Spinor field in cosmology with Lyra's geometry

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Introduction

Shortly after Einstein proposed his famous theory of gravity, Weyl (1918) in an attempt to unify gravitation and electromagnetic field, introduced a generalization of Riemannian Geometry. Weyl theory was not taken seriously as it contradicted some well-known observational result. Lyra (1951) proposed a modification of Riemannian geometry which bears a close resemblance of Weyl geometry. But unlike Weyl geometry, in Lyra's geometry the connection is metric preserving as in Riemannian geometry. In doing so he introduced a gauge function into the structureless manifold. This theory was further developed by Schibe (1952), Sen (1957, 1960), Halford (1970), Sen and Dunn (1971), Manoukian (1972), Send and Vanstone (1972), Hudgin (1973). Recently Casana and coauthors (2005), Shchigolev (2012) and many others have considered Lyra's geometry is cosmology.

Introduction: Aim and scope

In a number of papers (Saha) it was shown that spinor field is very sensitive to the gravitational one. In most cases there exist nontrivial non-diagonal components of energy-momentum tensor (EMT) which leads to the different types of restrictions both on the geometry of space-time and the spinor field itself. The aim for considering Lyra's geometry is to clarify whether it can remove or weaken the restrictions those occur in usual cases.

Introduction: Weyl's geometry

There exist a geodesic gauge in which length of a vector does not change under parallel transform, but in an arbitrary gauge it is assumed to change

$$d\xi = -\xi \phi_\mu dx^\mu, \tag{1}$$

where ϕ is a vector function characterizing the manifold. Thus the metrical connection of a Weyl manifold is characterized by two independent quantities $g_{\mu\nu}$ and ϕ_μ. If one makes a gauge transformation $\xi \to \bar{\xi} = \lambda \xi, \quad \lambda = \lambda(x)$ then

$$g_{\mu\nu} \to \bar{g}_{\mu\nu} = \lambda g_{\mu\nu}, \quad \bar{\phi}_\mu = \phi_\mu - \lambda_\mu/\lambda, \lambda_\mu = \partial\lambda/\partial x^\mu. \tag{2}$$

$$\bar{\Gamma}^\alpha_{\mu\nu} = \Gamma^\alpha_{\mu\nu} + \frac{1}{2}\left(\delta^\alpha_\mu \phi_\nu + \delta^\alpha_\nu \phi_\mu - g_{\mu\nu}\phi^\alpha\right), \quad \phi^\mu = g^{\mu\nu}\phi_\nu. \tag{3}$$

Lyra suggested a modification of Riemannian geometry which is also a modification of Weyl geometry. The metrical concept of gauge in Weyl geometry was modified by a structureless gauge function. The displacement vector between two neighboring points now has the components $\xi^\mu = x^0 dx^\mu$ where x^0 is a nonzero gauge function. A general transformation is given by

$$x^\mu \to x^{\mu'} = x^{\mu'}(x^\lambda), \quad (x^0; x^\mu) = (x^{0'}; x^{\mu'}) \tag{4}$$

$$A_\mu^{\mu'} = \partial x^{\mu'}/\partial x^\mu, \quad \det A_\mu^{\mu'} \neq 0. \tag{5}$$

A vector in Lyra's geometry transforms as

$$\xi^{\mu'} = \lambda A_\mu^{\mu'} \xi^\mu, \quad \delta\xi^\mu = -\tilde{\Gamma}^\mu_{\alpha\beta}\xi^\alpha x^0 dx^\beta, \quad \lambda = \bar{x}^0/x^0, \tag{6}$$

$$\tilde{\Gamma}^\mu_{\alpha\beta} = \Gamma^\mu_{\alpha\beta} - \frac{1}{2}\delta^\mu_\alpha \phi_\beta, \quad \tilde{\Gamma}^\mu_{\alpha\beta} \neq \tilde{\Gamma}^\mu_{\beta\alpha}, \quad \Gamma^\mu_{\alpha\beta} = \Gamma^\mu_{\beta\alpha}. \tag{7}$$

Introduction: Lyra's geometry

The transformation formulae for $\Gamma^{\mu}_{\alpha\beta}$ and ϕ_{α} are

$$\Gamma^{\tau}_{\alpha\beta} = \frac{\partial x^{\tau}}{\partial \bar{x}^{\rho}} \frac{\partial \bar{x}^{\mu}}{\partial x^{\alpha}} \frac{\partial \bar{x}^{\nu}}{\partial x^{\beta}} \bar{\Gamma}^{\rho}_{\mu\nu} + \frac{1}{x^0} \frac{\partial x^{\tau}}{\partial \bar{x}^{\rho}} \frac{\partial \bar{x}^{\rho}}{\partial x^{\alpha} \partial x^{\beta}}, \tag{8a}$$

$$\phi_{\alpha} = \frac{\partial \bar{x}^{\beta}}{\partial \bar{x}^{\alpha}} \bar{\phi}_{\beta}, \tag{8b}$$

under coordinate transformation and

$$\bar{\Gamma}^{\tau}_{\alpha\beta} = \lambda^{-1} \Gamma^{\rho}_{\mu\nu}, \quad \lambda = \bar{x}^0 / x^0 \tag{9a}$$

$$\bar{\phi}_{\alpha} = \lambda^{-1} \left(\phi_{\alpha} + \frac{1}{x^0} \frac{\partial \ln \lambda^2}{\partial x^{\alpha}} \right), \tag{9b}$$

under gauge transformation.

Introduction: Lyra's geometry

The interval in this case is given by

$$ds^2 = g_{\mu\nu} x^0 dx^\mu x^0 dx^\nu. \tag{10}$$

The parallel transport of length in Lyra geometry is integrable, i.e., $\delta(g_{\mu\nu}\xi^\mu\xi^\nu) = 0$ and the connection takes the form

$$\bar{\Gamma}^\alpha_{\mu\nu} = \frac{1}{x^0}\Gamma^\alpha_{\mu\nu} + \frac{1}{2}\left(\delta^\alpha_\mu \phi_\nu + \delta^\alpha_\nu \phi_\mu - g_{\mu\nu}\phi^\alpha\right), \tag{11}$$

which is similar to that of Weyl geometry except $1/x^0$.

The parallel transfer, hence the equation of motion

$$\frac{1}{x^0}\frac{\partial \xi^\alpha}{\partial \beta} + \tilde{\Gamma}^\alpha_{\nu\beta}\xi^\nu = 0, \tag{12}$$

is integrable if the components of the tensor

$$K^\lambda_{\mu\alpha\beta} = \frac{1}{(x^0)^2}\left[\frac{\partial(x^0\tilde{\Gamma}^\lambda_{\mu\beta})}{\partial x^\alpha} - \frac{\partial(x^0\tilde{\Gamma}^\lambda_{\mu\alpha})}{\partial x^\beta} + x^0\tilde{\Gamma}^\lambda_{\rho\alpha}x^0\tilde{\Gamma}^\rho_{\mu\beta} - x^0\tilde{\Gamma}^\lambda_{\rho\beta}x^0\tilde{\Gamma}^\rho_{\mu\alpha}\right], \tag{13}$$

vanish.

The foregoing expression can be expressed as follows

$$K^{\lambda}_{\mu\alpha\beta} = \star R^{\lambda}_{\mu\alpha\beta} + \frac{1}{2}\delta^{\lambda}_{\mu}\Phi_{\alpha\beta}, \tag{14}$$

where

$$\star R^{\lambda}_{\mu\alpha\beta} = \frac{1}{x^0}\left[\frac{\partial \Gamma^{\lambda}_{\mu\beta}}{\partial x^{\alpha}} - \frac{\partial \Gamma^{\lambda}_{\mu\alpha}}{\partial x^{\beta}}\right] + \Gamma^{\lambda}_{\rho\alpha}\Gamma^{\rho}_{\mu\beta} - \Gamma^{\lambda}_{\rho\beta}\Gamma^{\rho}_{\mu\alpha}$$
$$+ \frac{1}{2}\left(\breve{\phi}_{\alpha}\Gamma^{\lambda}_{\mu\beta} - \breve{\phi}_{\beta}\Gamma^{\lambda}_{\mu\alpha}\right), \tag{15}$$

$$\Phi_{\alpha\beta} = \frac{1}{x^0}\left[\frac{\partial \phi_{\alpha}}{\partial x^{\beta}} - \frac{\partial \phi_{\beta}}{\partial x^{\alpha}}\right] + \frac{1}{2}\left(\breve{\phi}_{\alpha}\phi_{\beta} - \breve{\phi}_{\beta}\phi_{\alpha}\right). \tag{16}$$

Einstein's field equation in Lyra's geometry in normal gauge, i.e., $x^0 = 1$ was found by Sen and can be written as

$$G_\mu^\nu + \frac{3}{2}\phi_\mu\phi^\nu - \frac{3}{4}\delta_\mu^\nu\phi_\alpha\phi^\alpha = \kappa T_\mu^\nu, \tag{17}$$

where ϕ_μ is the displacement vector. Let us consider ϕ_μ as a time-like vector field of displacement. Here

$$G_\mu^\nu = R_\mu^\nu - \frac{1}{2}\delta_\mu^\nu R. \tag{18}$$

Basic equations: Spinor field

Given the role that spinor field can play in the evolution of the Universe, question that naturally pops up is, if the spinor field can redraw the picture of evolution caused by perfect fluid and dark energy, is it possible to simulate perfect fluid and dark energy by means of a spinor field? Affirmative answer to this question was given in the a number of papers. We consider the spinor field Lagrangian given by

$$L_{\rm sp} = \frac{\imath}{2}\left[\bar\psi\gamma^\mu\nabla_\mu\psi - \nabla_\mu\bar\psi\gamma^\mu\psi\right] - m_{\rm sp}\bar\psi\psi - F, \tag{19}$$

where the nonlinear term F describes the self-interaction of a spinor field and can be presented as some arbitrary functions of invariants K that takes one of the following values $\{I,\, J,\, I+J,\, I-J\}$ generated from the real bilinear forms of a spinor field. We also consider the case $\psi = \psi(t)$ so that $I = S^2 = (\bar\psi\psi)^2$, & $J = P^2 = (\imath\bar\psi\gamma^5\psi)^2$.

Basic equations: Spinor field

The spinor field equations take the form

$$\imath\gamma^\mu\nabla_\mu\psi - m_{\rm sp}\psi - \mathcal{D}\psi - \imath\mathcal{G}\gamma^5\psi = 0, \quad (20)$$

$$\imath\nabla_\mu\bar{\psi}\gamma^\mu + m_{\rm sp}\bar{\psi} + \mathcal{D}\bar{\psi} + \imath\mathcal{G}\bar{\psi}\gamma^5 = 0. \quad (21)$$

where we denote $\mathcal{D} = 2SF_K K_I$ and $\mathcal{G} = 2PF_K K_J$ with $F_K = dF/dK$, $K_I = dK/dI$ and $K_J = dK/dJ$. In the Lagrangian (19) and spinor field equations (20) and (21) ∇_μ is the covariant covariant derivative of the spinor field so that $\nabla_\mu\psi = \partial_\mu - \Omega_\mu\psi$ and $\nabla_\mu\bar{\psi} = \partial\bar{\psi} + \bar{\psi}\Omega_\mu$ where

$$\Omega_\mu = \frac{1}{4}\bar{\gamma}_a\gamma^\nu\partial_\mu e_\nu^{(a)} - \frac{1}{4}\gamma_\rho\gamma^\nu\tilde{\Gamma}^\rho_{\mu\nu}. \quad (22)$$

Basic equations: Spinor field

The energy momentum tensor of the spinor field is given by

$$
\begin{aligned}
T_{\mu}{}^{\rho} &= \frac{\imath g^{\rho\nu}}{4}\left(\bar\psi\gamma_\mu\nabla_\nu\psi + \bar\psi\gamma_\nu\nabla_\mu\psi - \nabla_\mu\bar\psi\gamma_\nu\psi - \nabla_\nu\bar\psi\gamma_\mu\psi\right) - \delta_\mu^\rho L_{\rm sp}\\
&= \frac{\imath}{4} g^{\rho\nu}\left(\bar\psi\gamma_\mu\partial_\nu\psi + \bar\psi\gamma_\nu\partial_\mu\psi - \partial_\mu\bar\psi\gamma_\nu\psi - \partial_\nu\bar\psi\gamma_\mu\psi\right)\\
&- \frac{\imath}{4} g^{\rho\nu}\bar\psi\left(\gamma_\mu\Omega_\nu + \Omega_\nu\gamma_\mu + \gamma_\nu\Omega_\mu + \Omega_\mu\gamma_\nu\right)\psi\\
&- \delta_\mu^\rho\left(2KF_K - F(K)\right).
\end{aligned}
\tag{23}
$$

On account of spinor field equations (20) and (21) the spinor field Lagrangian takes the form $L_{\rm sp} = 2KF_K - F(K)$. The term in red is responsible for non-diagonal components. Thanks to spinor field equations the conservation of energy holds, i.e.,

$$
T^{\mu}_{\nu;\mu} = 0. \tag{24}
$$

The Bianchi type-I we take in the form

$$ds^2 = dt^2 - a_1^2 dx_1^2 - a_2^2 dx_2^2 - a_3^2 dx_3^2, \tag{25}$$

with a_1, a_2, a_3 being the functions of time only. Introducing the displacement vector $\phi_\mu = \{\beta(t), 0, 0, 0\}$ we find

$$\Omega_0 = \frac{1}{8}\beta, \tag{26a}$$

$$\Omega_1 = \frac{1}{2}\left(\dot{a}_1 - \frac{\beta a_1}{4}\right)\bar{\gamma}^1\bar{\gamma}^0, \tag{26b}$$

$$\Omega_2 = \frac{1}{2}\left(\dot{a}_2 - \frac{\beta a_2}{4}\right)\bar{\gamma}^2\bar{\gamma}^0, \tag{26c}$$

$$\Omega_3 = \frac{1}{2}\left(\dot{a}_3 - \frac{\beta a_3}{4}\right)\bar{\gamma}^3\bar{\gamma}^0. \tag{26d}$$

Basic equations: Spinor field equation

The spinor field equations in this case take the form

$$\frac{1}{x^0}\dot{\psi} + \frac{\dot{V}}{2V}\psi - \frac{\imath\beta}{2}\psi + \imath\bar{\gamma}^0\left(m_{\rm sp} + \mathcal{D}\right)\psi - \mathcal{G}\bar{\gamma}^0\bar{\gamma}^5\psi = 0, \tag{27a}$$

$$\frac{1}{x^0}\dot{\bar{\psi}} + \frac{\dot{V}}{2V}\bar{\psi} - \frac{\imath\beta}{4}\bar{\psi} - \imath\left(m_{\rm sp} + \mathcal{D}\right)\bar{\psi}\bar{\gamma}^0 - \mathcal{G}\bar{\psi}\bar{\gamma}^0\bar{\gamma}^5 = 0. \tag{27b}$$

Note that in Lyra's geometry the differential operator $\partial/\partial x^\mu$ is substituted by $(1/x^0)\partial/\partial x^\mu$. But in natural gauge with $x^0 = 1$ there is no need to write it. Since we will work in natural gauge, we omit it in out further calculations. Here we also define volume scale

$$V = a_1 a_2 a_3. \tag{28}$$

Basic equations: Spinor field equation

For the invariants we find

$$\dot{S}_0 - \frac{3}{4}\beta S_0 + 2\mathcal{G} A_0^0 = 0, \tag{29a}$$

$$\dot{P}_0 - \frac{3}{4}\beta P_0 + 2\left(m_{\rm sp} + \mathcal{D}\right) A_0^0 = 0, \tag{29b}$$

$$\dot{A}_0^0 - \frac{3}{4}\beta A_0^0 + 2\left(m_{\rm sp} + \mathcal{D}\right) P_0 - 2\mathcal{G} S_0 = 0, \tag{29c}$$

with the solution

$$S_0^2 - P_0^2 + {A_0^0}^2 = \exp[(3/2)\int \beta(t) dt], \tag{30}$$

where we define $S_0 = SV$, $P_0 = PV$, $A_0^0 = A^0 V$.

Basic equations: Energy momentum tensor of spinor field

From we then find following nontrivial components of EMT

$$T_0^0 = m_{\rm sp} S + \lambda F, \tag{31a}$$

$$T_1^1 = T_2^2 = T_3^3 = \lambda\left(F - 2KF_K\right). \tag{31b}$$

$$T_2^1 = \frac{\imath}{8}\frac{a_2}{a_1}\left(\frac{\dot a_1}{a_1} - \frac{\dot a_2}{a_2}\right) A^3, \tag{31c}$$

$$T_1^3 = \frac{\imath}{8}\frac{a_1}{a_3}\left(\frac{\dot a_3}{a_3} - \frac{\dot a_1}{a_1}\right) A^2, \tag{31d}$$

$$T_3^2 = \frac{\imath}{8}\frac{a_3}{a_2}\left(\frac{\dot a_2}{a_2} - \frac{\dot a_3}{a_3}\right) A^1, \tag{31e}$$

with $A^\mu = \bar\psi \bar\gamma^5 \bar\gamma^\mu \psi$ being the pseudo-vector. Note that although the SAC in this case differs from those without Lyra's geometry, the additional terms cancel out leaving the components of EMT unaltered.

Basic equations: Einstein equation

The diagonal components of Einstein's equations are

$$\frac{\ddot{a}_2}{a_2} + \frac{\ddot{a}_3}{a_3} + \frac{\dot{a}_2}{a_2}\frac{\dot{a}_3}{a_3} - \frac{3}{2}\beta^2 = \kappa\left(F - 2KF_K\right), \tag{32a}$$

$$\frac{\ddot{a}_3}{a_3} + \frac{\ddot{a}_1}{a_1} + \frac{\dot{a}_3}{a_3}\frac{\dot{a}_1}{a_1} - \frac{3}{2}\beta^2 = \kappa\left(F - 2KF_K\right), \tag{32b}$$

$$\frac{\ddot{a}_1}{a_1} + \frac{\ddot{a}_2}{a_2} + \frac{\dot{a}_1}{a_1}\frac{\dot{a}_2}{a_2} - \frac{3}{2}\beta^2 = \kappa\left(F - 2KF_K\right), \tag{32c}$$

$$\frac{\dot{a}_1}{a_1}\frac{\dot{a}_2}{a_2} + \frac{\dot{a}_2}{a_2}\frac{\dot{a}_3}{a_3} + \frac{\dot{a}_3}{a_3}\frac{\dot{a}_1}{a_1} + \frac{3}{2}\beta^2 = \kappa\left(m_{\rm sp} S + F\right), \tag{32d}$$

Note that though the spinor affine connections in this case differ from case without Lyra's geometry, additional terms cancel out. As a result the components of EMT remain same in both cases.

From (32) one finds the following solutions for metric functions:

$$a_i = X_i V^{1/3} \exp[Y_i \int (1/V) dt], \quad \prod_{i=1}^{3} X_i = 1, \quad \sum_{i=1}^{3} Y_i = 0. \qquad (33)$$

Thus we see that the metric functions are given in terms of V. Hence we have to find the volume scale as well. From (32) for volume scale we find

$$\ddot{V} = \frac{3\kappa}{2} \left[m_{\rm sp} S + \lambda (F - K F_K) \right] V. \qquad (34)$$

But we have to find β as well. Taking into account that the in case of spinor field $T^{\nu}_{\mu;\nu} = 0$ on account on Bianchi identity from (17) we find

$$\left(\frac{3}{2}\phi_\mu\phi^\nu - \frac{3}{4}\delta^\nu_\mu\phi_\alpha\phi^\alpha\right)_{;\nu} = 0, \qquad (35)$$

which in our case takes the form

$$V\dot{\beta} + \beta\dot{V} = 0 \quad \Longrightarrow \beta = \beta_0/V, \quad \beta_0 = \text{const.} \qquad (36)$$

Basic equations: Einstein equation

Now from (29) on account of (36) we obtain

$$S = \frac{C_0}{V} \exp\left[(3/4)\int V^{-1} dt\right], \quad S_0 = \text{const.}, \tag{37a}$$

$$K = \frac{C_0^2}{V} \exp\left[(3/2)\int V^{-1} dt\right], \quad K_0 = \text{const.} \tag{37b}$$

Note that for $K = \{J,\, I \pm J\}$ relations (37b) holds for massless spinor field only, whereas for $K = I$ it is true for both massive and massless spinor field.

Basic equations: Einstein equation

The RHS of the equation (34) depends on V only, hence can be solved. We do it numerically for different values of nonlinear term. In doing so we have considered a few cases that correspond to quintessence, Chapligyn gas, modified quintessence and modified chapligyn gas, respectively:

$$F(K) = \lambda K^{(1+W)/2} - mS, \quad W = \text{const.} - \text{quintessence}, \tag{38a}$$

$$F(K) = \left(A + \lambda K^{(1+\alpha)/2}\right)^{1/(1+\alpha)}, \quad A > 0, \quad 0 \le \alpha \le 1, \tag{38b}$$

$$\text{Chapligyn gas},$$

$$F(K) = \left[\frac{A}{1+W} + \lambda K^{(1+\alpha)(1+W)/2}\right]^{1/(1+\alpha)}, \tag{38c}$$

$$\text{modified Chapligyn gas}.$$

Basic equations: Einstein equation

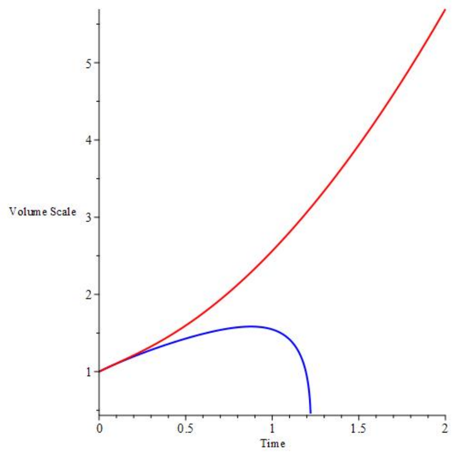

Рис.: Evolution of the volume scale for with (blue) and without (red) Lyra geometry for power law

Basic equations: Einstein equation

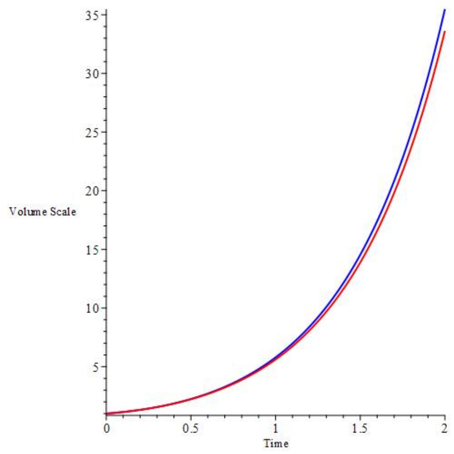

Рис.: Evolution of the volume scale for with (blue) and without (red) Lyra geometry for cosmological constant

Basic equations: Einstein equation

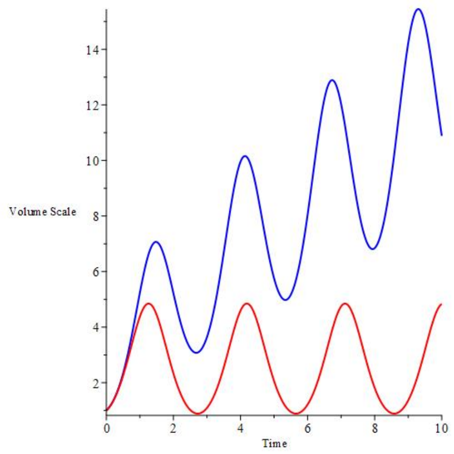

Рис.: Evolution of the volume scale for with (blue) and without (red) Lyra geometry for modified Chapligyn gas

Basic equations: Einstein equation

The non-diagonal components of the EMT leads to

$$\left(\frac{\dot a_1}{a_1} - \frac{\dot a_2}{a_2}\right) A^3 = 0, \tag{39a}$$

$$\left(\frac{\dot a_3}{a_3} - \frac{\dot a_1}{a_1}\right) A^2 = 0, \tag{39b}$$

$$\left(\frac{\dot a_2}{a_2} - \frac{\dot a_3}{a_3}\right) A^1 = 0. \tag{39c}$$

The foregoing system leads to three different cases:
(i) $A^1 = A^2 = A^3 = 0$. By virtue of Fierz identity in this case the spinor field becomes linear and massless;
(ii) $A^2 = A^3 = 0$ and $a_2 = a_3$ which gives rise to locally rotational symmetric Bianchi type-I (LRSBI) model;
(iii) $a_1 = a_2 = a_3$ i.e. the anisotropy vanishes and BI space-time becomes Friedmann-Lamaitre-Robertson-Walker (FLRW) one.

Concluding remarks

Within the scope of BI anisotropic cosmological model with Lyra's geometry we have studied the role of spinor filed in the evolution of the Universe. Though the spinor affine connections in this case differ from those without Lyra's geometry, the components of energy momentum tensor and the equation of volume scale remain the unaltered and the restrictions which occurs in usual cases remain the same. Nevertheless, the gauge function influences the solution through invariants of spinor field. These problems need additional attention. We hope to address it in near future.

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Thank You!!!!!!!!!