Supersymmetric 1D sigma models on coadjoint orbits

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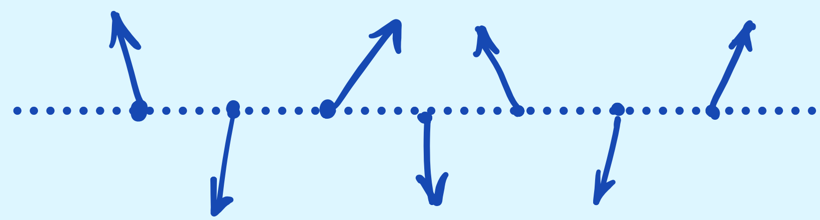
PMMP'25, Dubna, 11 Feb 2025

Russian Science Foundation
Grant № 22-72-10122

1. Some history & Geometry

A bit of history

SU(2) Spin chains



classical

$$H = \sum_{i\in\mathbb{Z}} \alpha\,(\vec{S}_i, \vec{S}_{i+1}) + \beta\,(\vec{S}_i, \vec{S}_{i+1})^2 + \dots$$

quantum

$$(\vec{S}, \vec{T}) \equiv \sum_{\alpha=1}^{3} S^\alpha \otimes T^\alpha$$

su_2 generators in some representation

$1970^{\underline{s}}$ Integrable spin chains

Baxter Yang
Faddeev Kulish
Sklyanin Takhtajan

spin-$\frac{1}{2}$, $\beta=0$: Heisenberg Bethe '1931

spin-1, $\beta=-\alpha$: Takhtajan-Babujian '1982

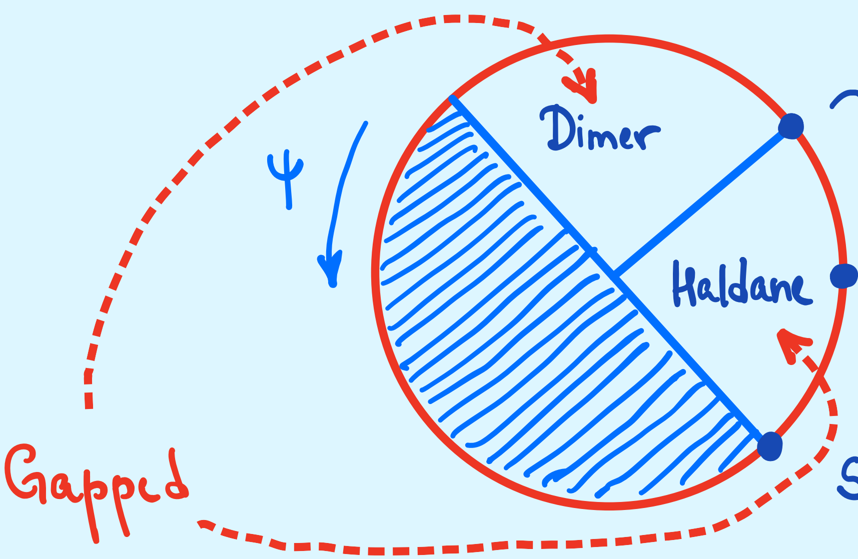
Gapped or gapless?

- The above two spin chains soluble by Bethe ansatz are gapless

- So maybe always so?

NO! Haldane '1983 Mapping to a $S^2 \simeq \mathbb{CP}^1$ sigma model

"Phase diagram" $\frac{\beta}{\alpha} = -\mathrm{tg}\,\psi$



~ Takhtajan-Babujian (spin-1 integrable)

Spin-1 Heisenberg Affleck '1986

SU(3) fundamental-fund. integrable

Haldane meets symplectic geometry

Path integral for spin chain: $\mathcal{Z} \equiv \int D\varphi \; e^{-S[\varphi]}$

$S[\varphi]$ = classical action

geometric quantization / coherent states

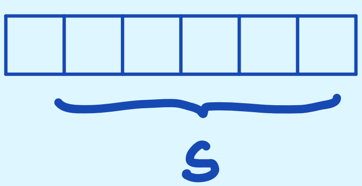
Berezin '1975

Perelomov '1977

$SU(2)$:

$\varphi \in S^2 \simeq \mathbb{CP}^1 \quad \begin{pmatrix} z_1 \\ z_2 \end{pmatrix}$

One spin $\frac{s}{2}$



$$S_{classical} = \int dt \; \frac{is}{2} \frac{\sum_i \bar{z}_i \dot{z}_i - \dot{\bar{z}}_i z_i}{\sum_j \bar{z}_j z_j}$$

A

$dA = p\, \Omega_{FS}$

Two spins

$V_s := \underbrace{\boxed{}\boxed{}\boxed{}\boxed{}\boxed{}\boxed{}}_{s/2}$ and $\underbrace{\boxed{}\boxed{}\boxed{}\boxed{}\boxed{}\boxed{}}_{s/2}$

$\begin{pmatrix} z_1^{(1)} \\ z_2^{(1)} \end{pmatrix}$ $\qquad$ $\begin{pmatrix} z_1^{(2)} \\ z_2^{(2)} \end{pmatrix}$

$$\tilde{S}_{classical} = S^{(1\ spin)} + S^{(2\ spin)} + S^{(int)}$$

spin-spin coupling

$$H = \sum_{\alpha=1}^{3} S_1^{\alpha} \otimes S_2^{\alpha} \in \mathrm{End}\,(V_s \otimes V_s) \qquad \text{quantum}$$

$\downarrow$

$$H_{classical} = s^2\, \vec{n}_1 \vec{n}_2 = s^2 \left(2\, \frac{\left| z^{(1)\dagger} \cdot z^{(2)} \right|^2}{|z^{(1)}|^2\, |z^{(2)}|^2} - 1 \right)$$

Antiferromagnetic \ Néel vacuum

$$\mathrm{Min}\left(H_{\text{classical}}\right): \qquad \vec{n}_2 = -\vec{n}_1$$

$$\downarrow \qquad\qquad z^{(1)\dagger} \circ z^{(2)} = 0$$

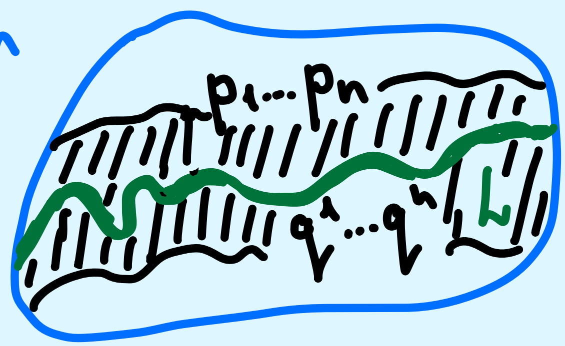
$$S^2 \hookrightarrow S^2 \times S^2$$

Key point: embedding is Lagrangian

$$\Omega_{FS}^{(1)} + \Omega_{FS}^{(2)} \Big|_{S^2} = 0$$

'Weinstein 1971: In a neighborhood of $L \subset (N^{2n}, \Omega)$

N^{2n}



$$\Omega = \sum_{i=1}^{n} dp_i \wedge dq^i$$

The sigma model

Near L : $H_{classical} = \text{const.} + s^2 \frac{1}{2} g^{ij} p_i p_j + s^2 O(p^3)$

$$\Rightarrow S_{classical} = \int dt \left(s \sum p_i \dot{q}^i - s^2 \frac{1}{2} g^{ij}(q) p_i p_j + s^2 O(p^3) \right)$$

Limit $p \to \frac{p}{s}$, $s \to \infty$ $\Rightarrow$ sigma model

$$\boxed{\begin{aligned} & S_{classical} \longmapsto \int dt \; \frac{1}{2} g_{ij}(q) \dot{q}^i \dot{q}^j \\ & \text{Metric} \qquad g_{ij} = \Omega_{ik} \left(\frac{\partial^2 H_{classical}}{\partial x^2} \right)^{-1}_{km} \Omega_{mj} \end{aligned}}$$

'DB 2012

Back to spin chains

Generalize $\mathbb{CP}^1 \hookrightarrow \mathbb{CP}^1 \times \mathbb{CP}^1$ $\quad z^{(1)+} \circ z^{(2)} = 0$

DB '2011

$$\mathcal{F}_3 = \frac{U(3)}{U(1)^3} \hookrightarrow \mathbb{CP}^2 \times \mathbb{CP}^2 \times \mathbb{CP}^2$$

Flag manifold

$$H_{\text{classical}} = \sum_{A<B} d^2_{AB} \frac{|z^{(A)+} \circ z^{(B)}|^2}{|z^{(A)}|^2 \, |z^{(B)}|^2}$$

Minima: $\quad z^{(1)+} \circ z^{(2)} = z^{(1)+} \circ z^{(3)} = z^{(2)+} \circ z^{(3)} = 0$

SU(n) spin chains → generalized Haldane conjectures

'Affleck et al. 2017 'Affleck, DB, Wamer 2022

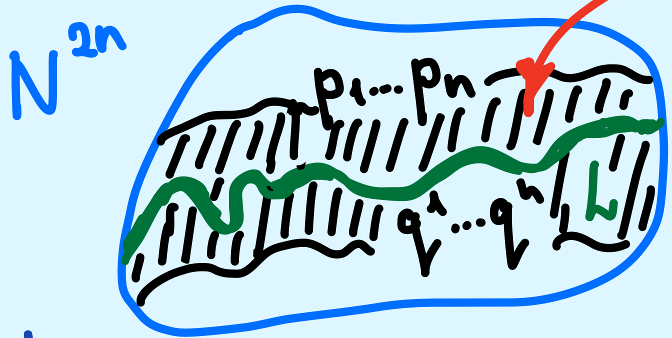
Field theory avatars: discrete 't Hooft anomalies

'Tanizaki, Sulejmanpasic 2018
'Ohmori, Seiberg, Shao

Some more geometry

DB, Kuzovchikov 2024

Recall



How big is this neighborhood?

$\mathbb{CP}^1 \hookrightarrow \mathbb{CP}^1 \times \mathbb{CP}^1$

Consider the divizor $D \subset \mathbb{CP}^1 \times \mathbb{CP}^1 : \{\vec{n}_1 = \vec{n}_2\}$

Then $\left(\mathbb{CP}^1 \times \mathbb{CP}^1 / D,\ s\left(\Omega_{FS}^{(1)} + \Omega_{FS}^{(2)}\right)\right)$

$\simeq$ open subset in $T^*\mathbb{CP}^1$

$\uparrow$ symplectomorphic

$\lim\limits_{s \to \infty} \simeq T^*\mathbb{CP}^1, \quad H_{classical} = H_{geodesic}$

Construct the map: $\mathbb{CP}^1 \times \mathbb{CP}^1 / D \longmapsto T^*\mathbb{CP}^1$

$$\begin{array}{ccc} \uparrow & & \cong \\ (\vec{n}_1, \vec{n}_2) & & \{\vec{n}^2 = 1, \vec{p}\vec{n} = 0\} \subset \mathbb{R}^6 \end{array}$$

$$\boxed{\vec{n} = \frac{1}{\sqrt{2(1-\vec{n}_1\vec{n}_2)}}(\vec{n}_1 - \vec{n}_2); \quad \vec{p} = \frac{1}{\sqrt{2(1-\vec{n}_1\vec{n}_2)}}\vec{n}_1 \times \vec{n}_2 \qquad H = \vec{p}^{\,2} = \frac{1+\vec{n}_1\vec{n}_2}{2} \leftarrow \text{'spin chain'} \qquad \text{geodesic flow}}$$

Generalization: $\mathcal{N} := \underbrace{\mathbb{CP}^{n-1} \times \cdots \times \mathbb{CP}^{n-1}}_{n \text{ times}}$, divisor

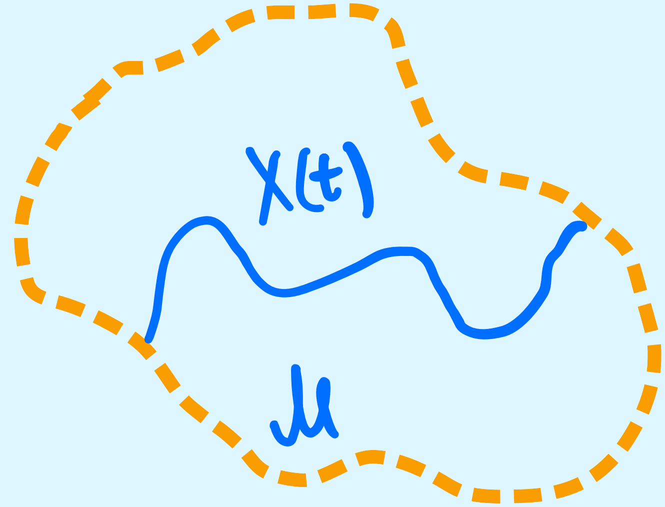
$\mathbb{CP}^{n-1} \sim z^{(1)}$, $\mathbb{CP}^{n-1} \sim z^{(n)}$

$D = \{\mathrm{Det}(z^{(1)}, \ldots, z^{(n)}) = 0\}$

$\mathcal{N}/D \simeq$ open subset in $T^*\mathcal{F}_n$

2. 1D sigma models, Representations & SUSY extensions

1D sigma models



$$(\mathcal{M} \mid G, A)$$

Manifold Metric Gauge field

$$L = \int dt\, \frac{1}{2} G_{IJ}(x)\, \dot{x}^I \dot{x}^J - \int dt\, A_I(x)\, \dot{x}^I + \text{SUSY}$$

Classical: (Magnetic) geodesics

Novikov, Schmelzer '1981
Kordyukov, Taimanov '2020

Quantum: (Generalized) Laplacians Δ

Bochner, Dolbeault, de Rham

Integrable?

compact (semi) simple

$\mathcal{M}$ = Homogeneous G/H

- Symmetric space: YES

  Geodesic $g(t) = e^{At} g(0)$, $A \in \mathrm{Lie}(G)$.

- Non-symmetric: YES for NORMAL metric

  $\mathrm{Lie}(G) = \mathfrak{h} \oplus \mathfrak{m}$, $\mathfrak{m} = \mathfrak{h}^{\perp}$

  $g \in G \Rightarrow ds^2 = -\langle (g^{-1}dg)_{\mathfrak{m}}, (g^{-1}dg)_{\mathfrak{m}} \rangle$

- In general, open question

Manakov '1976

Miščenko, Fomenko '1978

Thimm '1981

Bolsinov Jovanović '2004

Arvanitoyeorgos Souris '2016

Target space

$\mathcal{M}$ = Homogeneous $\dfrac{SU(n)}{H}$

symplectic
complex
Kähler

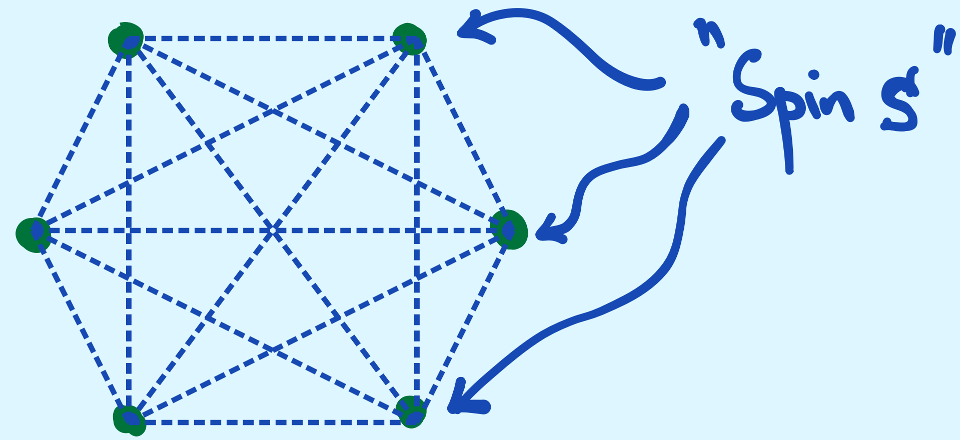
(Co) adjoint orbit $\quad g\Lambda g^{-1} \quad \Lambda \in su_n, \; g \in SU(n)$

$$\Lambda = \begin{pmatrix} \lambda_1 & & & & & & & & \\ & \ddots & & & & & & & \\ & & \lambda_1 & & & & & & \\ & & & \lambda_2 & & & & & \\ & & & & \ddots & & & & \\ & & & & & \lambda_2 & & & \\ & & & & & & \ddots & & \\ & & & & & & & \lambda_m & \\ & & & & & & & & \ddots \\ & & & & & & & & & \lambda_m \end{pmatrix} \begin{array}{l} n_1 \\ n_2 \\ n_m \end{array}$$

$$\boxed{\dfrac{SU(n)}{S(U(n_1)\times\cdots\times U(n_m))}}$$

Flag manifold

SUSY Spin chain



$$\Big\downarrow H_s$$

Exact truncation of Δ

* Oscillator variables | Nicolai '1976-77

* Witten index of spin chain | Witten '1982, Alvarez-Gaume '1983

$\rightsquigarrow$ index theorems on flags

$\mathbb{CP}^1 \simeq S^2$

Recall $\mathbb{CP}^1 \hookrightarrow \mathbb{CP}^1 \times \mathbb{CP}^1$

Spectrum: $E_\ell = \ell(\ell+1)$, $\ell = 0, 1, 2, \dots$

Y_ℓ^m spherical harmonics

$$V_{2\ell} := \underbrace{\boxed{}\boxed{}\boxed{}\boxed{}\boxed{}\boxed{}}_{2\ell}$$

su_2 representation

Truncate to first '$s+1$' harmonics $\ell = 0, 1, \dots, s$

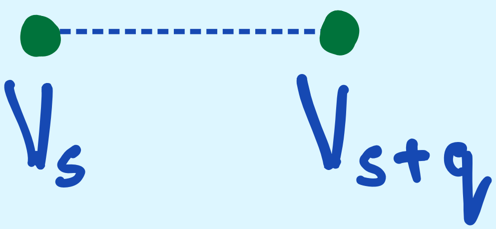
$$\boxed{\bigoplus_{\ell=0}^{s} V_{2\ell} = V_s \otimes V_s}$$

$\Rightarrow$ 'Spin chain' 

Schwinger–Wigner oscillators

q = monopole charge

'Spin chain'



Tamm '1931
Wu–Yang '1976

$$\text{Hamiltonian} = \text{Casimir} = \sum_{\alpha=1}^{3} S_{1\alpha} S_{2\alpha} \equiv (S_1, S_2)$$

$$S_{A\alpha} = a^{+}_{Ai} (\sigma_\alpha)_{ij}\, a_{Aj} \qquad [a_{Ai}, a^{+}_{Bj}] = \delta_{AB}\, \delta_{ij}$$

$$a^{+}_1 \circ a_1 = s \,, \qquad a^{+}_2 \circ a_2 = s + q$$

Hamiltonian

$$\boxed{H = (a_1^\dagger \cdot a_2)\,(a_2^\dagger \cdot a_1)}$$

Let $q \geq 0$. Construct ground state with $H=0$:

$$|\text{ground state}\rangle = \sum_{i_1,\dots,i_q=1}^{2} \psi_{i_1\dots i_q}\,(a_2)^\dagger_{i_1}\cdots(a_2)^\dagger_{i_q}\,\times$$

$$\times \left(\varepsilon_{lm}\,(a_1)^\dagger_l\,(a_2)^\dagger_m\right)^s |0\rangle$$

Representation $\underbrace{\boxed{}\boxed{}\boxed{}\boxed{}\boxed{}\boxed{}}_{q}$

1D sigma models with $\mathcal{N}=2$ SUSY

- 2a model / de Rham

'Hull 1999
'Ivanov, Smilga 2012

X^A = real superfields

$$\mathcal{L} = \int d^2\theta \left[G_{AB}\, D X^A \bar{D} X^B - W(X) \right]$$

$$Q \leftrightarrow e^{-W} d e^{W}$$

- 2b model / Dolbeault

Z^A = complex chiral superfields

$$\mathcal{L} = \int d^2\theta \left[G_{AB}\, D Z^A \bar{D} \bar{Z}^B - W(Z, \bar{Z}) \right]$$

$$Q \leftrightarrow e^{-W} \partial e^{W}$$

SUSY extension: harmonic oscillator

2a model, $\mathcal{L} = \int d^2\theta \left[DX\bar{D}X - \frac{1}{2}X^2 \right]$

$H_{susy} = \{Q, Q^+\}$ 'Nicolai 1976

Let a, a^+ = bosonic ψ, ψ^+ = fermionic

$[a, a^+] = 1$ $\{\psi, \psi^+\} = 1$

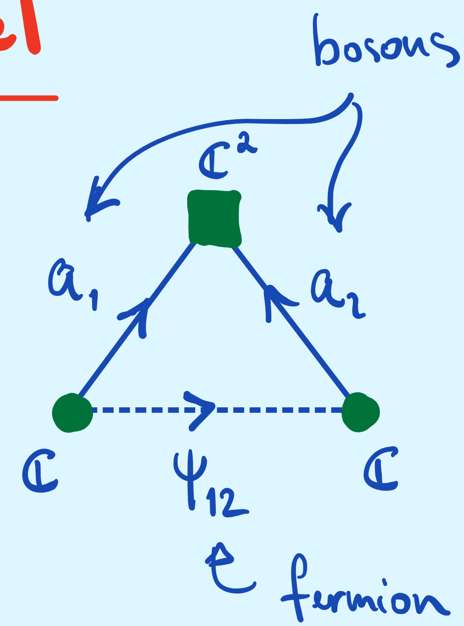
Set $Q = a^+\psi$, $Q^+ = a\psi^+$

$\Rightarrow H_{susy} = a^+a + \psi^+\psi$

SUSY extension $\mathcal{N}=2$ / D-model

Now set $$Q = \alpha_{12}\, \psi_{12}\, a_1^\dagger \cdot a_2$$

real parameter



Constraints:

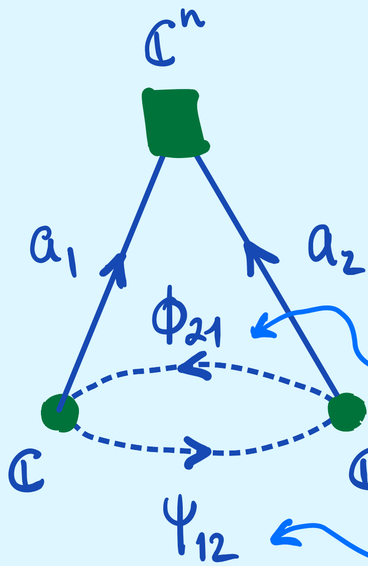
$$C_1 = a_1^\dagger \cdot a_1 + \psi_{12}^\dagger \psi_{12} - s = 0$$

$$C_2 = a_2^\dagger \cdot a_2 - \psi_{12}^\dagger \psi_{12} - (s+q) = 0$$

$$\boxed{[C_1, Q] = [C_2, Q] = 0}$$

H_{susy} = Truncation of Dolbeault Δ on $\mathbb{CP}^1$

Kähler-de Rham $\mathcal{N}=4$ / K-model



$$Q_1 = \alpha_{12}\, \psi_{12}\, a_1^\dagger \circ a_2$$

$$Q_2 = \alpha_{12}\, \phi_{21}^\dagger\, a_1^\dagger \circ a_2$$

SU(2) doublet

twice as many fermions: ψ_{12}, ϕ_{21}

No monopoles!

Constraints: $$C_1 = a_1^\dagger \circ a_1 + \psi_{12}^\dagger \psi_{12} - \phi_{21}^\dagger \phi_{21} - s = 0$$

$$C_2 = a_2^\dagger \circ a_2 - \psi_{12}^\dagger \psi_{12} + \phi_{21}^\dagger \phi_{21} - s = 0$$

$$\{Q_A, Q_B\} = 0, \quad \{Q_A, Q_B^\dagger\} = \delta_{AB} H$$

Truncation of de Rham Δ on $\mathbb{CP}^1$

D-model in N=2 superspace

Superderivatives $D, \bar{D} \longmapsto D^2 = \bar{D}^2 = 0$

Superfields $A_{1i} = a_{1i} + \ldots,\ A_{2i} = a_{2i} + \ldots$

$$\bar{D}\begin{pmatrix} A_1 \\ A_2 \end{pmatrix} = \underbrace{\begin{pmatrix} \Lambda_1 & \varkappa\Psi_{12} \\ 0 & \Lambda_2 \end{pmatrix}}_{:=\mathcal{B}} \begin{pmatrix} A_1 \\ A_2 \end{pmatrix}$$

Coupling

"Superconnection"

Flatness: $\bar{D}\mathcal{B} - \mathcal{B}\mathcal{B} = 0$

Ivanov
Krivonos '1997
Toppan

$$L = \int d^2\theta \left[\bar{A}_1 \cdot A_1 + \bar{A}_2 \cdot A_2 + \bar{\Psi}_{12}\Psi_{12} \right] + \text{FI terms}$$

$$+ \left(s \int d\theta\, \Lambda_1 + (s+q) \int d\theta\, \Lambda_2 + \text{c.c.} \right)$$

"Free" Lagrangian

K-model in $\mathcal{N}=2$ superspace

Same fields A_{1i}, A_{2i}

Coupling

$$\bar{D}\begin{pmatrix} A_1 \\ A_2 \end{pmatrix} = \begin{pmatrix} \Lambda_1 & \varkappa\Psi_{12} \\ \varkappa\Phi_{21} & \Lambda_2 \end{pmatrix}\begin{pmatrix} A_1 \\ A_2 \end{pmatrix}$$

Only $\Lambda_1+\Lambda_2$ is chiral: $\bar{D}(\Lambda_1+\Lambda_2)=0$.

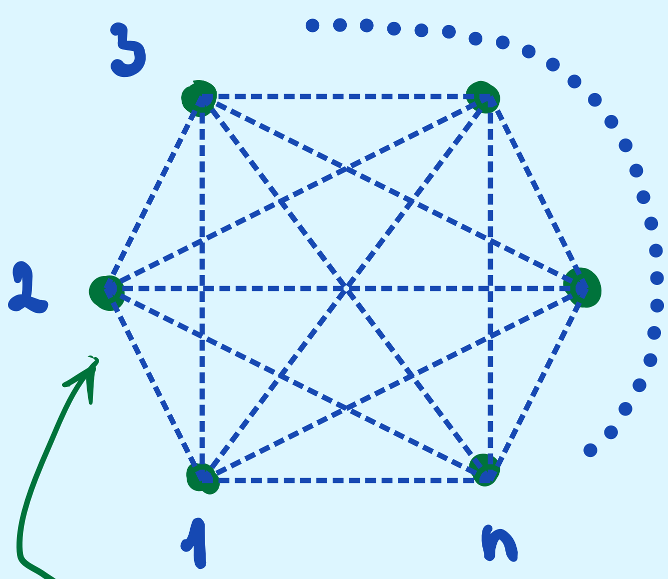
$$L = \int d^2\theta\left[\bar{A}_1\circ A_1 + \bar{A}_2\circ A_2 + \bar{\Psi}_{12}\Psi_{12} + \bar{\Phi}_{21}\Phi_{21}\right] +$$

$$+\ s\left(\int d\theta\,(\Lambda_1+\Lambda_2) + \text{c.c.}\right)$$

Single FI term

Flags: the spin chain

DB '2024
Kuzovchikov



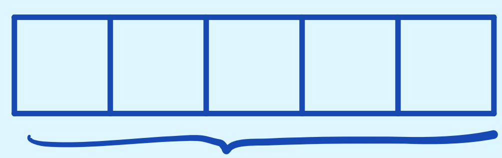
$$\mathcal{F}_n := \frac{SU(n)}{S(U(1)^n)} \hookrightarrow \left(\mathbb{CP}^{n-1}\right)^{\times n}$$

$(n=2:\ \mathbb{CP}^1)$

$$\boxed{H = \sum_{A<B} d_{AB}^2\, (S_A, S_B)}$$

$\frac{n(n-1)}{2}$ parameters

Representations $V_{s_1}, V_{s_2} \ldots$



$s_i = s + q_i$ magnetic charges

Metric
$$\boxed{ds^2 = \sum_{A<B} \frac{1}{d_{AB}^2} \left| \bar{u}_A \circ du_B \right|^2}$$

$\bar{u}_A \circ u_B = \delta_{AB}$

Supercharges

cubic terms needed for $Q^2=0$

D-model:

$$Q = \sum_{A<B} \alpha_{AB}\, \psi_{AB}\, a_A^\dagger \circ a_B - \sum_{A<B<C} \frac{\alpha_{AB}\alpha_{BC}}{\alpha_{AC}} \psi_{AB}\psi_{BC}\psi_{AC}^\dagger$$

K-model:

$$\xi_{AB} := \begin{pmatrix} \psi_{AB} \\ \phi_{BA}^\dagger \end{pmatrix}$$

SU(2) doublets

$$Q = \sum_{A<B} \alpha_{AB}\, \xi_{AB}\, a_A^\dagger \circ a_B + \sum_{A<B<C} \left(\frac{\alpha_{AB}\alpha_{AC}}{\alpha_{BC}} \xi_{AB} (\xi_{AC}^\dagger \xi_{BC}) - \frac{\alpha_{AC}\alpha_{BC}}{\alpha_{AB}} \xi_{BC} (\xi_{AC}^\dagger \xi_{AB}) \right)$$

SUSY algebra $\Rightarrow$ Kähler constraint $\quad \frac{1}{\alpha_{AC}^2} = \frac{1}{\alpha_{AB}^2} + \frac{1}{\alpha_{BC}^2} \quad A<B<C$

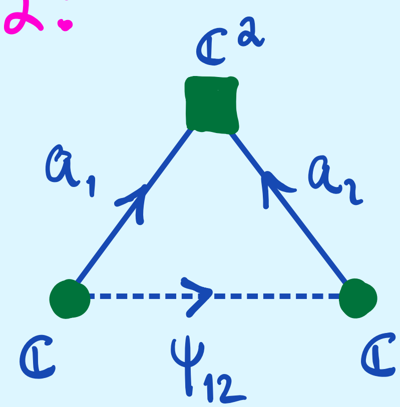
The Witten index / D-model

$$W = \mathrm{STr}\left(g\, e^{-\beta H}\right), \quad g \in SU(n)$$

Independent of β: $\beta \to \infty$ $\quad W = \mathrm{STr}(g)\big|_{H=0}$

$\beta \to 0$ $\quad W = \mathrm{STr}(g)\big|_{\text{constrained Fock space}}$

$n=2$:



$$C_1 = a_1^\dagger \circ a_1 + \psi_{12}^\dagger \psi_{12} - s = 0$$

$$C_2 = a_2^\dagger \circ a_2 - \psi_{12}^\dagger \psi_{12} - (s+q) = 0$$

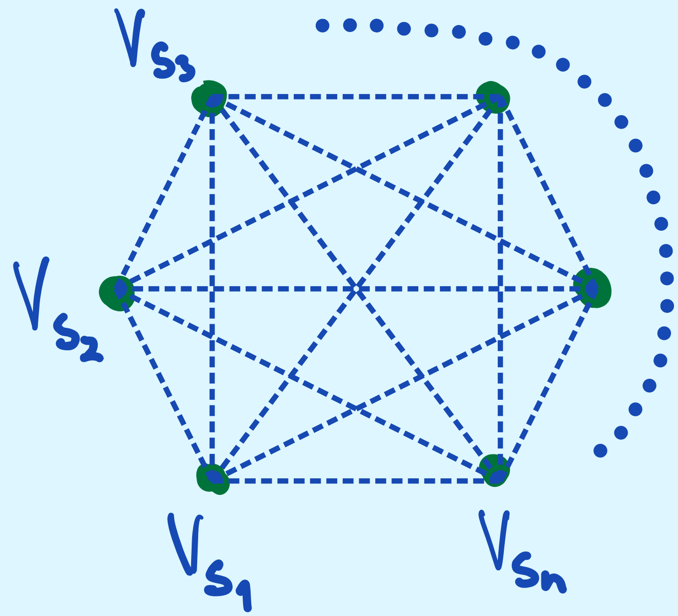
Fermion number 0: $V_s \otimes V_{s+q}$ Fermion number 1: $V_{s-1} \otimes V_{s+q+1}$

$$W = \chi_s \chi_{s+q} - \chi_{s-1} \chi_{s+q+1} = \chi_q$$

Independent of s (of the truncation)

$q+1$ zero energy states

Index: general case



$$S_A = S + q_A$$

$$W = \mathrm{STr}(g)\Big|_{\substack{\text{constrained} \\ \text{Fock space}}} = \,?$$

1) Oscillator partition function

$$Z(t|\lambda) = \mathrm{STr}\Big(\underbrace{\prod_{i=1}^{n} t_i^{J_i}}_{=g} \prod_{j=1}^{n} \lambda_j^{c_j} \Big) =$$

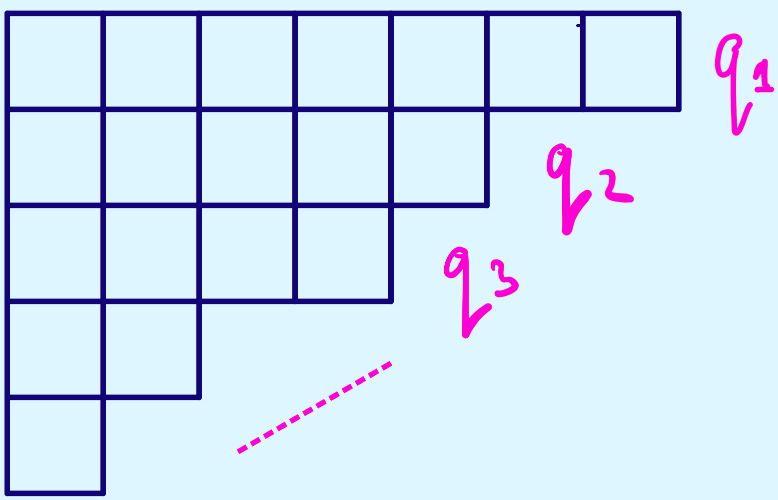
constraints

$u(1)^n \subset u(n)$ generators

$$= \prod_{i,j=1}^{n} \frac{1}{1 - t_i \lambda_j} \times \prod_{k<l} \left(1 - \frac{\lambda_k}{\lambda_l} \right) \frac{1}{\lambda_1^{p_1} \lambda_2^{p_2} \lambda_3^{p_3}}$$

2) Set $C_j = 0$ by residues

$$W = \oint \frac{d\lambda_1}{2\pi i \lambda_1} \cdots \oint \frac{d\lambda_n}{2\pi i \lambda_n} Z(t|\lambda) =$$ Weyl formula for $\chi_{q_1,\dots,q_n}$



Borel-Bott-Weil theorem

K-model: $W =$ Euler characteristic $= n!$

Conclusion & Outlook

Yamaguchi '1979
Kuwabara '1988

* Computing spectra of (magnetic) $\nabla\Delta$ on flags

  $\equiv$ Diagonalizing spin chains

  (infinite spin limit $p \mapsto \infty$)

* SUSY extension: nonlinear chiral multiplets

  Ivanov, Krivonos, Toppan '1997

* Index theorems via oscillator partition function

* Is this an integrable problem?
  Explicit solution?

  | DB, Kuzovchikov '2024

* Generalization to ∞-dim. groups
  (loop groups etc.)

* Relation to 2D sigma models
  (Gross-Neveu models, ...)

  | DB '2020+

THANK
YOU!