formalism** Dirac, Pauli, Schwinger, Fronsdal, Berends, ...
- **Light-front formalism** Bengtsson, Bengtsson, Brink; Metsaev, ...
- **Superspace for SUSY models** Buchbinder, Ivanov, Kuzenko, Zaigraev ...
- **BRST covariant formalism** Buchbinder, Pashnev, Bengtsson ...
- **Frame-like formalism** Deser, Lebedev group, Zinoviev ...
- **Unfolded dynamics** Lebedev group, Iazeolla, Sundell ...

Unfolded Dynamics

First-order form of differential equations

$$\dot{q}^i(t) = \varphi^i(q(t)) \qquad \text{initial values:} \quad q^i(t_0)$$

Unfolded dynamics: multidimensional generalization

$$\frac{\partial}{\partial t} \to \mathsf{d}\,, \qquad q^i(t) \to W^\Omega(x) = \theta^{\underline{n}_1} \dots \theta^{\underline{n}_p} W^\Omega_{\underline{n}_1 \dots \underline{n}_p}(x)$$

$$\mathsf{d}\mathbf{W}^{\boldsymbol{\Omega}}(\mathbf{x}) = \mathbf{G}^{\boldsymbol{\Omega}}(\mathbf{W}(\mathbf{x}))\,, \qquad \mathsf{d} = \theta^{\underline{n}}\partial_{\underline{n}} \qquad \textbf{MV} \quad \textbf{1988}$$

$G^\Omega(W)$: **function of "supercoordinates"** W^Ω

$$G^\Omega(W) = \sum_{n=1}^{\infty} f^\Omega{}_{\Phi_1 \dots \Phi_{\underline{n}}} W^{\Phi_1} \dots W^{\Phi_{\underline{n}}}$$

Covariant first-order differential equations

$d > 1$: **Compatibility conditions**

$$G^\Phi(W) \frac{\partial G^\Omega(W)}{\partial W^\Phi} = 0$$

Universal Equations

Unfolded equations are universal if the compatibility condition holds independentlay of space-time dimension, not using that any p-form with $p > d$ is zero. In this case equations acquire the form

$$\mathrm{d}F(W(x)) = \mathcal{Q}(F(W(x)))\,, \qquad \mathcal{Q} := G^{\Omega}(W)\frac{\partial}{\partial W^{\Omega}}$$

$\mathcal{Q}$ is a homological vector field in the target space with coordinates W^{Ω} obeying the nilpotency condition

$$\mathcal{Q}^2 = 0 \qquad \text{compatibility condition on} \quad G^{\Omega}(W)$$

Analogy with Hamiltonian formalism

$$\dot{F}(q(t)) = \{H\,, F(q(t))\}\,, \qquad q \quad \text{are phase space coordinates}$$

Related concepts: L_∞, A_∞, $\mathcal{Q}$-manifolds, etc

Properties

- General applicability
- Manifest (HS) gauge invariance
- Invariance under diffeomorphisms
- Clear group-theoretical interpretation of fields and equations in terms of modules and cohomology of the symmetry algebra $\mathfrak{s}$

  Background fields: flat connection of $\mathfrak{s}$

  Fields: $\mathfrak{s}$-modules

  Equations: covariant constancy conditions
- Local degrees of freedom are in zero-forms $C^i(x_0)$ at any $x = x_0$

  (as $q(t_0)$) infinite-dimensional module dual to the space of single-particle states: $C^i(x_0)$ moduli of solutions

HS Multiplets and Vertices

HS fields are described by the generating functions of certain auxiliary variables Y: one-forms $\omega(Y|x)$ and zero-forms $C(Y|x)$

The problem: consistent non-linear corrections 1988 in the local frame

$$\mathsf{d}_x\omega = -\omega * \omega + \Upsilon(\omega,\omega,C) + \Upsilon(\omega,\omega,C,C) + \ldots\,,$$

$$\mathsf{d}_x C = -[\omega, C]_* + \Upsilon(\omega,C,C) + \ldots$$

Spin-local vertices are local for any finite subset of fields

Gelfond, MV 2018

Projectively compact spin-local vertices are spin-local both in the space-time coordinates and in the Y coordinates 2023

Spinor Versus Tensor Formulations

Spinor model in $d = 4$: auxiliary variables $Y_A = (y_\alpha, \bar{y}_{\dot{\alpha}})$

$A = 1 - 4$ spinor index

Tensor model in any dimension d: auxiliary variables Y_i^A

$A = 0, \ldots, d$: AdS_d vector index $i = 1, 2$: $sp(2)$ vector index

$$[Y_i^A, Y_j^B]_* = \eta^{AB} \epsilon_{ij}$$

$$T^{AB} = -T^{BA} := \frac{1}{2} Y^{iA} Y_i^B \; : \qquad o(d-1, 2)$$

$$t_{ij} = t_{ji} := Y_i^A Y_{jA} \; : \qquad sp(2)$$

Space of Functions

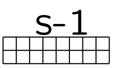
Components of the fields

$$\omega(Y|x) = \sum_{l=0}^{\infty} \omega_{A_1 \dots A_l\,, B_1 \dots B_l}(x) Y_1^{A_1} \dots Y_1^{A_l} Y_2^{B_1} \dots Y_2^{B_l}$$

$sp(2)$ **invariance**

$$D(t_{ij}) = 0 \; : \qquad [\omega\,, t_{ij}] = 0$$

implies that they are valued in all two-row rectangular Young diagrams



of $o(d-1,2)$

The components are made traceless by the ideal factorisation

$$t_{ij} * g^{ij} = g^{ij} * t_{ij} \sim 0$$

In the new approach both conditions are implemented by the

BRST formalism

Adjoint BRST Approach

Let T_α be generators of a Lie (super)algebra g

$$[T_\alpha\,, T_\beta]_\pm = f^\gamma_{\alpha\beta} T_\gamma$$

BRST operator

$$Q := c^\alpha T_\alpha - \frac{1}{2} f^\gamma_{\alpha\beta} c^\alpha c^\beta b_\gamma \,, \qquad Q^2 = 0$$

$$[c^\alpha\,, b_\beta]_\pm = \delta^\alpha_\beta\,, \qquad [c^\alpha\,, c^\beta]_\pm = 0\,, \qquad [b_\alpha\,, b_\beta]_\pm = 0$$

Q usually acts on a left module V generated from the vacuum $|0\rangle$

$$b_\alpha|0\rangle = 0\,.$$

Elements $v \in V$ are b–independent

$$v = v(c)|0\rangle$$

In the construction of this talk Q acts via graded commutator

$$Q(a) := [Q\,, a]_\pm\,, \qquad \forall a \in A\,.$$

Factorisation via Gauge Transformation

Important novelty: mixing between b_α-dependent and b_α-independent sectors: let

$$\xi = \xi^\beta b_\beta$$

For c^α, b_α-independent $\xi^\beta \in A$ the transformation

$$\delta a = Q(\xi)$$

in the c^α , b_β-independent sector has the form

$$Q(\xi)\Big|_{b=0} := \{T_\beta\,, \xi^\beta\}\,.$$

That just describes the $sp(2)$ ideal factorisation.

In the sector linear in C^α the Q invariance condition

$$Q(a) = 0$$

implies invariance of a under the adjoint action of g.

Nonlinear $4d$ System

How to find nonlinear corrections to HS equations? The efficient trick MV 1992 reduces the problem to De Rham cohomology with respect to additional spinor variables $Z^A = (z^\alpha, \bar{z}^{\dot{\alpha}})$ in presence of Klein operators K

$$\omega(Y;K|x) \longrightarrow W(Z;Y;K|x)\,, \qquad C(Y;K|x) \longrightarrow B(Z;Y;K|x)\,, \quad Y^A = (y^\alpha, \bar{y}^{\dot{\alpha}})$$

Nonlinear HS Equations

$$\begin{cases} \mathsf{d}_x \mathcal{W} + \mathcal{W} \star \mathcal{W} = i(\theta^A \theta_A + (\eta\gamma + \bar{\eta}\bar{\gamma}) \star B) \\ \mathsf{d}_x B + \mathcal{W} \star B - B \star \mathcal{W} = 0 \end{cases}$$

determine Z_A-dependence in terms of "initial data" $\omega(Y;K|x)$ and $C(Y;K|x)$

$S(Z;Y;K|x) = \theta^A S_A(Z;Y;K|x)$ is a connection along Z^A $(\theta^A \equiv dZ^A)$

η: complex coupling constant

$$\gamma := \theta^\alpha \theta_\alpha k\kappa\,, \qquad \bar{\gamma} := \bar{\theta}^{\dot{\alpha}} \bar{\theta}_{\dot{\alpha}} \bar{k}\bar{\kappa}$$

Klein operators $K = (k, \bar{k})$ generate chirality automorphisms

$$kf(A) = f(\tilde{A})k\,, \quad A = (a_\alpha\,, \bar{a}_{\dot{\alpha}}) : \quad \tilde{A} = (-a_\alpha\,, \bar{a}_{\dot{\alpha}})$$

Inner Klein operators: $\kappa = \exp i z_\alpha y^\alpha\,, \bar{\kappa} = \exp i \bar{z}_{\dot{\alpha}} \bar{y}^{\dot{\alpha}}\,, \kappa \star f = \tilde{f} \star \kappa\,, \qquad \kappa \star \kappa = 1$

Extensions and Reductions

A_∞ **structure: Extension of** $W, B \to W^i{}_j\,, B^i{}_j \in A$ **any associative algebra Various YM groups are possible. There are always spin 2 and spin 0 fields in the singlet representation**

(Anti)self-dual reduction $(\eta = 0)$ $\bar{\eta} = 0$ **1992**

In this case only self-dual or anti self-dual components of the HS field strengths couple with HS connections. (Anti)self-dual theory is far simpler than the full one since it is free of current interactions.
Recall that the stress tensor for spin-one fields is

$$T_{\alpha\beta,\dot{\alpha}\dot{\beta}} = C_{\alpha\beta}\bar{C}_{\dot{\alpha}\dot{\beta}}$$

If $C_{\alpha\beta}$ **or** $\bar{C}_{\dot{\alpha}\dot{\beta}}$ **decouples, there is no stress tensor contribution.**
Self-dual HS theory often called chiral is non-unitary (complex) since η **and** $\bar{\eta}$ **are complex conjugated in Minkowski signature**
Reduction with $\eta = \bar{\eta} = 0$ **is even nicer. This is the free HS theory.**

HS Theories in any d within BRST Formalism

The BRST operator of $osp(1,2)$ **is**

$$Q := c^{ij}\tau_{ij} + c^i\tau_i - (c^i{}_n c^{jn} + \frac{1}{4}c^i c^j)b_{ij} - 2c^{ij}c_i b_j$$

$osp(1,2)$ **ghosts** c^{ij}, c^i, b_{ij} **and** b_j

$$\{c^{ij}, b_{nm}\} = \delta^i_n\delta^j_m + \delta^i_m\delta^j_n\,, \qquad [c^i, b_j] = \delta^i_j\,.$$

So defined BRST charge allows one to introduce the total differential

$$\mathsf{d} := \mathsf{d}_Z + \mathsf{d}_\psi + Q + \dots\,, \qquad \mathsf{d}_Z := \theta^i_A\frac{\partial}{\partial Z^i_A}\,, \qquad \mathsf{d}_\psi := \lambda^A\frac{\partial}{\partial \psi_A}$$

With the collective variables $\mathcal{Y} := \{\theta^A_i, \lambda^A, Z^A_i, Y^A_i, \psi^A, K, c^{ij}, b_{ij}\}$ **HS equations are formulated in terms of the fields** $\mathcal{A} = (\mathcal{W}, \mathcal{B})$

$$\mathcal{A} = \mathcal{A}_{11}(\mathcal{Y}) * \Pi_1 * F + \mathcal{A}_{22}(\mathcal{Y}; \phi, c^i, b_i) * \Pi_2 + \mathcal{A}_{12}(\mathcal{Y}; \phi_+, c^i, b_i) * \delta(\phi_-) * \bar{\Theta}$$
$$+\Theta * \delta(\phi_+) * \mathcal{A}_{21}(\mathcal{Y}; \phi_-, c^i, b_i)\,,$$

$$\{\Theta, \bar{\Theta}\} = 1\,, \qquad \Pi_1 = \Theta\bar{\Theta}\,, \qquad \Pi_2 = \bar{\Theta}\Theta$$

A_{11} : $A-$**model**, A_{22} : $B-$**model**, A_{12} **and** A_{21}-**fermions**

HS Equations

Nonlinear HS equations take the form analogous to that of $4d$ HS theory

$$\mathcal{W}*\mathcal{W}=\frac{1}{2}(\theta^i_A\theta^A_i+4g\Lambda^{-1}\theta^i\theta_i*K*\mathcal{K}*F(\mathcal{B})),\qquad [\mathcal{W},\mathcal{B}]_*=0$$

with

$$u_i:=V_AU^A_i\,,\qquad V^AV_A=1\,,\qquad Kf(U)=f(\tilde{U})K\,,\qquad \tilde{U}^A:=U^A-2V^AV_BU^B$$

$$f(Y,\phi)*g(Y,\phi)=(2\pi)^{-2(M+2)}\int d^{2(M+2)}Sd^{2(M+2)}Td^{M+2}\alpha d^{M+2}\beta$$
$$\exp(2(\alpha^A\beta_A-S^A_jT^j_A))f(Z+S,Y+S,\psi+\alpha,\phi+\alpha)g(Z-T,Y+T,\psi-\beta,\phi+\beta)$$

Defining nonzero Weyl-Clifford commutation relations

$$[Y^A_i,Y^B_j]_*=\varepsilon_{ij}\eta^{AB}\,,\qquad [Z^A_i,Z^B_j]_*=-\varepsilon_{ij}\eta^{AB}\,,$$

$$\{\phi^A,\phi^B\}_*=\eta^{AB}\,,\qquad \{\psi^A,\psi^B\}_*=-\eta^{AB}\,.$$

Gauge Transformations

$$\delta\mathcal{W} = [\epsilon, \mathcal{W}]_* \,, \qquad \delta\mathcal{B} = [\epsilon\,, \mathcal{B}]_* \,.$$

For

$$\epsilon = \varepsilon + \xi^{ij}_{\mathcal{W}} b_{ij}$$

with c, b-independent parameters ε and $\xi^{ij}_{\mathcal{W}}$ reproduce usual HS gauge transformations and the factorization transformations, that factor out terms proportional to τ_{ij} in W.

Gauge symmetry responsible for the ideal factorization in the $\mathcal{B}$-sector

$$\delta\mathcal{B} = [\mathcal{W}\,, \xi_{\mathcal{B}}]_* \,, \qquad \delta\mathcal{W} = 2g\Lambda^{-1}\gamma * \xi_{\mathcal{B}} \,, \qquad \xi_{\mathcal{B}} = \xi^{ij}_{\mathcal{B}} b_{ij}$$

makes sense in the BRST-extended equations due to the Q term in $\mathcal{W}$.

Conjecture: The $\xi_{\mathcal{W},\mathcal{B}}$ gauge symmetries are responsible for nontrivial coupling constants (vertices) in the theory provided that $\xi_{\mathcal{W},\mathcal{B}}$ are demanded to be projectively-compact spin-local

Metsaev's Vertex Classification

Metsaev (2007) has shown that at $d > 4$ there exist many independent HS vertices (conserved currents) with

$$s_1 + s_2 + s_3 - 2s_{min} \leq N^{max}_{der} \leq s_1 + s_2 + s_3$$

This suggests that there should be two coupling constants in $d = 4$ and infinitely many independent coupling constants in $d > 4$ HS theory

This counting matches the results obtained recently with Yuri Tatarenko 2405.02452 for the $d = 4$ theory

But where are the coupling constants of the HS theories at $d > 4$?

Nontrivial coupling constants are conjectured to result from the locality restrictions on the ideal factorisation parameters in the BRST extended version of the A-model equations and their SHS extension.

Conclusion

New supersymmetric HS gauge theory in any dimension is constructed that unifies bosonic A and B models

A conjecture on the origin of the infinite number of independent coupling constants associated with the vertices classified by Metsaev from the relaxed ideal factorisation conditions is put forward

A new construction based on the BRST technique is developed that greatly simplifies the form of the equations and can be analysed within the differential homotopy approach

The BRST technique leads to the on-shell version
not to the off-shell?!

Consequences

Analysis of HS gauge theory has a potential to affect the paradigm of holographic corresondence replacing the gauge-gravity correspondence by the conformal gravity - gravity correspondence.

The BRST extension of the HS equations has deep analogy with the BRST description of String Theory: analogy between $sp(2) \sim sl_2$ and Virasoro algebra has been noticed just after the original HS theory in d dimensions was proposed in 2003 .

Having developed the BRST formalism gives a hope to find a direct relation of the HS theory with String Theory

Q: $2d$ conformal field theory