Surprises in supersymmetric quantum field theory (lecture)

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Review lecture: "Discussing structure of effective action in supersymmetric quantum field theory: some surprising (from my point of view) aspects".

Quantum description of fields with a large number manifest and hidden, global and local symmetries.

- Chiral effective potential in $4D$, $\mathcal{N} = 1$ supersymmetric theoris
- Non-holomirphic effective potential in $4D$, $\mathcal{N} = 4$ supersymmetric Yang-Mills theory
- Structure of divergences in the $6D$, $\mathcal{N} = (1,1)$ supersymmetric Yang-Mills theory

Effective action in quantum field theory from a bird's eye view

- Scalar field theory
  Generating functionals

$$Z[J] = \int \mathcal{D}\varphi \, e^{\frac{i}{\hbar}(S[\varphi] + \int d^4x J(x)\varphi(x))}$$

$$Z[J] = e^{\frac{i}{\hbar} W[J]}, \; \Phi = \frac{W[J]}{\delta J(x)}, \; \Gamma[\Phi] = (W[J] - \int d^4x J(x)\Phi(x))_{J=J(x|\Phi)}$$

Quantum effective action $\Gamma[\Phi] = S[\Phi] + \bar{\Gamma}[\Phi], \;\; \bar{\Gamma}[\Phi] = \Sigma_{L=1}^{\infty} \hbar^L \bar{\Gamma}_L[\Phi].$
Loop expansion:

$$e^{\frac{i}{\hbar}\bar{\Gamma}[\Phi]} = \Big(\int \mathcal{D}\varphi e^{\frac{i}{2}(S_2[\Phi]\varphi^2 + \Sigma_{n=3}^{\infty} \frac{\hbar^{\frac{n}{2}-1}}{n!} S_n[\Phi]\varphi^n} \Big)_{one\, particle\, irreducible}.$$

$$S_n[\Phi]\varphi^n \equiv \int d^4x_1 \ldots d^4x_n \frac{\delta^n S[\Phi]}{\delta\Phi(x_1)\ldots\delta\Phi(x_n)} \varphi(x_1)\ldots\varphi(x_n)$$

- Gauge theory: background field method

1. Gauge theory:

a. *Field* φ^i,

b. *Action* $S[\varphi]$,

c. *Classical gauge transformations* $\delta\varphi^i = R^i{}_\alpha \xi^\alpha$,

d. *Classical gauge invariance* $\frac{\delta S[\varphi]}{\delta\varphi^i} R^i{}_\alpha = 0$.

2. Effective action for gauge theory:

a. *Effective action is constructed on the base of Faddeev-Popov prescription,*

b. *Manifestly covariant in background field special gauge fixing conditions are used, which depend on the background gauge field* Φ^i *and the quantum gauge field* φ^i *(it takes some art to do this in each concrete theory),*

c. *Loop expansion for effective action is constructed in the form similar with scalar field theory,*

d. *Effective action is gauge invariant under the classical gauge transformations* $\frac{\delta\Gamma[\varphi]}{\delta\varphi^i} R^i{}_\alpha = 0$.

- Supersymmetry is an extension of special relativity symmetry. In its essence, the supersymmetry is a special relativity symmetry extended by the symmetry between bosons and fermions
- From mathematical point of view the relativistic symmetry is expressed in terms of Poincaré group with the generators P_m and J_{mn} satisfying the known commutation relations

$$\begin{aligned} [P_r, P_s] &= 0, \\ [J_{rs}, P_m] &= i(\eta_{rm} P_s - \eta_{sm} P_r), \\ [J_{mn}, J_{rs}] &= i(\eta_{mr} J_{ns} - \eta_{ms} J_{nr} + \eta_{ns} J_{mr} - \eta_{nr} J_{ms}) \end{aligned} \tag{1}$$

η_{mn} is the Minkowski metric.

- Extension of special relativity in four dimensions means extension of the Poincare algebra by the generators (supercharges) ${Q^{i}}_{\alpha}$, $\bar{Q}_{i\dot{\alpha}}$, $i = 1, 2, ..., \mathcal{N}$
- The relations among the supercharges (Poincaré superalgebra) are given in terms of anticommutators

$$
\begin{aligned}
\{{Q^{i}}_{\alpha}, {Q^{j}}_{\beta}\} &= \epsilon_{\alpha\beta} Z^{ij} \\
\{\bar{Q}_{i\dot{\alpha}}, \bar{Q}_{j\dot{\beta}}\} &= \epsilon_{\dot{\alpha}\dot{\beta}} \bar{Z}_{ij} \\
\{{Q^{i}}_{\alpha}, \bar{Q}_{j\dot{\alpha}}\} &= 2{\delta^{i}}_{j} {\sigma^{m}}_{\alpha\dot{\alpha}} P_m
\end{aligned}
\quad (2)
$$

Z^{ij}, $\bar{Z}_{ij}$ are the central charges (further they are assumed to be zeros); $\alpha = 1, 2$; $\dot{\alpha} = \dot{1}, \dot{2}$; $\epsilon_{\alpha\beta}$, $\epsilon_{\dot{\alpha}\dot{\beta}}$ are the invariant tensors of the Lorentz group, $\sigma_m = (\sigma_0, \sigma_i)$.

- Supersymmetric field model means a field model invariant under the above superalgebra. Since the supercharges are the spin $s = \frac{1}{2}$ Lorentz group spinors one can expect that any supersymmetric field model must contain both bosonic and fermionic fields

- Minkowski space coordinates x^m have the same tensor structure as the generators of space-time translations P^m. Analogously ones introduce the additional spinor coordinates $\theta^{i\alpha}$ and $\bar{\theta}^{i\dot{\alpha}}$ associated with the supercharges ${Q^i}_\alpha$ and $\bar{Q}_{i\dot{\alpha}}$. The additional coordinates have the fermionic structure as well as the supercharges and anticommute among themselves.
- Manifold parameterized by the commuting (bosonic) coordinates x^m and the anticommuting (fermionic) coordinates $\theta^{i\alpha}$, $\bar{\theta}^{i\dot{\alpha}}$ is called (conventional or general or standard) superspace
- Function defined on superspace is called superfield
- Since the fermionic coordinates are anticommuting, any superfield is no more then polynomial in these coordinates. The coefficients of such a polynomial are the conventional bosonic and fermionic fields on Minkowski space. All these coefficients are called the component fields of the superfield.
- Consider the supersymmetry transformations (supertranslations): $\theta^{i\alpha} \to \theta^{i\alpha} + \epsilon^{i\alpha}$, $\bar{\theta}^{i\dot{\alpha}} \to \bar{\theta}^{i\dot{\alpha}} + \bar{\epsilon}^{i\dot{\alpha}}$, $x^m \to x^m + \delta x^m$. The $\epsilon^{i\alpha}$ and $\bar{\epsilon}^{i\dot{\alpha}}$ are the anticommuting transformation parameters, δx^m are expressed in special form through the fermionic coordinates and the anticommuting parameters. The supertranslations define the supersymmetry transformations of the component fields.

$\mathcal{N} = 1$ supersymmetric field models

- $\mathcal{N} = 1$ superspace: coordinates $(x^m, \theta^\alpha, \bar{\theta}^{\dot{\alpha}})$.
- *Chiral scalar superfield* $\Phi(x, \theta, \bar{\theta}) = e^{i(\theta\sigma^m\bar{\theta})\partial_m}\Phi(x, \theta)$. *Component content* $\Phi(x, \theta) = A(x) + \theta^\alpha\psi_\alpha(x) + F(x)\theta^2$
- *Antichiral scalar superfield* $\bar{\Phi}(x, \theta, \bar{\theta}) = e^{-i(\theta\sigma^m\bar{\theta})\partial_m}\bar{\Phi}(x, \bar{\theta})$. *Component content* $\bar{A}(x, \bar{\theta}) = \bar{A}(x) + \bar{\theta}_{\dot{\alpha}}\bar{\psi}^{\dot{\alpha}}(x) + \bar{F}(x)\bar{\theta}^2$
- *Supercovariant derivatives* ∂_m, D_α, $\bar{D}_{\dot{\alpha}}$

$$D_\alpha = \partial_\alpha + i(\sigma^m)_{\alpha\dot{\alpha}}\bar{\theta}^{\dot{\alpha}}\partial_m,$$

$$\bar{D}_{\dot{\alpha}} = -\partial_{\dot{\alpha}} - i\theta^\alpha(\sigma^m)_{\alpha\dot{\alpha}}\partial_m.$$

- *Basic properties of chiral and antichiral superfields*

$$\bar{D}_{\dot{\alpha}}\Phi = 0,$$

$$D_\alpha\bar{\Phi} = 0.$$

- Superfield model (Wess-Zumino model)

$$S[\Phi,\bar\Phi]=\int d^4xd^2\theta d^2\bar\theta\bar\Phi(x,\theta,\bar\theta)\Phi(x,\theta,\bar\theta)+(\int d^4xd^2\theta W(\Phi)+c.c.)$$

$W(\Phi)=\frac{m}{2}\Phi^2+\frac{\lambda}{3!}\Phi^3$. Manifest supersymmetry.

- Component form of Wess-Zumino model

$$S=\int d^4x(-\partial^m\bar A\partial_m A-\frac{i}{2}\psi^\alpha\sigma^m{}_{\alpha\dot\alpha}\partial_m\bar\psi^{\dot\alpha}+\bar FF+F(mA+\frac{\lambda}{2}A^2)+$$

$$+\bar F(m\bar A+\frac{\lambda}{2}\bar A^2)-\frac{1}{4}(m+\lambda A)\psi^\alpha\psi_\alpha-\frac{1}{4}(m+\lambda\bar A)\bar\psi_{\dot\alpha}\bar\psi^{\dot\alpha})$$

Non-manifest supersymmetry.

- *Real scalar superfield* $V(x,\theta,\bar\theta)$. *Component content* $V(x,\theta,\bar\theta) = A(x) + \theta^\alpha\psi_\alpha(x) + \bar\theta_{\dot\alpha}\bar\psi^{\dot\alpha}(x) + \theta^2F(x) + \bar\theta^2\bar F(x) + (\theta\sigma^m\bar\theta)A_m(x) + \bar\theta^2\theta^\alpha\lambda_\alpha(x) + \theta^2\bar\theta_{\dot\alpha}\bar\lambda^{\dot\alpha}(x) + \theta^2\bar\theta^2D(x)$
- Superfield model ($\mathcal{N}=1$ supersymmetric Yang-Mills theory)

$$S_{SYM}[V] = \frac{1}{2g^2}\int d^4xd^2\theta\,\mathrm{tr}\,(W^\alpha W_\alpha)$$

Superfield V takes the values in Lie algebra of gauge group, $W_\alpha = -\frac{1}{8}\bar D^2(e^{-2V}D_\alpha e^{2V})$ (W_α is a superfield strength), $\bar D^2 = \bar D_{\dot\alpha}\bar D^{\dot\alpha}$,

- *Gauge transformations*

$$e^{2V'} = e^{i\bar\Lambda}e^{2V}e^{-i\Lambda}.$$

Manifest supersymmetry.

- Component form of supersymmetric Yang-Mills theory

$$S_{SYM} = \frac{1}{g^2}\int d^4xtr(-\frac{1}{4}G^{mn}G_{mn} - i\lambda^\alpha\sigma^m_{\alpha\dot\alpha}\nabla_m\bar\lambda^{\dot\alpha} + 2D^2)$$

Non-manifest supersymmetry.

Chiral-antichiral superpopagators

$$G(z,z')=\frac{1}{\Box-m^2}\hat{O}(D,\bar{D})\delta^4(x-x')\delta^4(\theta-\theta')$$

Gauge superfield superpropagator

$$D(z,z')=-\frac{1}{\Box}\delta^4(x-x')\delta^4(\theta-\theta')$$

Grassmann delta-functions

$$\delta^4(\theta-\theta')=\delta^2(\theta-\theta')\bar{\delta}^2(\bar{\theta}-\bar{\theta}')=(\theta-\theta')^2(\bar{\theta}-\bar{\theta}')^2.$$

Property of Grassmann delta-functions

$$\delta^4(\theta-\theta')_{|\theta'=\theta}=0!$$

- The SUSY models can be formulated in terms of bosonic and fermionic component fields. Supersymmetry is hidden. It is not very convenient in quantum field theory: supersymmetry algebra is open, very many Feynman diagrams, miraculous cancelations.
- Superfield formulation: manifest SUSY, comparatively small number of supergraphs, origin of miraculous cancelations. Problem: how to formulate the of $\mathcal{N}$-extended SUSY models in terms of unconstrained $\mathcal{N}$-extended superfields. General solution for arbitrary $\mathcal{N}$ is unknown.
- Why we want to get a formulation in terms of unconstrained superfields: **(a)**. In conventional field theory the fields must be the functionally independent arguments of action. Otherwise there is no Lagrangian formulation. **(b)**. Conventional quantum field theory is constructed in terms of unconstrained fields. Manifest supersymmetry, provided by superfield formulation, allows us to control the calculations. Supersymmetry algebra is automatically closed. **(c)**. Superfield formulation provides the simple enough ways to construct the various superinvariants that allows to describe the possible structure of contributions to effective action or to S-matrix.

$\mathcal{N} = 1$ non-renormalization theorem

Power of superfield formulation for $\mathcal{N} = 1$ supersymmetric theories-non-renormalization theorem: any L-loop contribution to effective action is written in the form of single integral over $d^4\theta = d^2\theta d^2\bar{\theta}$ (over full superspace not over chiral ($d^2\theta$) or antichiral ($d^2\bar{\theta}$) subspaces):

$$\int d^4p_1 \ldots d^4p_L d^4\theta \mathcal{F}(p_1, \ldots, p_L, \theta).$$

This relation completely determines the possible counterterms in the $\mathcal{N} = 1$ supergauge theories up to the coefficients. Complete explanations of miraculous cancelations of possible divergences in component approach.

- Main point of prove is a structure of the superpropagators. All of them contain the delta-function of anticommuting coordinates that allows us to perform integration over these coordinates in explicit form.
- This allows us (also using the power counting) to write down immediately all possible supersymmetric counterterms in explicit form.
- This allows us to find in some case the possible finite supersymmetric contributions to effective action practically up to numerical coefficients.

Chiral effective potential

At first glance it seems that the non-renormalization theorem forbids chiral quantum corrections to the classical chiral potential. In fact, such corrections exist and do not contradict the non-renormalization theorem.

How it is possible?

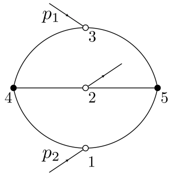
- The non-renormalization theorem does not forbid space-time non-local contributions to the effective action. The divergences are local, but the finite contributions are in general non-local.
- There is an identity for chiral superfields that transforms an integral over the full superspace into an integral over a chiral subspace.

$$\int d^4xd^4\theta u(\Phi)(-\frac{D^2}{4\Box})v(\Phi) = \int d^4xd^2\theta u(\Phi)v(\Phi).$$

- Chiral superfield propagators contain D and $\bar{D}$ covariant derivatives, besides the operator $\Box$ in the denominators for massless theory. Thus, one can expect that in principle, the chiral quantum corrections are admissible in massless case and under condition that last loop integration is non-divergent. Otherwise, the space-time non-locality is impossible.
- These considerations are confirmed by direct calculations.

Direct two-loop calculations in Wess-Zumini model.

- First chiral quantum correction arises at two loops from the following supergraph



External lines are the background chiral superfields.

- Two-loop correction

$$W^{(2)} \sim |\lambda|^4 W_{class}$$

- Although the theory contains divergences and requires renormalization, two-loop chiral quantum corrections is finite in terms of bare coupling constant and fields.

Chiral effective potential

General chiral superfield model. Direct two-loop calculations

- General chiral superspace model

$$S[\bar{\Phi},\Phi] = \int d^4x d^4\theta K(\bar{\Phi},\Phi) + \int d^4x d^2\theta W_{class}(\Phi) + \int d^4x d^2\bar{\theta}\bar{W}_{class}(\bar{\Phi}).$$

  $K(\bar{\Phi},\Phi)$ is a real function (Káhler potential). Four-dimensional sigma-model. Non-renormalizable theory.

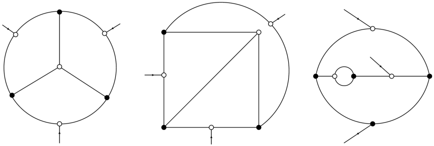
- Two-loop chiral quantum correction

$$W^{(2)} \sim \bar{W}^{'''}_{class}(0)(\frac{W^{''}_{class}(\Phi)}{K^2_{\Phi\bar{\Phi}}(0,\Phi)})^3.$$

- In spite of non-renoromalizability, two-loop chiral correction is finite in terms of bare parameters and fields.

Direct three-loop calculations in Wess-Zumini model

- Three-loop chiral quantum corrections are given by the following supergraphs



External lines are background chiral superfields.

- Two first supergraphs ate finite, third contains one-loop divergence due to divergent one-loop subgraph. Last integration is finite in accordance with qualitative analysis. After renormalization, one gets

$$W^{(3)} \sim |\lambda|^6 W_{class}.$$

I.L.B, R.M. Iakhibaev,D.I. Kazakov, D.M. Tolkachev, 2025.

- Chiral effective potential in gauge theories?

Superfield Lagrangian formulation of $\mathcal{N}=2$ supersymmetric theories as well as of higher $\mathcal{N}$ faces the fundamental problems in comparison with $\mathcal{N}=1$ case.

- The simplest $\mathcal{N}=2$ multiplets are the hypermultiplet and vector multiplet.
- On shell the hypermultiplet contains four scalar fields and two spinor fields.
- All these fields are the components of the superfield $q^i(z)$, where $z=(x^m, \theta^i{}_\alpha, \bar{\theta}^{i\dot{\alpha}})$ and $i=1,2$, under constraints

$$D^{(i}_\alpha q^{j)} = 0,\ \bar{D}^{(i}_{\dot{\alpha}} q^{j)} = 0.$$

  The above constraints put all the component fields on free mass shell. The superfield Lagrangian formulation is impossible.
- On shell the vector multiplet contains the vector field, a doublet of spinor fields, and a complex scalar field. All these fields can be include to some superfield satisfying constraints and off shell Lagrangian formulation in terms of this superfield faces the difficulties.
- Reason of difficulties is that there are too many anticommuting coordinates in $\mathcal{N}=2$ superfields.

Breakthrough was made in the pioneering works by A. Galperin, E. Ivanov, S. Kalitsyn, V. Ogievetsky and E. Sokatchev where the quite new $\mathcal{N}=2$ superspace has been introduced. This approach was also generalized to construct $\mathcal{N}=3$ supergauge theory.

Solution looked extremely quite paradoxical. Instead of finding a way to directly reduce the number of fermion coordinates, it was proposed to increase the number of bosonic coordinates. However, after this, the desired reduction in the number of fermionic coordinates became possible.

- The standard $\mathcal{N}=2$ superspace with coordinates $z^M = (x^m, \theta^i_\alpha, \bar{\theta}^i_{\dot{\alpha}})$ is extended by the commuting (harmonic) variables $u^\pm_i$ $(i = 1, 2)$ parameterizing two-sphere. $\mathcal{N}=2$ harmonic superspace with coordinates (z, u).
- Supercovariant derivatives D^i_α, $\bar{D}^i_{\dot{\alpha}}$ are decomposed into $D^\pm_\alpha = u^\pm_i D^i_\alpha$, $\bar{D}^\pm_{\dot{\alpha}} = u_i{}^\pm \bar{D}^i_{\dot{\alpha}}$, $(D^i_\alpha = \frac{\partial}{\partial\theta^\alpha_i} + ...)$.
- The harmonic derivatives are introduces:

$$D^{\pm\pm} = u^{\pm i}\frac{\partial}{\partial u^{\mp i}},\ D^0 = u^{+i}\frac{\partial}{\partial u^{+i}} - u^{-i}\frac{\partial}{\partial u^i}$$

- Superfields in harmonic superspace are characterized by $U(1)$ charge, $\Phi^{(q)}(z, u)$, $D^0\Phi^{(q)} = q\Phi^{(q)}$.

- Impose the supersymmetric conditions on the superfields

$$D^+_\alpha \Phi^{(q)} = 0,\ \bar{D}^+_{\dot{\alpha}} \Phi^{(q)} = 0.$$

Such superfields depend on coordinates $(x^m_A, \theta^{+\alpha}, \bar{\theta}^{+\dot{\alpha}}, u) = (\zeta_A, u)$. Analytic superfields. The number of fermionic coordinates has decreased to their number in $\mathcal{N} = 1$ case. The price for this is an increase of bosonic coordinates.

- Construction of off shell $\mathcal{N} = 2$ supersymmetric models in terms of analytic superfields.
Hypermultiplet theory in terms of unconstrained analytic superfield q^+ with action $S_q[q^+]$ and super Yang-Mills theory in terms of unconstrained analytic superfield V^{++}, taking the values in the Lie algebra of gauge group, with action $S_{SYM}[V^{++}]$. The dependence on harmonics disappears when moving to the component formulation after using the equations of motion and gauge conditions.

- Hypermultiplet coupling to gauge multiplet is constructed

- Action of $\mathcal{N} = 4$ SYM theory in terms of $\mathcal{N} = 2$ superfields

$$S[V^{++}, q^{+}] = -\frac{1}{4g^2} \int d^8\zeta_L tr W^2 - \frac{1}{2} \int d\zeta^{(-4)} tr q^{+a} (D^{++} + igV^{++}) q^{+}_a,$$

  where W is a chiral superfield strength constructed from V^{++}. The superfields q^{+} and V^{++} belong to adjoint representation of gauge group. $d^8\zeta_L = d^4x d^2\theta^{+} d^2\theta^{-} du$, $d\zeta^{(-4)} = d^4x d^4\theta^{+} du$.

- Manifest gauge invariance, manifest $\mathcal{N} = 2$ supersymmetry and hidden $\mathcal{N} = 2$ on-shell supersymmetry rotating W and q^{+}.
- Field content of $4D$, $\mathcal{N} = 4$ SYM theory: *one vector field, three complex scalars, four Majorana fermions. All the fields are in adjoint representation. Global symmetry group (R-symmetry)* $SU(4)$.

Different formulations of $\mathcal{N} = 4$ SYM theory:

1. Component formulation. All supersymmetries are hidden.

2. Formulation in terms of $\mathcal{N} = 1$ superfields. One supersymmetry is manifest, three are hidden.

3. Formulation in terms of $\mathcal{N} = 2$ harminic superfields. Two supersymmetries are manifest, two are hidden.

4. Formulation in terms of $\mathcal{N} = 3$ harminic superfields. Three supersymmetries are manifest, one is hidden.

The harmonic superspace formulation is the best we have at the moment.

Finiteness of $\mathcal{N} = 4$ SYM theory

Analysis of divergences in $\mathcal{N} = 2$ harmonic superspace formulation.
Steps:

- Background superfield formulation. Effective action is manifestly $\mathcal{N} = 2$ supersymmetric and is manifestly invariant under the classical gauge transformations.
- $\mathcal{N} = 2$ non-renormalization theorem: any L-loop contribution to effective action is written in the form of single integral over full $\mathcal{N} = 2$ superspace. Proof is similar with $\mathcal{N} = 1$ theories.
- Power counting using background superfield formulation: the divergences are absent beyond one loop.
- One-loop divergences are directly calculated and canceled out.
- $4D$, $\mathcal{N} = 4$ SYM theory is UV finite quantum field theoretical model and hence, is conformal invariant quantum gauge theory.

- Consider $\mathcal{N}=4$ SYM theory formulated in terms of $\mathcal{N}=2$ harmonic superfields as interacting theory of $\mathcal{N}=2$ vector multiplet anf hypermultiplet.
- In general, effective action $\Gamma[V^{++},q^{+}]$. Theory is formulated within background field method that provides invariance under the classical gauge transformations. Begin with effective action in vector multiplet sector. Due to gauge invariance, the effective action is a functional of $\mathcal{N}=2$ superfield strength W. Assume $SU(2)$ gauge group spontaneously broken down to $U(1)$ (Coulomb branch). W is Abelian strength superfield.
- Approximation of slowly varying in space-time superfields. Effective action is expressed through effective Lagrangian.

$$\Gamma[V^{++}]=\int d^4xd^8\theta\mathcal{L}_{eff}.$$

$$\mathcal{L}_{eff}=\mathcal{H}(W,\bar{W})+\ldots.$$

  $\mathcal{H}$ is called non-holomorphic effective potential. Bosonic sector $\sim F^4$. Analogous to leading term in Heisenberg-Euler effective action in quantum electrodynamics.
- Dimensions: $[W]=1$, $[\mathcal{H}]=0$, $\mathcal{H}=\mathcal{H}(\frac{W}{\Lambda},\frac{\bar{W}}{\Lambda})$.

Non-holomorphic effective potential

- Due to conformal invariance of the theory, the effective action does not depend on scale Λ

$$\Lambda \frac{d}{d\Lambda} \int d^4x d^8\theta \mathcal{H}(\frac{W}{\Lambda}, \frac{\bar{W}}{\Lambda}) = 0.$$

- General solution is

$$\mathcal{H}(\frac{W}{\Lambda}, \frac{\bar{W}}{\Lambda}) = c \ln \frac{W^2}{\Lambda^2} \ln \frac{\bar{W}^2}{\Lambda^2} + \ldots.$$

- c is arbitrary numerical constant, should be found from quantum field calculations. It was done many authors in one-loop approximation.
- There are qualitative arguments that non-holomorphic potential gets neither perturbative non-perturbative contribution beyond one-loop (E. Witten, N. Seiberg, 1995; M. Dine, N. Seiberg, 1997). This was confirmed later by direct two-, three- and four-loop calculations.

Power of harmonic superspace: complete non-holomorphic effective potential

The complete low-energy action depends on all the fields of $\mathcal{N}=4$ vector multiplet, that is on V^{++} and q^+. Use the hidden $\mathcal{N}=2$ supersymmetry. Exact result (I.L.B, E.A. Ivanov, 2002):

$$\Gamma[W,\bar{W},q^+] = \int d^{12}z du[\mathcal{H}(W,\bar{W}) + \mathcal{L}_q(X)],$$

$$\mathcal{L}_q(X) = c[(X-1)\frac{\ln(1-X)}{X} + Li_2(X) - 1],$$

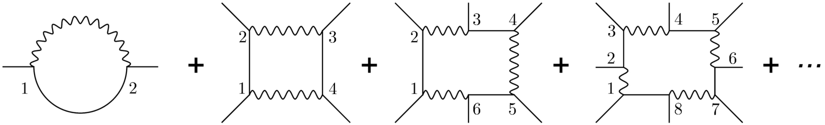
$$X = -\frac{q^{ia}q_{ia}}{W\bar{W}},$$

where $Li_2(X)$ is the Euler dilogarithm. Leading low-energy effective action is exactly found.
Bosonic sector

$$\Gamma \sim \int d^4x \frac{F^4}{(|\varphi|^2+f^2)^2},$$

where the denominator is $SU(4)$ invariant square of scalars from $\mathcal{N}=4$ vector multiplet.

Power of harmonic superspace: complete non-holomorphic effective potential, harmonic supergraph calculations.



One-loop supergraphs for complete non-holomorphic effective potential. External lines are background hypermultiplet, propagators exactly depend on constant W. Result corresponds to analysis on the base of hidden supersymmetry (I.L.B, E.A. Ivanov, A.Yu. Petrov, 2003).

Study of supersymmetric field theories in various dimensions related to superstring theory.

Specific feature of the superstring theory is existence of so called D-branes which are the $D+1$ dimensional surfaces in the ten-dimensional space-time. In the low-energy limit the D-brane is associated with $D+1$-dimensional extended supersymmetric gauge theory. Therefore, study of low-energy limit of superstring theory can be related to extended supersymmetric field theory in various dimensions.

$D3$-brane is associated with $D4, \mathcal{N}=4$ SYM theory. $D5$-brane is associated with $D6, \mathcal{N}=(1,1)$ SYM theory.

Extension of $D4, \mathcal{N}=4$ SYM theory to higher dimensions.

Higher-dimensional supergauge theories. Basic Motivation. General Problems and Results

Some problems of higher dimensional supersymmetric gauge theories.

1. Describing the quantum structure of six-dimensional supersymmetric gauge theories dimensionally reduced from superstrings.

2. Description of the interacting multiple $M5$-branes.

- Hypothetic M-theory is characterized by two extended objects: $M2$-brane and $M5$-brane in eleven dimensional space.
- The field description of interacting multiple $M2$-branes is given by Bagger-Lambert-Gustavsson (J. Bagger, N. Lambert, 2007; 2008. A. Gustavsson, 2009) theory which is $3D$, $\mathcal{N} = 8$ supersymmetric gauge theory.
- Lagrangian description of the interacting multiple $M5$-branes is not constructed so far.

3. Problem of miraculous cancelation of some on-shell divergences in higher dimensional maximally supersymmetric gauge theories (theories with 16 supercharges). All these theories are non-renormalizable by power counting.

- Field limit of superstring amplitude shows that $6D, \mathcal{N} = (1,1)$ SYM theory is on-shell finite at one-loop (M.B. Green, J.H. Schwarz, L. Brink, 1982).
- Analysis based on on-shell supersymmetries, gauge invariance and field redefinitions (P.S. Howe, K.S. Stelle, 1984, 2003; G. Bossard, P.S. Howe, K.S. Stelle, 2009).
- Direct one-loop and two-loop component calculations (mainly in on-shell and in bosonic sector (E.S. Fradkin, A.A. Tseytlin, 1983; N. Marcus, A. Sagnotti, 1984, 1985.)
- Direct calculations of on-shell scattering amplitudes in $6D, \mathcal{N} = (1,1)$ theory up to five loops and in $D8, 10$ theories up to four loops (L.V. Bork, D.I. Kazakov, M.V. Kompaniets, D.M. Tolkachev, D.E. Vlasenko, 2015).

Results: On-shell divergences in maximally extended $6D$ SYM theory start at three loops. One-shell divergences in $8D$ and $10D$ SYM theories start at one loop.

The problems we are dealing with is aimed at studying the off-shell divergence structure. To understand, what is a reason that the divergences at one and two loops are proportional to the classical equations of motion and why, starting from three loops, this is violated.

Preservation of manifest supersymmetry: off-shell superfield formulation. Best formulation for $6D$ supersymmetric gauge theories is a harmonic superspace approach. In the case of $\mathcal{N} = (1,1)$ theory it is provides explicit off-shell $\mathcal{N} = (1,0)$ supersymmetry and hidden on-shell $\mathcal{N} = (0,1)$ supersymmetry.

Preservation of classical gauge invariance in quantum theory: harmonic superfield background field method.

Preservation of explicit gauge invariance and $\mathcal{N} = (1,0)$ supersymmetry at all steps of loop calculations: superfield proper-time methods.

I.L.B., E.A. Ivanov, B.C. Merzlikin, K.V. Stepanyantz (2016 - 2024).

$6D, \mathcal{N} = (1,1)$ SYM theory, basic superfields

$6D$ superalgebra is described by two independent supercharges. The simplest representations corresponds to $\mathcal{N} = (1,0)$ and $\mathcal{N} = (0,1)$ supersymmetries. In this sense, the maximally extended rigid supergauge theory is the $\mathcal{N} = (1,1)$ SYM theory.

$\mathcal{N} = (1,0)$ harmonic superspace:
Bosonic (commuting) coordinates x^M, $(M = 0,1,2,3,4,5)$; $u^{\pm i}\,(i = 1,2)$.
Fermionic (anticommuting) coordinates θ^a_i, $(a = 1,2,3,4)$.
Analytic subspace $\zeta = ({x_A}^M,\, \theta^{\pm\, a},\, u^{\pm\, i})$.
$\theta^{\pm\, a} = \theta^a_i u^{\pm\, i}$.

- $\mathcal{N} = (1,0)$ harmonic superfields. The construction is very similar to $D4$, $\mathcal{N} = 2$ supersymmetric theory. Two basic multiplets: hypemultiplet and vector multiplet.
- Hypermultiplet is described by analytic superfield $q^+(\zeta)$. *On-shell field contents: scalar field $f^i(x)$ and the spinor field $\psi_a(x)$)*
- Vector multiplet is described by analytic superfield V^{++}. *On-shell field contents: vector field and spinor field.*
- Theory of $\mathcal{N} = (1,0)$ non-Abelian vector multiplet coupled to hypermultiplet (E.I. Ivanov, A.V. Smilga, B.M. Zupnik, 2005).

$\mathcal{N} = (1,1)$ SYM theory can be formulated in terms of $\mathcal{N} = (1,0)$ harmonic superfields as the $\mathcal{N} = (1,0)$ vector multiplet coupled to hypermultiplet in adjoint representation. The theory is manifestly $\mathcal{N} = (1,0)$ supersymmetric and possesses the extra hidden $\mathcal{N} = (0,1)$ supersymmetry.

- Action

$$S[V^{++}, q^{+}] = S_{SYM}[V^{++}] + S_{HYPER}[q^{+}, V^{++}]$$

- The action is manifestly $\mathcal{N} = (1,0)$ supersymmetric.
- The action is invariant under the transformations of extra hidden $\mathcal{N} = (0,1)$ supersymmetry

$$\delta V^{++} = \epsilon^{+} q^{+}, \qquad \delta q^{+} = -(D^{+})^{4}(\epsilon^{-} V^{--})$$

where the transformation parameter $\epsilon^{\pm}_{A} = \epsilon_{aA}\theta^{\pm A}$.

General strategy

- We start with harmonic superfield formulations of vector multiplet coupled to hypermultiplet.
- Effective action is formulated in the framework of the harmonic superfield background field method. It provides manifest $\mathcal{N} = (1, 0)$ supersymmetry and gauge invariance of effective action under the classical gauge transformations.
- Effective action can be calculated on the base of superfield proper-time technique. It provides preservation of manifest $\mathcal{N} = (1, 0)$ supersymmetry and manifest gauge invariance at all steps of calculations.
- The effective action can also be calculated perturbatively on the base of Feynman diagrams in superspace (supergraph technique).
- One-loop analysis. We study the model, where the $\mathcal{N} = (1, 0)$ vector multiplet interacts with hypermultiplet in the arbitrary representation of the gauge group. Then, we assume in the final result for one-loop divergences, that this representation is adjoint what corresponds to $\mathcal{N} = (1, 1)$ SYM theory. Finite one-loop effective action without renormalization.
- Two-loop analysis. All the possible divergences can be listed, using the the superfield power counting and then they can be calculated in the framework of the background field method.

Manifestly covariant one-loop calculation

Calculating the one-loop divergences of superfield functional determinants is carried out in the framework of proper-time technique (superfield version of Schwinger-De Witt technique). Such technique allows us to preserve the manifest gauge invariance and manifest $\mathcal{N} = (1, 0)$ supersymmetry at all steps of calculations.

General scheme of calculations

- Proper-time representation

$$Tr ln \hat{O} \sim Tr \int_0^\infty \frac{d(is)}{(is)^{1+\varepsilon}} e^{is\hat{O}_1} \delta(1,2)|_{2=1}$$

- Here s is the proper-time parameter and ε is a parameter of dimensional regularization.
- Typically the $\delta(1,2)$ contains $\delta^8(\theta_1 - \theta_2)$, which vanishes at $\theta_1 = \theta_2$
- Operator $\hat{O}$ is associated with quadratic part of action and depends on background field. This operator contains some number of spinor derivatives D^+_a, D^-_a which act on the Grassmann delta-functions $\delta^8(\theta_1 - \theta_2)$ and can kill them. Non-zero result will be only if all these δ-functions are killed.
- To get divergences, only these terms are taking into account which have the pole $\frac{1}{\varepsilon}$ after integration over proper-time. The other terms generate the finite contributions.

One-loop divergences

Results of calculations

$$\Gamma^{(1)}_{div}[V^{++}, Q^{+}] = \frac{C_2 - T(R)}{3(4\pi)^3\varepsilon} tr \int d\zeta^{(-4)} du (F^{++})^2 -$$

$$-\frac{2if^2}{(4\pi)^3\varepsilon} \int d\zeta^{(-4)} du \widetilde{Q}^{+m} (C_2 \delta_m{}^n - C(R)_m{}^n) F^{++} Q^{+}{}_n.$$

- The quantities $C_2, T(R), C(R)$ are defined as follows

$$tr(T^A T^B) = T(R)\delta^{AB}$$

$$tr(T^A_{Adj} T^B_{Adj}) = f^{ACD} f^{BCD} = C_2 \delta^{AB}$$

$$(T^A T^A)_m{}^n = C(R)_m{}^n.$$

- In $\mathcal{N} = (1,1)$ SYM theory, the hypermultiplet is in the same representation as the vector multiplet. Then $C_2 = T(R) = C(R)$. Then $\Gamma^{(1)}_{div}[V^{++}, Q^{+}] = 0$!

Procedure of calculations: gauge multiplet sector

- Two-loop divergences are calculated within background field method and proper-time technique like in one-loop case.
- We begin with only gauge multiplet background.
- Power counting shows that the only possible two-loop divergent contribution in the gauge superfield sector has the following structure

$$\Gamma^{(2)}_{\rm div}[V^{++}] = a \int d\zeta^{(-4)}\, du\, {\rm tr}\left(F^{++}\,\widehat{\Box}\,F^{++}\right)$$

  with some constant a, which diverges after removing a regularization. F^{++} is a left hand side of classical equation of motion.
- Within background field method, the two-loop contributions to superfield effective action are given by two-loop vacuum harmonic supergraphs with background field dependent lines.
- The background field dependent propagators (lines) are represented by proper-time integrals.
- Constant a in principle should have the following structure $a = \frac{d_1}{\varepsilon} + \frac{d_2}{\varepsilon^2}$ with arbitrary real parameters $d_1\, d_2$.

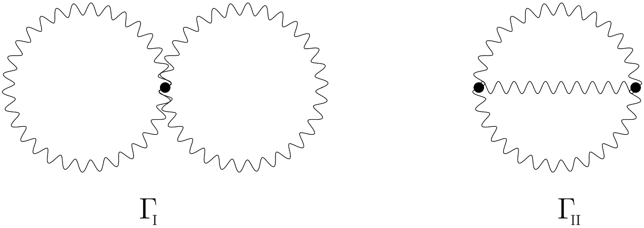


Figure: Two-loop Feynman supergraphs with gauge self-interactions vertices.

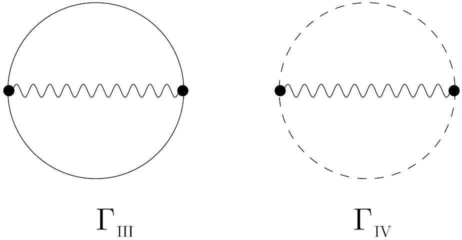


Figure: Two-loop Feynman supergraphs with hypermultiplet and ghosts vertices.

Procedure of calculations

- One can prove that the in the case under consideration the only two-loop divergent contribution comes from the '∞' supergraph.
- Contribution of this supergraph contains the product of two Green functions $G^{(2,2)}(z_1,u_1;z_2,u_2)$ at $z_1 = z_2$.
- Divergent part of such Green function can be calculated and has the form $\sim \frac{1}{\varepsilon}F^{++}$. Therefore $G^{(2,2)}(z_1,u_1;z_2,u_2)|_{z_1=z_2} \sim \frac{1}{\varepsilon}F^{++} + g^{++}$ where g^{++} is some finite functional.
- It means that full two loop contribution of the '∞' supergraph looks like

$$b\int d\zeta^{(-4)}\,du\,(\frac{1}{\varepsilon}F^{++}+g^{++})\;\widehat{\Box}\;(\frac{1}{\varepsilon}F^{++}+g^{++}).$$

  with some constant b. Therefore there are two types of contributions, one containing $\frac{1}{\varepsilon}$ and another one containing $\frac{1}{\varepsilon^2}$.
- The terms with simple pole $\frac{1}{\varepsilon}$ has the form $\sim \frac{1}{\varepsilon}F^{++}\;\widehat{\Box}\;g^{++}$.
- However, the power counting tells us that the two loop divergence has the form $\sim F^{++}\;\widehat{\Box}\;F^{++}$. Therefore, we must assume that $g^{++}=0$ or $g^{++}\sim F^{++}$.

Results of calculations in gauge multiplet sector

- Further we consider only the case $g^{++} = 0$.
- In this case, the divergent part of two-loop effective action has the form

$$\Gamma^{(2)}_{div} = \frac{8f^2}{(4\pi)^6 \varepsilon^2} (C_2)^2 \mathrm{tr} \int d\zeta^{(-4)}\, du\, F^{++}\, \widehat{\Box}\, F^{++},$$

where $F^{++} = 0$ is the classical equation of motion in the case when the hypermultiplet is absent.

- Coefficient c_2 looks like

$$c_2 = \frac{8f^2}{(4\pi)^6 \varepsilon^2} (C_2)^2.$$

Field redefinition and cancelation of divergences

- Consider the off-shell transformation of the superfield V^{++} in the classical action $V^{++} \to V^{++} - a\,\widehat{\Box}\,F^{++}$.
- The corresponding transformation of the classical action is $\delta S = -a \int d\zeta^{(-4)}\,du\,\mathrm{tr}\,F^{++}\,\widehat{\Box}\,F^{++}$. That allows to cancel completely off-shell the two-loop divergence of the effective action in the gauge multiplet sector.
- Thus, one can state that the theory under consideration is off-shell finite at one- and two-loops (at least in gauge multiplet sector).

Two-loop divergences

Hypermultiplet dependence of the two-loop divergences: indirect analysis.

- The hypermultiplet-dependent contribution to two-loop divergences can be obtained by the straightforward quantum computations of the two-loop effective action taking into account the hypermultiplet background.
- The general form of hypermultiplet dependent divergences can in principle be found without direct calculations, assuming the invariance of the effective action under the hidden $\mathcal{N} = (0,1)$ supersymmetry.
- The result has an extremely simple form

$$\Gamma^{(2)}_{\rm div}[V^{++}, q^{+}] = a \int d\zeta^{(-4)}\, du\, {\rm tr}\, E^{++}\, \widehat{\Box}\, E^{++},$$

  where $E^{++} = F^{++} + \frac{i}{2}[q^{+A}, q^{+}_{A}]$ is the left hand side of classical equation of motion for vector multiplet superfield coupled to hypermultiplet.
- Two-loop divergences vanish on-shell as expected.

Hypermultiplet dependence of the two-loop divergences: direct calculations:

$$\Gamma^{(2)}_{\rm div}[V^{++}, q^+] = \frac{f^2 (C_2)^2}{8(2\pi)^6 \epsilon^2} \int d\zeta^{(-4)}\, du\, {\rm tr}\, E^{++}\, \widehat{\Box}\, E^{++}$$

$+$terms proportional to e.o.m for hypermultiplet.

Aspects higher derivative field theories (e.g. review by A.V. Smilga, 2017). Superfield quantum formulation (I.L.B, E.A. Ivanov, B.C. Merzlikin, K.V. Stepanyantz, 2020).

The six-dimensional $\mathcal{N}=(1,0)$ supersymmetric higher-derivative gauge theory describes a self-interacting non-Abelian gauge multiplet (E.A. Ivanov, A.V. Smilga, B.M. Zupnik, 2005). Action

$$S_0 = \pm\frac{1}{2g^2}\mathrm{tr}\int d\zeta^{(-4)}du\,(F^{++})^2\,,$$

The action is invariant under the gauge transformation

$$\delta_\lambda V^{\pm\pm} = -D^{\pm\pm}\lambda - i[V^{\pm\pm},\lambda]\,, \qquad \delta_\lambda F^{++} = i[\lambda, F^{++}]\,.$$

with the Hermitian analytic superfield parameter λ
Bosonic component sector

$$S \sim \frac{1}{g^2}\mathrm{tr}\int d^6x\,(\nabla^M F_{MN})^2,$$

F_{MN} is standard Yang-Mills strength. Coupling constant g is dimensionless.

General scheme:

- Background field method
- Power counting. Theory is renormalizable.
- One-loop effective action in terms of functional determinants on superspace.
- Superfield proper-time technique
- One-loop divergences

$$\Gamma^{(1)}_{div} = -\frac{11}{3}\frac{C_2}{(4\pi)^3\varepsilon}\mathrm{tr}\int d\zeta^{(-4)}du(F^{++})^2\,.$$

- Manifestly $\mathcal{N} = (1,0)$ supersymmetric result. Bosonic and fermionic sectors are included.
- Coincides with earlier component calculations (E.A. Ivanov, A.V. Smilga, B.M. Zupnik, 2005; L. Casarin, A.A. Tseytlin, 2019).
- Asymptotic freedom at some sign in classical action. Another sign leads to null-charge problem.

Some aspects of supersymmetric quantum field theory were discussed:

- Chiral effective potential in $4D$, $\mathcal{N} = 1$ supersymmetric theories has considered.
- The construction of a non-holomorphic effective potential in $4D$, $\mathcal{N} = 4$ supersymmetric Yang-Mills theory, depending on all fields of the multiplet $\mathcal{N} = 4$ has considered.
- Structure of divergences in the $6D$, $\mathcal{N} = (1, 1)$ supersymmetric Yang-Mills theory has considered and its one-loop finiteness shown.

THANK YOU VERY MUCH!