Perturbative versus NonPerturbative renormalisation

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FRG in a nutshell

Perturbative RG NonPerturbative RG

The Euclidean generating functional of correlation functions is given by

$$\mathscr{Z}[J] = \frac{1}{\mathscr{N}} \int \mathscr{D}\varphi \exp\left\{-S[\varphi] + \int_x J(x)\,\varphi(x)\right\} . \tag{2.1}$$

Here $\mathscr{N}$ is a normalisation factor, $J(x)$ are the sources and $\int_x = \int d^4x$. The path integral contains divergences as usual. These divergences need to be regularised and renormalised. In (2.1) we are assuming that this has already been done, e.g., by a cutoff regularisation and thus the generating

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quantum effective action. We denote the generating functional with suppressed IR modes by

$$\mathscr{Z}_k[J] = \frac{1}{\mathcal{N}} \int \mathscr{D}\varphi_{p^2\geq k^2} \exp\left\{-S[\varphi] + \int_x J(x)\,\varphi(x)\right\}, \tag{2.18}$$

where the subscript $p^2 \geq k^2$ indicates that we only include momentum modes above the scale k. For $k \to 0$, we get back the full generating functional $\mathscr{Z}_{k=0} = \mathscr{Z}$. Such a restriction of the path integral does not preserve the symmetries of most QFTs. We come back to this issue in Sec. 2.7.

The full suppression of the IR modes leads to the Wegner-Houghton equation [6]. A more general approach is to introduce a function that smoothly suppresses these modes. Thus, we define

$$\int \mathscr{D}\varphi_{p^2\geq k^2} = \int \mathscr{D}\varphi \exp\{-\Delta S_k[\phi]\}, \tag{2.19}$$

where

$$\Delta S_k[\phi] = \frac{1}{2}\int_p \phi(p) R_k(p^2) \phi(-p). \tag{2.20}$$

Here we have defined $\int_p = \int d^4p/(2\pi)^4$ and R_k is the regulator function that suppresses modes with $p^2 \lesssim k^2$ but leaves modes with $p^2 \gtrsim k^2$ unaffected. This can be viewed as a momentum-dependent mass term. The regulator function is required to have three properties

Regulator

- Suppression of IR modes:

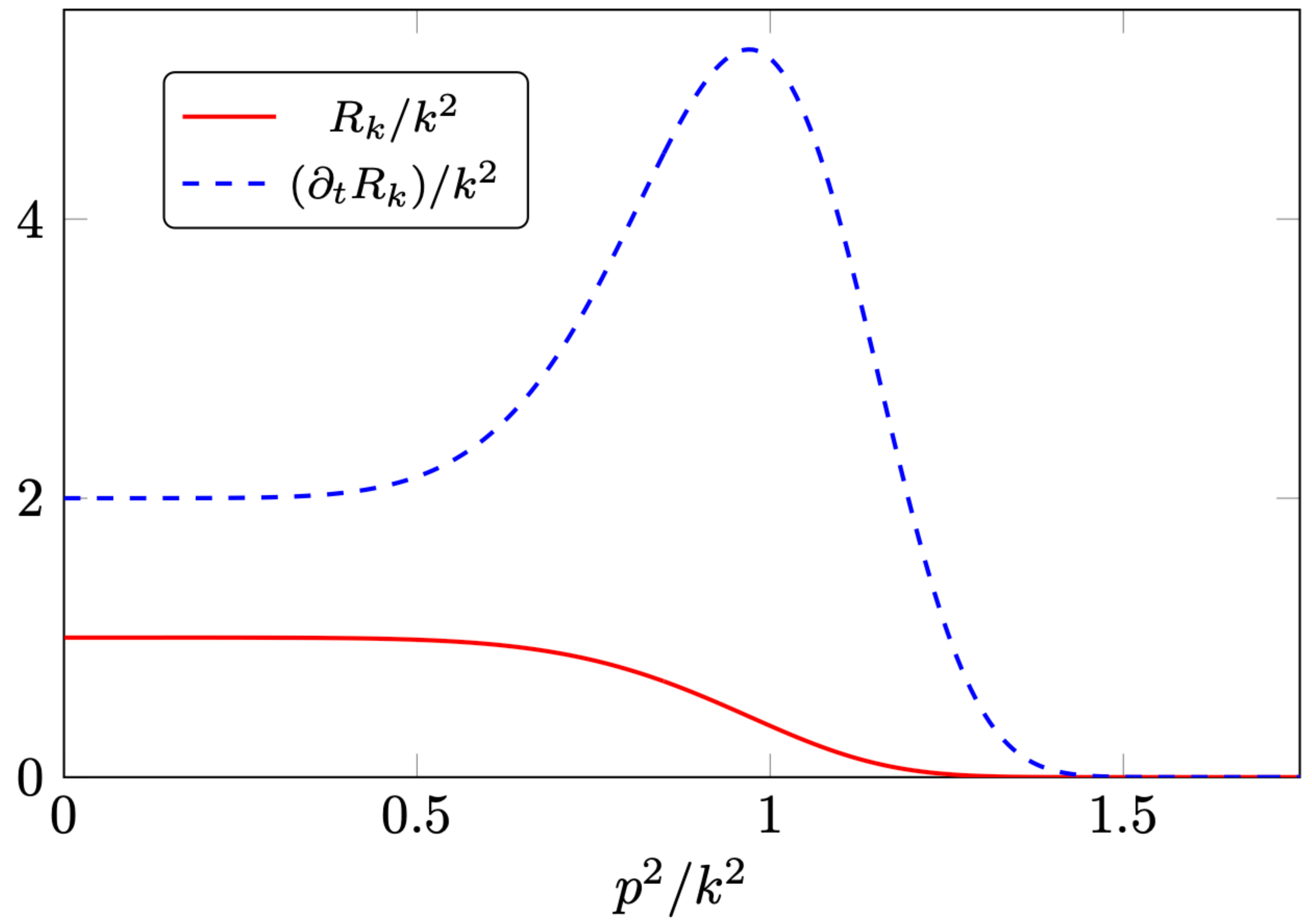
$$\lim_{p^2 \to 0} R_k(p^2) > 0\,.$$

- Physical limit to ensure that $\mathscr{Z}_{k=0} = \mathscr{Z}$:

$$\lim_{k \to 0} R_k(p^2) = 0\,.$$

- UV-limit to ensure that $\Gamma_{k=\Lambda} = S$:

$$\lim_{k \to \Lambda \to \infty} R_k(p^2) \to \infty\,.$$



A frequent parameterisation of the regulator is

$$R_k(p^2) = p^2 r(p^2/k^2)\,, \tag{2.24}$$

where r is the dimensionless shape function of the regulator. A common choice for the shape function is the Litim-type regulator [60, 61]

$$r_{\text{Litim}}(x) = \left(\frac{1}{x} - 1\right)\Theta(1-x)\,. \tag{2.25}$$

This shape function has the advantage that it often provides analytical flow equations. For numerical purposes, the Litim-type shape function is less advantageous since it is not smooth. The exponential shape function is an example for a smooth shape function

$$r_{\text{exp}}(x) = \frac{e^{-x^{2n}}}{x}\,. \tag{2.26}$$

Schwinger functional, is introduced

$$\mathscr{W}[J] = \ln \mathscr{Z}[J]\,. \tag{2.3}$$

The connected n-point functions are generated with functional derivatives with respect to the source

$$\frac{\delta^n \mathscr{W}[J]}{\delta J(x_1)\dots\delta J(x_n)} \equiv \mathscr{W}^{(n)}[J] = \langle \varphi(x_1)\dots\varphi(x_n)\rangle_{J,\mathrm{c}}\,. \tag{2.4}$$

Here we have introduced the notation $\mathscr{W}^{(n)}$ for n functional derivatives. How can we see that the Schwinger functional generates only connected correlation functions? As an example, we look at the propagator

$$\begin{aligned}\frac{\delta^2 \mathscr{W}[J]}{\delta J(x_1)\delta J(x_2)} &= \frac{\delta^2 \ln \mathscr{Z}[J]}{\delta J(x_1)\delta J(x_2)} = \frac{\delta}{\delta J(x_1)}\frac{1}{\mathscr{Z}[J]}\frac{\delta \mathscr{Z}[J]}{\delta J(x_2)} \\ &= \frac{1}{\mathscr{Z}[J]}\frac{\delta^2 \mathscr{Z}[J]}{\delta J(x_1)\delta J(x_2)} - \frac{1}{\mathscr{Z}[J]^2}\frac{\delta \mathscr{Z}[J]}{\delta J(x_1)}\frac{\delta \mathscr{Z}[J]}{\delta J(x_2)} \\ &= \langle \varphi(x_1)\varphi(x_2)\rangle_J - \langle \varphi(x_1)\rangle_J \langle \varphi(x_2)\rangle_J \\ &= \langle \varphi(x_1)\varphi(x_2)\rangle_{J,\mathrm{c}} \equiv G(x_1,x_2)\,. \end{aligned} \tag{2.5}$$

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$$\Delta S_k[\phi] = \frac{1}{2}\int_p \phi(p) R_k(p^2)\phi(-p)\,.$$

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$$\mathscr{Z}_k[J] = \frac{1}{\mathscr{N}}\int \mathscr{D}\varphi_{p^2\geq k^2} \exp\left\{-S[\varphi] + \int_x J(x)\,\varphi(x)\right\}, \tag{2.18}$$

Let us now turn back to the scale-dependent generating functional $\mathscr{Z}_k$. We are interested in a flow equation for $\mathscr{Z}_k$ and thus we take a derivative of (2.18) with respect to the RG time t. The term ΔS_k is the only term that is scale-dependent and thus

$$\begin{aligned}\partial_t \mathscr{Z}_k[J] &= \frac{1}{\mathscr{N}}\int \mathscr{D}\varphi\,(-\partial_t \Delta S_k[\varphi]) \exp\left\{-S[\varphi] - \Delta S_k[\varphi] + \int_x J(x)\,\varphi(x)\right\} \\ &= -\langle \partial_t \Delta S_k[\varphi]\rangle\, \mathscr{Z}_k[J]\,. \end{aligned} \tag{2.27}$$

Another convenient way to express this flow equation is to replace the field by a derivative with respect to the source, $\varphi = \delta/\delta J$. Then we obtain

$$\partial_t \mathscr{Z}_k[J] = -\left(\partial_t \Delta S_k\left[\frac{\delta}{\delta J}\right]\right)\mathscr{Z}_k[J] = -\frac{1}{2}\int_p \frac{\delta^2 \mathscr{Z}_k[J]}{\delta J(p)\delta J(-p)}\partial_t R_k(p^2)\,. \tag{2.28}$$

This is already a useful formulation of the flow equation for the generating functional. As we can see, this is an integro-differential equation, the flow of $\mathscr{Z}_k$ depends on $\mathscr{Z}_k^{(2)}$. Importantly, we do not need to solve a path integral to obtain $\mathscr{Z} = \mathscr{Z}_{k=0}$.

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We now switch to a flow equation for the Schwinger functional $\mathscr{W}_k = \ln \mathscr{Z}_k$. Again, the full Schwinger functional is obtained in the limit $k \to 0$, $\mathscr{W}_{k=0} = \mathscr{W}$. We multiply (2.28) with $1/\mathscr{Z}_k$ and use

$$\begin{aligned}\frac{\delta^2 \mathscr{W}_k}{\delta J(x_1)\delta J(x_2)} &= \frac{1}{\mathscr{Z}_k} \frac{\delta^2 \mathscr{Z}_k}{\delta J(x_1)\delta J(x_2)} - \frac{1}{\mathscr{Z}_k^2} \frac{\delta \mathscr{Z}_k}{\delta J(x_1)} \frac{\delta \mathscr{Z}_k}{\delta J(x_2)} \\ &= \frac{1}{\mathscr{Z}_k} \frac{\delta^2 \mathscr{Z}_k}{\delta J(x_1)\delta J(x_2)} - \frac{\delta \mathscr{W}_k}{\delta J(x_1)} \frac{\delta \mathscr{W}_k}{\delta J(x_2)}, \end{aligned} \tag{2.29}$$

as well as $\partial_t \mathscr{W}_k = \frac{1}{\mathscr{Z}_k}\partial_t \mathscr{Z}_k$. The flow equation is then given by

$$\partial_t \mathscr{W}_k[J] = -\frac{1}{2}\int_p \left[\frac{\delta^2 \mathscr{W}_k}{\delta J(p)\delta J(-p)} + \frac{\delta \mathscr{W}_k}{\delta J(p)} \frac{\delta \mathscr{W}_k}{\delta J(-p)}\right] \partial_t R_k(p^2). \tag{2.30}$$

The Polchinski equation [7] is a flow equation for the Schwinger functional and it can be obtained from (2.30) by amputating the legs from the connected correlation functions. We turn now to the

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An even more efficient way to store the information of a quantum theory is the effective action, which is the Legendre transformation of the Schwinger functional with respect to the mean field

$$\Gamma[\phi] = \sup_J \left\{ \int_x J(x)\,\phi(x) - \mathscr{W}[J] \right\} = \int_x J_{\text{sup}}(x)\,\phi(x) - \mathscr{W}[J_{\text{sup}}] \,. \tag{2.6}$$

In the second expression we have picked out a configuration of sources, which maximises the Legendre transform. This supremum of the source is a function of the mean field $J_{\text{sup}}[\phi]$. The effective action generates one-particle irreducible (1PI) n-point functions. 1PI means that the corresponding Feynman diagram cannot be cut into two diagrams by the cut of a single internal line. These 1PI correlation functions are generated from the effective action by functional differentiation with respect to the mean field

$$\Gamma^{(n)}[\phi] \equiv \frac{\delta^n \Gamma[\phi]}{\delta\phi(x_1)\cdots\delta\phi(x_n)} = \langle \varphi(x_1)\cdots\varphi(x_n)\rangle_{\text{1PI}} \,. \tag{2.7}$$

So far we have only claimed that the effective action generates 1PI diagrams. We illustrate this property again in an inductive way. Let us start with the fact that conjugate variable of the source in (2.6) is indeed the mean field

$$\phi(x) = \left.\frac{\delta \mathscr{W}[J]}{\delta J(x)}\right|_{J_{\text{sup}}} = \left.\frac{1}{\mathscr{Z}[J]}\frac{\delta \mathscr{Z}[J]}{\delta J(x)}\right|_{J_{\text{sup}}} = \langle \varphi(x)\rangle_{J_{\text{sup}}} \,. \tag{2.8}$$

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from (2.30) by amputating the legs from the connected correlation functions. We turn now to the flow equation for the scale-dependent effective action Γ_k. For this we use a modified Legendre transform compared to (2.6)

$$\Gamma_k[\phi] = \sup_J \left\{ \int_x J(x)\phi(x) - \mathscr{W}_k[J] - \Delta S_k[\phi] \right\}. \tag{2.31}$$

It is a choice to include the term ΔS_k into the Legendre transform. We only need to guarantee that for $k=0$ the original Legendre transform (2.6) is restored, which is indeed the case since $\Delta S_{k=0} = 0$. We will see that the choice to include ΔS_k in the Legendre transform results in a simpler flow equation. Eq. (2.31) implies that $\Gamma_k + \Delta S_k$ is the Legendre transform of $\mathscr{W}_k$. Thus, the relations (2.8) and (2.9) are modified and now read

$$\frac{\delta(\Gamma_k + \Delta S_k)}{\delta\phi(x)} = J_{\text{sup}}[\phi(x)]\,, \qquad \left.\frac{\delta \mathscr{W}_k}{\delta J(x)}\right|_{J_{\text{sup}}} = \phi(x)\,. \tag{2.32}$$

By taking one derivative of the effective action with respect to the mean field, we obtain the quantum equation of motion

$$\frac{\delta\Gamma[\phi]}{\delta\phi(x)} = J_{\text{sup}}(x) + \sup_J \left\{ \int_y \frac{\delta J(y)}{\delta\phi(x)} \underbrace{\left(\phi(y) - \frac{\delta\mathscr{W}[J]}{\delta J(y)}\right)}_{=0} \right\} = J_{\text{sup}}(x)\,. \tag{2.9}$$

We turn now to the two-point function, where we will find that the quantum propagator is the inverse of the 1PI two-point function. Anticipating that result, we compute

$$\begin{aligned}\int_y \frac{\delta^2\mathscr{W}}{\delta J(x_1)\delta J(y)} \frac{\delta^2\Gamma}{\delta\phi(y)\delta\phi(x_2)} &= \int_y \frac{\delta}{\delta J(x_1)}\left[\frac{\delta\mathscr{W}}{\delta J(y)}\right] \frac{\delta}{\delta\phi(y)}\left[\frac{\delta\Gamma}{\delta\phi(x_2)}\right] \\ &= \int_y \frac{\delta\phi(y)}{\delta J(x_1)} \frac{\delta J(x_2)}{\delta\phi(y)} = \delta(x_1 - x_2)\,.\end{aligned} \tag{2.10}$$

This proves the relation

$$\mathscr{W}^{(2)}(x_1,x_2) = G(x_1,x_2) = \left(\Gamma^{(2)}(x_1,x_2)\right)^{-1}\,. \tag{2.11}$$

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By taking one derivative of the effective action with respect to the mean field, we obtain the quantum equation of motion

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Consequently, also the relation to the quantum propagator G_k, see (2.11), is modified

$$G_k(p,-p) = \frac{\delta^2\mathscr{W}_k}{\delta J(p)\delta J(-p)} = \left(\frac{\delta^2(\Gamma_k + \Delta S_k)}{\delta\phi(p)\delta\phi(-p)}\right)^{-1} = \frac{1}{\Gamma_k^{(2)} + R_k}(p,-p)\,.$$

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For the higher n-point functions, we need the relation between a derivative with respect to the source and with respect to the mean field

$$\frac{\delta}{\delta J(x)} = \int_y \frac{\delta \phi(y)}{\delta J(x)} \frac{\delta}{\delta \phi(y)} = \int_y \frac{\delta \mathscr{W}^{(1)}(y)}{\delta J(x)} \frac{\delta}{\delta \phi(y)} = \int_y G(x,y) \frac{\delta}{\delta \phi(y)} . \tag{2.12}$$

This relation allows us to derive

$$\langle \varphi(x_1) \cdots \varphi(x_n) \rangle_c \equiv \mathscr{W}^{(n)} = \prod_{i=1}^{n-1} \left(\int_{x_i'} G(x_i, x_i') \frac{\delta}{\delta \phi(x_i')} \right) \phi(x_n) . \tag{2.13}$$

In (2.13) it is important to notice that the propagator is still a function of the mean field ϕ and that derivatives of the propagator generate the 1PI three-point function

$$\frac{\delta}{\delta \phi(x_1)} G(x_2, x_3) = \frac{\delta}{\delta \phi(x_1)} \left(\Gamma^{(2)}(x_2, x_3) \right)^{-1} = - \int_{y_1, y_2} G(x_2, y_1) \Gamma^{(3)}(x_1, y_1, y_2) G(x_3, y_2) . \tag{2.14}$$

Evaluating (2.13) leads to the explicit representations of the connected correlation functions in terms of 1PI correlation functions. For $n = 3$ this leads to

$$\mathscr{W}^{(3)} = - \int_{y_1, y_2, y_3} G(x_1, y_1) G(x_2, y_2) G(x_3, y_3) \Gamma^{(3)}(y_1, y_2, y_3) . \tag{2.15}$$

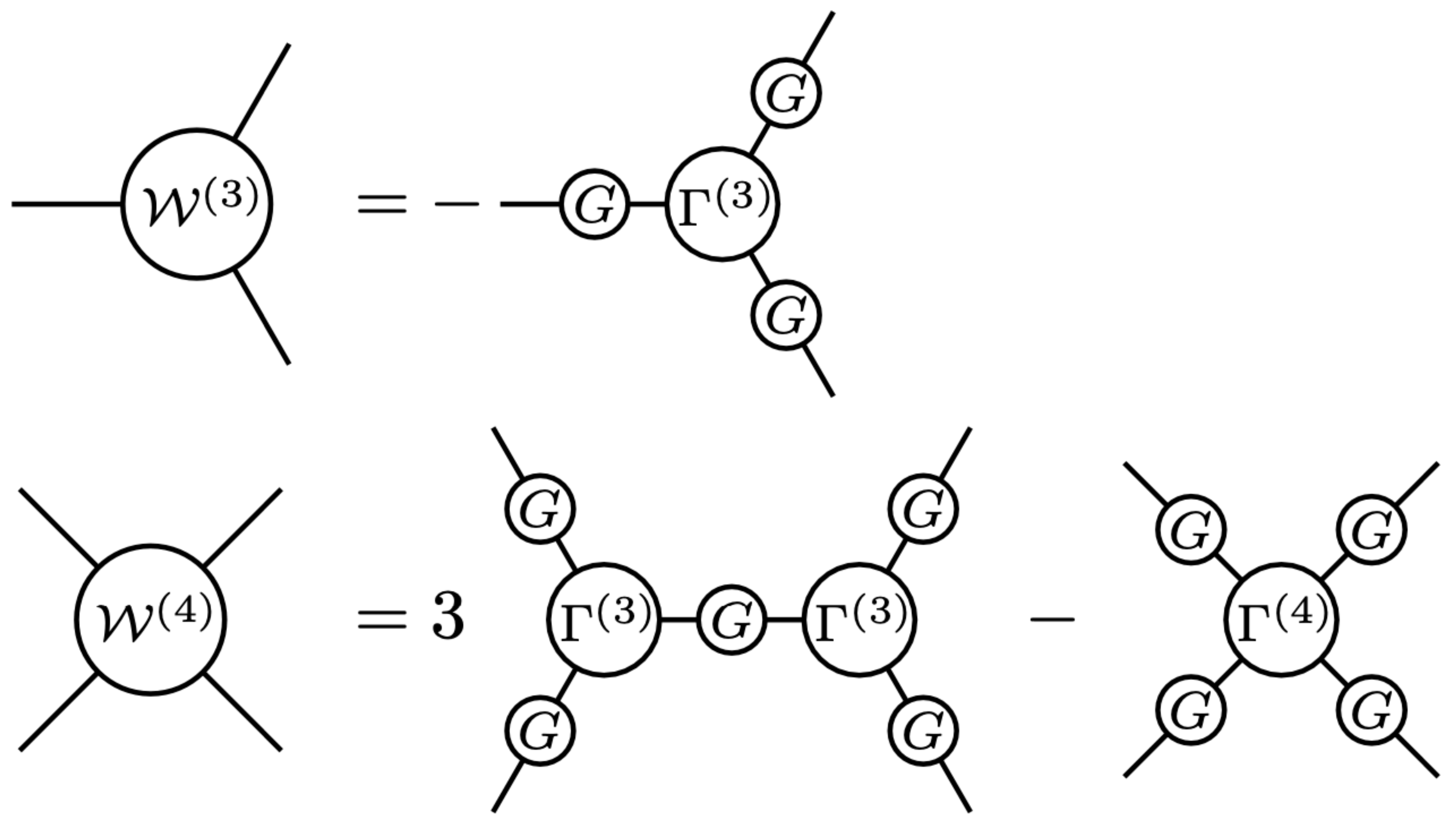

Figure 1: Diagrammatic representation of $\mathscr{W}^{(3)}$ and $\mathscr{W}^{(4)}$ in terms of G, $\Gamma^{(3)}$, and $\Gamma^{(4)}$. The first diagram in the second equation summarised the s, t, and u channel, indicated by the factor 3.

Perturbative RG

The effective action is a very powerful object and so far we have discussed its derivation from the generating functional. For a computation in terms of a path integral, we consider its exponential and the relation to the generating functional (2.1). We obtain

$$
\begin{aligned}
e^{-\Gamma[\phi]} = e^{-\int_x J_{\rm sup}(x)\phi(x)+\mathscr{W}[J_{\rm sup}]} \\
&= e^{-\int_x \frac{\delta\Gamma[\phi]}{\delta\phi(x)}\phi(x)} \int \mathscr{D}\varphi\, e^{-S[\varphi]+\int_x J_{\rm sup}(x)\varphi(x)} \\
&= \int \mathscr{D}\varphi' e^{-S[\phi+\varphi']+\int_x \varphi'(x)\frac{\delta\Gamma[\phi]}{\delta\phi(x)}} \,.
\end{aligned} \tag{2.16}
$$

In the last line we performed a shift of the integration variable $\varphi \to \varphi' + \phi$. This is a path integral, where the integrand depends on $\delta\Gamma/\delta\phi$. It can only be solved for very simple cases. The representation in (2.16) is nonetheless useful as it allows to discuss the symmetries of a theory on the quantum level. A systematic approximation scheme of (2.16) is the vertex expansion

$$
\Gamma[\phi] = \sum_{n=0}^{\infty} \frac{1}{n!} \int_{x_1,\dots,x_n} \Gamma^{(n)}[\phi=0](x_1,\dots,x_n)\phi(x_1)\cdots\phi(x_n)\,, \tag{2.17}
$$

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$$\partial_t \mathscr{W}_k[J] = -\frac{1}{2}\int_p \left[\frac{\delta^2 \mathscr{W}_k}{\delta J(p)\delta J(-p)} + \frac{\delta \mathscr{W}_k}{\delta J(p)}\frac{\delta \mathscr{W}_k}{\delta J(-p)}\right]\partial_t R_k(p^2)\,. \tag{2.30}$$

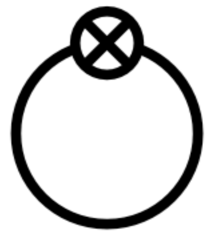
$$\Gamma_k[\phi] = \sup_J \left\{\int_x J(x)\phi(x) - \mathscr{W}_k[J] - \Delta S_k[\phi]\right\}. \tag{2.31}$$

$$\frac{\delta(\Gamma_k + \Delta S_k)}{\delta\phi(x)} = J_{\text{sup}}[\phi(x)]\,, \qquad \left.\frac{\delta \mathscr{W}_k}{\delta J(x)}\right|_{J_{\text{sup}}} = \phi(x)\,. \tag{2.32}$$

$$G_k(p,-p) = \frac{\delta^2 \mathscr{W}_k}{\delta J(p)\delta J(-p)} = \left(\frac{\delta^2(\Gamma_k + \Delta S_k)}{\delta\phi(p)\delta\phi(-p)}\right)^{-1} = \frac{1}{\Gamma_k^{(2)} + R_k}(p,-p)\,. \tag{2.33}$$

We take now a scale derivative of (2.31) and use (2.30), (2.32), and (2.33). The flow of the scale-dependent effective action is then given by

$$\begin{aligned}
\partial_t \Gamma_k[\phi] &= -\partial_t \mathscr{W}_k[J] - \partial_t \Delta S_k[\phi] + \int_x \partial_t J(x) \left.\underbrace{\left[\phi(x) - \frac{\delta \mathscr{W}_k[J]}{\delta J(x)}\right]}_{=0}\right|_{J=J_{\text{sup}}[\phi]} \\
&= \frac{1}{2}\int_p \left[G_k(p,-p) + \phi(p)\phi(-p)\right]\partial_t R_k(p^2) - \partial_t \Delta S_k \\
&= \frac{1}{2}\int_p \frac{1}{\Gamma_k^{(2)} + R_k}(p,-p)\partial_t R_k(p^2) \\
&= \frac{1}{2}\text{STr}\left[\frac{1}{\Gamma_k^{(2)} + R_k}\partial_t R_k\right] =
\end{aligned}$$

 . (2.34)

This is the Wetterich equation in its most compact form. In the last step, we have generalised our derivation and introduced the super trace, STr. The super trace sums over all discrete indices, such as Lorentz and gauge indices, and integrates over continuous indices, such as space or momentum. It further includes a minus sign for Grassmann valued fields, such as fermions or ghosts. In (2.34), we have also introduced a diagrammatic representation of the Wetterich equation. The solid line stands for the quantum propagator and the cross represents a regulator insertion. We use this diagrammatic notation also later in these notes. Note that all quantities in this equation are fully dressed, i.e., all quantities are formulated in term of the scale-dependent effective action and not in terms of the bare action.

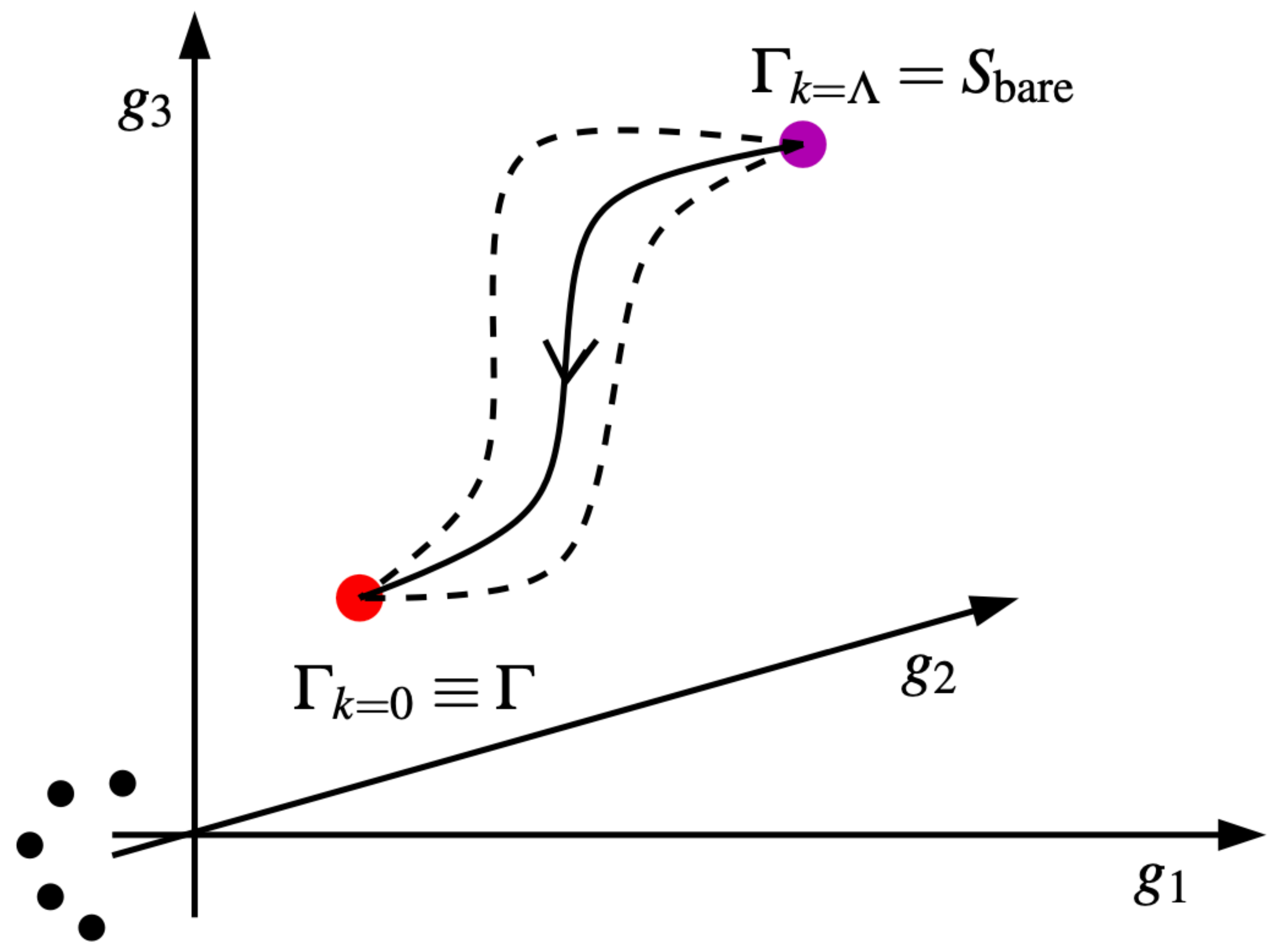
NonPerturbative RG

- By construction, the limits of the scale-dependent effective action are given by the bare action in the UV, $\Gamma_{k=\Lambda} = S$, and by the quantum effective action in the IR, $\Gamma_{k=0} = \Gamma$. The latter

- The properties of the regulator guarantee the correct limits of Γ_k, but the regulator serves more purposes. The derivative of the regulator is peaked around $p^2 \approx k^2$, see Fig. 2. This implements that momentum shells around $p^2 \approx k^2$ are integrated out. Furthermore, the Wetterich equation is inherently finite due to the regulator in the UV as well as in the IR

$$\frac{1}{\Gamma_k^{(2)} + R_k} \quad \longleftrightarrow \quad \text{IR finiteness,}$$
$$\partial_t R_k \quad \longleftrightarrow \quad \text{UV finiteness.}$$

- We can interpret a solution to (2.34) as trajectory between the bare action and the quantum effective action in theory space. The theory space is the infinite-dimensional space of all couplings. The couplings are the prefactors of all operators that are compatible with the symmetry of the theory. We display a sketch of the theory space in Fig. 3.



We have introduced the RG scale k just as a tool to interpolate between the bare action and the quantum effective action.

- We can expand (2.34) in loop orders and by that retain perturbation theory. We expand the scale-dependent effective action with $\Gamma_k = S + \Gamma_{k,\text{1-loop}}$. The Wetterich equation is 1-loop on the right-hand side and thus at 1-loop order we get

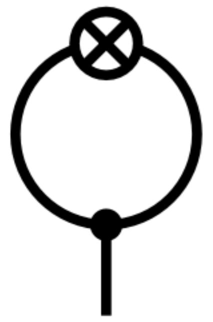
$$\partial_t \Gamma_{k,\text{1-loop}} = \frac{1}{2}\mathrm{Tr}\left[\frac{1}{S^{(2)} + R_k}\partial_t R_k\right] = \partial_t \frac{1}{2}\mathrm{Tr}\left[\ln(S^{(2)} + R_k)\right] , \tag{2.35}$$

where we used in the last step that $S^{(2)}$ is not k dependent. Now it follows that

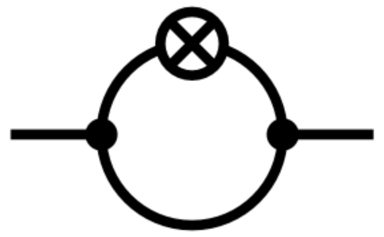
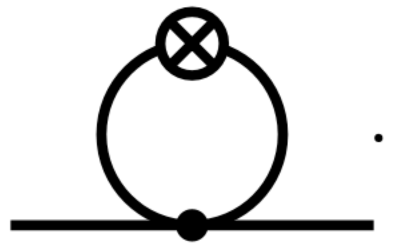
$$\Gamma_{\text{1-loop}} = \frac{1}{2}\mathrm{Tr}\ln S^{(2)} + \text{const.} , \tag{2.36}$$

which is the standard formula for the 1-loop effective action.

(2.34) still carry field dependence, for instance, $G_k[\phi] = (\Gamma_k^{(2)}[\phi] + R_k)^{-1}$. Consequently, a field derivative acts on the propagator with $\delta/\delta\phi\, G_k = -\Gamma_k^{(3)} G_k \Gamma_k^{(3)}$. The flow equation for the one-point function is given by

$$\partial_t \Gamma_k^{(1)} = -\frac{1}{2}\mathrm{Tr}\left[G_k \Gamma_k^{(3)} G_k \partial_t R_k\right] =$$

$, \qquad (2.37)$

while the flow equation for the two-point function reads

$$\partial_t \Gamma_k^{(2)} = -\frac{1}{2}\mathrm{Tr}\left[G_k(\Gamma_k^{(4)} - 2\Gamma_k^{(3)} G_k \Gamma_k^{(3)}) G_k \partial_t R_k\right] =$$

$-\frac{1}{2}$

$. \qquad (2.38)$

Perturbative RG

quantum level. A systematic approximation scheme of (2.16) is the vertex expansion

$$\Gamma[\phi] = \sum_{n=0}^{\infty} \frac{1}{n!} \int_{x_1,\dots,x_n} \Gamma^{(n)}[\phi = 0](x_1,\dots,x_n)\phi(x_1)\cdots\phi(x_n)\,, \tag{2.17}$$

which we also use later in the FRG context. Inserting (2.17) into (2.16) and comparing the field monomials leads to an infinite tower of integro-differential equations known as Dyson-Schwinger equations [51–53]. This tower can be truncated to a finite amount of equations and, for example,

NonPerturbative RG

vertex expansion. The vertex expansion of the scale-dependent effective reads

$$\Gamma_k[\phi] = \sum_{n=0}^{\infty} \frac{1}{n!} \int_{x_1,\dots,x_n} \Gamma_k^{(n)}[\phi = 0](x_1,\dots,x_n)\phi(x_1)\cdots\phi(x_n)\,. \tag{2.39}$$

Plugging this ansatz into (2.34) leads to an infinite tower of coupled differential equations, with the first ones precisely given by (2.37) and (2.38). However, an important difference is that the $\Gamma_k^{(n)}$

Beta functions

The Wetterich equation is not only a tool to compute the quantum effective action, it also allows for the computation of beta functions. We expand the scale-dependent effective action in operators with scale-dependent couplings

$$\Gamma_k[\phi] = \sum_i \bar{g}_i(k) \mathscr{O}_i(\phi) \,. \tag{2.40}$$

The $\bar{g}_i(k)$ are precisely the couplings that span the infinite-dimensional theory space, depicted in Fig. 3. The scale derivative of the couplings result in the respective beta functions $k\,\partial_k\,\bar{g}_i(k) = \partial_t\,\bar{g}_i(k) = \beta_{\bar{g}_i}$. By taking the scale derivative of (2.40), we obtain

$$\partial_t \Gamma_k[\phi] = \sum_i \beta_{\bar{g}_i} \mathscr{O}_i(\phi) \,, \tag{2.41}$$

Example: the anharmonic oscillator

The action for the anharmonic oscillator is given by

$$S = \int \mathrm{d}\tau \left(\frac{1}{2}\dot{x}^2 + \frac{1}{2}\omega^2 x^2 + \frac{1}{24}\lambda x^4 \right) . \tag{2.53}$$

The dot indicates a derivative with respect to τ. We consider only the case with $\omega^2 > 0$ and $\lambda > 0$. The symmetry-breaking case with $\omega^2 < 0$ was investigated with the FRG in [92, 93]. We are interested in the ground state energy in the non-perturbative regime, i.e., for large $\lambda \gg 1$. We display the computation with the FRG and with perturbation theory.

NonPerturbative RG

FRG We have to make an ansatz for the scale-dependent effective action. We choose the truncation

$$\Gamma_k[x] = \int \mathrm{d}\tau \left(\frac{1}{2}\dot{x}^2 + V_k(x) \right) . \tag{2.54}$$

(2.54), we compute the second derivative with respect to x, which is given by

$$\frac{\delta \Gamma_k[x]}{\delta x(\tau_1)\delta x(\tau_2)} = \left(-\partial_{\tau_1}^2 + V_k''(x)\right) \delta(\tau_1 - \tau_2) . \tag{2.55}$$

We furthermore perform a Fourier transformation from τ to p and obtain

$$\Gamma_k^{(2)}[x] = p^2 + V_k''(x) . \tag{2.56}$$

Now we need to choose the regulator function. As explained in (2.25), the Limit-type regulator is advantageous since it allows for analytic flow equations. For the anharmonic oscillator, it is even an optimised regulator [60, 61]. It is given by

$$R_k = (k^2 - p^2)\Theta(k^2 - p^2) . \tag{2.57}$$

From this we can compute the scale derivative of the regulator

$$\partial_t R_k = 2k^2\Theta(k^2 - p^2) + 2k^2(k^2 - p^2)\delta(k^2 - p^2) = 2k^2\Theta(k^2 - p^2) , \tag{2.58}$$

where we have used that $x\,\delta(x) = 0$. From (2.56) and (2.57), we obtain the full propagator

$$G_k = \frac{1}{\Gamma_k^{(2)} + R_k} = \frac{1}{p^2 + (k^2 - p^2)\Theta(k^2 - p^2) + V''(x)} = \begin{cases} \frac{1}{k^2 + V''(x)} & \text{for } p^2 \leq k^2 \\ \frac{1}{p^2 + V''(x)} & \text{for } p^2 \geq k^2 . \end{cases} \tag{2.59}$$

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We have now all ingredients to write down the full flow equation. We are only interested in the ground-state energy and thus we consider only vanishing external momentum. This leads us directly to a flow equation for the effective potential

$$\partial_t V_k(x) = \frac{1}{2}\int_{-\infty}^{\infty} \frac{\mathrm{d}p}{2\pi} \frac{2k^2\Theta(k^2-p^2)}{k^2+V_k''(x)} = \frac{1}{\pi}\frac{k^3}{k^2+V''(x)}\,. \tag{2.60}$$

purposes to expand the potential in a polynomial

$$V_k(x) = \tilde{E}_k + \frac{1}{2}w_k^2 x^2 + \frac{1}{24}\lambda_k x^4 + \ldots\,. \tag{2.61}$$

$$\partial_k E_k = \frac{1}{\pi}\left(\frac{k^2}{k^2+w_k^2} - 1\right)\,. \tag{2.63}$$

From (2.60) and (2.61), we can easily obtain the other flow equations

$$\partial_k w_k^2 = -\frac{1}{\pi}\frac{k^2}{\left(k^2+w_k^2\right)^2}\lambda_k\,, \tag{2.64}$$

$$\partial_k \lambda_k = \frac{6}{\pi}\frac{k^2}{\left(k^2+w_k^2\right)^3}\lambda_k^2\,. \tag{2.65}$$

NonPerturbative RG

The corresponding integrals can be solved analytically, for example, with Mathematica. We plug these results into (2.63) and integrate them down, which again can be done analytically. The result is

$$E_0 = -\int_0^\infty \mathrm{d}k\, \partial_k E_k = \int_0^\infty \mathrm{d}k\, \frac{1}{\pi}\left(1 - \frac{k^2}{k^2 + w_k^2}\right)$$

$$= \frac{1}{2}w + \frac{3}{4}\left(\frac{\lambda}{24w^3}\right)w - \frac{3}{16\pi}(8\pi^2 + 29)\left(\frac{\lambda}{24w^3}\right)^2 w$$

Let us compare this result to ordinary perturbation theory

$$E_{0,\mathrm{PT}} = \frac{1}{2}w + \frac{3}{4}\left(\frac{\lambda}{24w^3}\right)w - \frac{21}{8}\left(\frac{\lambda}{24w^3}\right)^2 w + \ldots$$

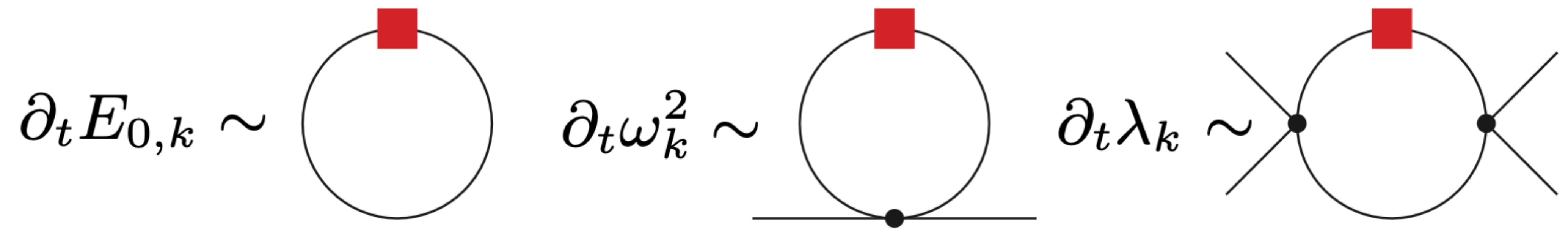

Fig. 4. Diagrammatic representation of Eqs. (37)-(39). The diagrams look similar to one-loop perturbative diagrams with all internal propagators and vertices being fully dressed quantities. One internal line always carries the regulator insertion $\partial_t R_k$ (filled box). (One further diagram for $\partial_t \lambda_k$ involving a 6-point vertex is dropped, as in Eq. (39).)

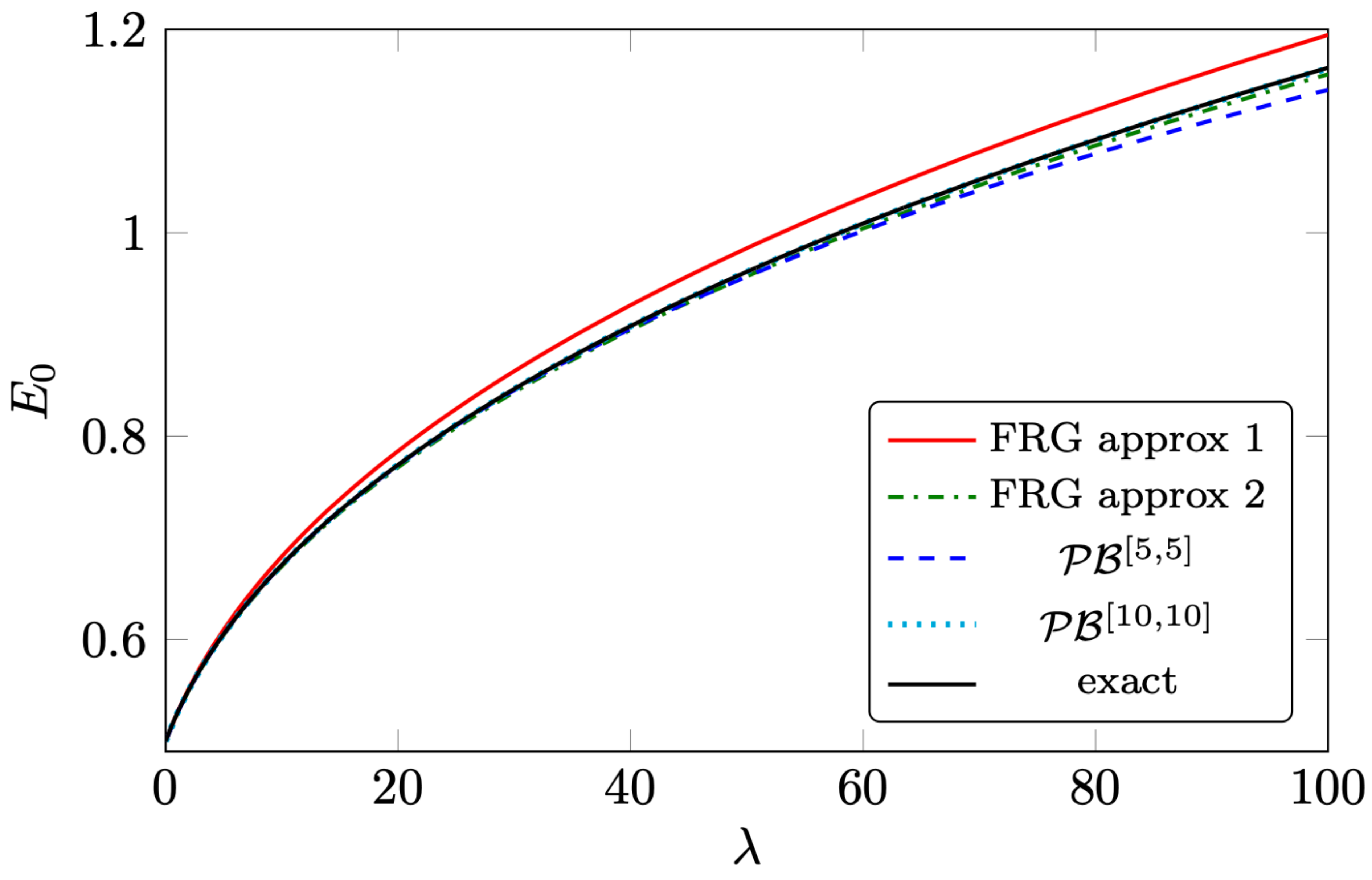

Figure 4: The energy of the ground state of the anharmonic oscillator as a function of λ for $\omega = 1$. FRG approx 1 is based on a numerical integration of (2.63) and (2.64), while FRG approx 2 additionally includes (2.65). We compare this to the Padé approximants of the Borel transform $\mathscr{P}\mathscr{B}^{[n,m]}$, where n is the degree of the polynomial of the numerator, m of the denominator, and $n+m$ is the order of the perturbation series needed for this approximant. The exact solution stems from a well-converged numerical diagonalisation of Hamilton operator in terms of ladder operators.

Thank you for your attention!