Factorization of infrared singularities and Steinmann relations

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Contents

The study of infrared singularities in gauge theories has a long history.
It began with the "infrared catastrophe" (divergences in radiative corrections at low photon frequencies) in quantum electrodynamics (QED).
The cancellation of divergences with account of emission of any number of soft photons was demonstrated by Bloch and Nordsieck in an approximate model.
F. Bloch and A. Nordsieck, Phys. Rev. **52** (1937), 54-59.
In real QED, this cancellation was demonstrated in
D. R. Yennie, Steven C. Frautschi, H.!Suura, Annals Phys. **13** (1961) 379-452.
Note that in QED with massless electrons the singularities much stronger, since besides the divergences at low photon frequencies, so called collinear or mass singularities (divergences of integrals at sero angles between momenta of electrons and emitted by them photons) appear.

In this case, summation over the number of emitted photons in the final state is insufficient to cancel the divergences; averaging over all energy-degenerate initial states is also required

T. Kinoshita, J. Math. Phys. **3** (1962) 650–677.

T. D. Lee and M. Nauenberg, Phys. Rev. **133 B** (1964) 1549–B1562.

In quantum chromodynamics (QCD), the situation with divergence cancellation is even more complicated, even if quarks are considered massive, since gluons, whose masslessness is required by gauge invariance, have a colour charge and therefore emit. For our purposes, we only need formulas for factoring virtual corrections, and we will not discuss the problem of divergence cancellation in what follows. We will also use the term ”infrared divergences” not only for divergences in frequency of radiation but also in radiation angle, as is usually done for brevity.

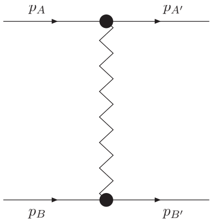
The study of the analytical properties of scattering amplitudes has an even longer history. The analytical properties of elastic scattering amplitudes have been well known since the middle of the last century, when a consistent theory of strong interactions did not exist, and the main tools for their description were dispersion relations
N. N. Bogolyubov, B. V. Medvedev and M. K. Polivanov, UCRL-TRANS-499(L).
Since there was no acceptable field theory of strong interactions before the advent of QCD, the analytic properties of amplitudes were studied in axiomatic quantum field theory.
A. S. Wightman, Phys. Rev. **101** (1956) 860-866.
H. Lehmann, K. Symanzik, W. Zimmermann, Nuovo Cim. **1** (1955) 205-225, **6** (1957) 319-332.
N. N. Bogolyubov, A. A. Logunov and I. T. Todorov, Nauka, 1969.

These properties are important for construction of the Regge-Gribov theory of complex moments J
Gribov, V. N., The theory of complex angular momenta: Gribov lectures on theoretical physics, 2003
Cambridge Monographs on Mathematical Physics Cambridge University Press.

In this theory, the asymptotics of the scattering amplitude $A(s,t)$ for $s \to \infty$ and a fixed t is determined by the position of the poles (called Reggeons) in the j-plane of the partial wave $A_l(t)$ analytically continued to complex j. The analytic properties of the scattering amplitudes allow continuation from either even or odd l, so reggeons have an additional quantum number compared to particles, the signature. The contribution of a reggeon with trajectory $\alpha \equiv \alpha(t)$ and signature σ to the amplitude of the process $AB \to A'B'$ is given by the expression

$$\mathcal{A}_{AB}^{A'B'} = \Gamma_{AA'}(t) s^{\alpha} \xi_{\alpha} \Gamma_{BB'}(t), \qquad (1)$$

where $\Gamma_{AA'}(t)$ and $\Gamma_{AA'}(t)$ are the vertices of the reggeon-particle interaction, $s = (p_A + p_B)^2, \;\; t = (p_A - p'_A)^2$, $\xi_\alpha = \frac{e^{-i\pi\alpha} + \sigma}{\sin \pi\alpha}$ – signature factor and is presented by the picture



The important thing is that the vertices of the reggeon-particle interaction are real in the region of physical momentum transfer t, so that the analytical properties of the amplitudes are ehibited explicitly in the expression (1).

The situation with the amplitudes of many-particle processes is not so good, although the increasing role of multiple production processes in strong interactions with increasing energy was recognized already in those days, see, for example,
K. A. Ter-Martirosyan, Asymptotic behaviour of essentially inelastic cross sections, Nucl.Phys, Vol. 68 (1965) p. 591-608.
and investigation of their analytical properties was started already in the sixties of the last century.
Multiparticle amplitudes are necessary for:

- direct description of processes with a large multiplicity (whose role increases with the energy of colliding particles)
- calculating amplitudes with a smaller number of particles using unitarity relations.

In both cases, their analytical properties are important.

At present, the generally accepted and widely used is the statement of absence of simultaneous discontinuities of multiparticle amplitudes in energy invariants of overlapping channels. To justify this statement, the Steinmann relations are used.

O. Steinmann, Uber den Zusammenhang zwischen den Wightmanfunktionen und den retardierten Kommutatoren, 1960, Helv. Phys. Acta , Vol. 33, p. 267-298;

O. Steinmann, Wightmanfunktionen und den retardierten Kommutatoren, 1960, Helv. Phys. Acta , Vol. 33, p. 347-362.

However, this statement is not correct and using the Steinmann relations to prove it is illegal. In the case of infrared singular parts of the amplitudes, existence of simultaneous discontinuities in overlapping channels is quite natural. But it is not limited to such parts, and is not limited to the existence of infrared singularities at all, but occurs also in their absence.

According to [Yennie:1961ad], the amplitudes of processes with an arbitrary number of particles with momenta p_i (all momenta are considered incoming) are represented as

$$A(\{p_i\}) = \exp\left\{ -\sum_{i<j} Q_i Q_j V\left(p_i, p_j\right) \right\} A_{ns}(\{p_i\}) \,,$$

$$V\left(p_i, p_j\right) = -\frac{e^2}{2}\int \frac{d^4k}{i(2\pi)^4}\frac{1}{k^2-\lambda^2+i0}\left(\frac{2p_i-k}{k^2-2(kp_i)+i0}+\frac{2p_j+k}{k^2+2(kp_j)+i0}\right)^2, \tag{2}$$

where $Q_i = 1$ for an electron (positron) in the initial (final) state and $Q_i = -1$ for an electron (positron) in the final (initial) state, λ –introduced to regularize the infrared divergence of the "photon mass", and the amplitude $A_{ns}(\{p_i\})$ is finite at $\lambda \to 0$.

The expression for $V(p_i, p_j)$ integrated over d^4k is well known. Its infrared singular part is quite simple, especially in the high-energy region of interest to us:

$$V_{sing}(p_i, p_j) \simeq \frac{\alpha}{2\pi}\left(\ln\left(\frac{-s_{ij}}{m^2}\right) - 1\right) \ln\left(\frac{m^2}{\lambda^2}\right) , \qquad (3)$$

where $s_{ij} = (p_i + p_j)^2$.
In quantum chromodynamics (QCD), factorization is complicated by the non-Abelian nature of the theory, which leads to both additional singularities and to a matrix structure of the emission vertices. By now, many papers have been published in which these singularities have been studied in QCD amplitudes. The standard set of references on this topic includes the papers

Stefano Catani, The Singular behavior of QCD amplitudes at two loop order, Phys. Lett. B, 427:161–171, 1998 ,

George F. Sterman and Maria E. Tejeda-Yeomans, Multiloop amplitudes and resummation, Phys. Lett. B, 552:48–56, 2003 ,

Lance J. Dixon, Lorenzo Magnea, and George F. Sterman, Universal structure of subleading infrared poles in gauge theory amplitudes, JHEP, 08:022, 2008 ,

Thomas Becher and Matthias Neubert, On the Structure of Infrared Singularities of Gauge-Theory Amplitudes, JHEP, 06:081, 2009, Erratum: JHEP 11, 024 (2013) ,

Einan Gardi and Lorenzo Magnea, Factorization constraints for soft anomalous dimensions in QCD scattering amplitudes, JHEP, 03:079, 2009 .

As usual in QCD, the analysis is carried out with the regularization of divergences (both ultraviolet and infrared) by the space-time dimension $D = 4 + 2\epsilon$. At present, factorization formulas representing amplitudes as a product $\mathcal{Z}\,\mathcal{H}$ are considered well established, where $\mathcal{H}$ is the so-called hard amplitude, which has no singularities in ϵ, and all singularities are contained in the factor $\mathcal{Z}$. This representation is valid in both QCD and QED, for both massless and massive electrons. In the latter case, it is equivalent to the representation of [Yennie:1961ad] with the regularization of infrared divergences by the photon mass λ. But unlike QED, in QCD the factors $\mathcal{Z}$ and $\mathcal{H}$ have a matrix structure.

For amplitudes with total number n of participating partons the factorization is written as

$$\mathcal{M}_n\left(\frac{p_i}{\mu},\alpha_s(\mu^2)\right) = \mathcal{Z}_n\left(\frac{p_i}{\mu},\alpha_s(\mu^2)\right)\mathcal{H}_n\left(\frac{p_i}{\mu},\alpha_s(\mu^2)\right) . \quad (4)$$

Here $\mathcal{H}$ is a colour vector, which is finite as $\epsilon \to 0$, and represents a matching condition, to be determined order by order in perturbation theory after the subtraction of divergent contributions. The infrared operator $\mathcal{Z}_n$, on the other hand, is an $r \times r$ matrix in colour space, generating all infrared and collinear singularities of the amplitude; it satisfies a (matrix) renormalization group equation, whose general solution can be written in the form

$$\mathcal{Z}_n\left(\frac{p_i}{\mu},\alpha_s(\mu^2)\right) = \mathcal{P}\exp\left[\frac{1}{2}\int_0^{\mu^2}\frac{d\lambda^2}{\lambda^2}\,\Gamma_n\left(\frac{p_i}{\lambda},\alpha_s(\lambda^2)\right)\right] , \quad (5)$$

where $\Gamma_n\left(\frac{p_i}{\lambda}, \alpha_s(\lambda^2)\right)$ is the soft anomalous dimension matrix and $\mathcal{P}$ denotes path ordering in colour space. For massless particles, up to two loops, the *n*-parton soft anomalous dimension matrix has a remarkably simple "dipole" form, proportional to the one-loop result, regardless of the number of partons involved. The "dipole formula" looks as

$$\Gamma_n^{\rm dip}\left(\frac{p_i}{\lambda}, \alpha_s(\lambda^2)\right) = \frac{1}{4}\gamma_K\left(\alpha_s(\lambda^2)\right)\sum_{(i,j)}\ln\left(\frac{-s_{ij}}{\lambda^2}\right)\mathbf{T}_i\cdot\mathbf{T}_j - \sum_{i=1}^{n}\gamma_i\left(\alpha_s(\lambda^2)\right), \qquad (6)$$

where $\mathbf{T}_i$ are the colour group generators for the particle i.

All dependence on on the coupling constant is contained in the anomalous dimensions

$$\gamma_K\left(\alpha_s(\lambda^2)\right), \quad \gamma_i\left(\alpha_s(\lambda^2)\right), \tag{7}$$

The dipole formula is exact at least up to two loops.
The anomalous dimensions can be extracted from form factor data.
Note that in higher orders the "dipole formula" for Γ_n must be supplemented by contributions of higher "multipoles". These are contributions containing sums over combinations of k particles with $k > 2$. The quadrupole correction first appearing in three loops was calculated relatively recently.

Ø. Almelid, C. Duhr and E. Gardi, Phys. Rev. Lett. **117** (2016) no.17, 172002

All poles in ϵ are generated through the integration of the d-dimensional running coupling down to vanishing scale, $\lambda \to 0$.

$$\int_0^{\mu^2} \frac{d\lambda^2}{\lambda^2} f(\alpha_s(\lambda^2)) = \int_0^{\alpha_s(\mu^2)} \frac{dx}{\beta(x,\epsilon)} \frac{f(x)}{x} \; , \tag{8}$$

$$\beta(x,\epsilon) = -\epsilon - \sum_{n=1}^{\infty} \beta_{n-1} \left(\frac{x}{4\pi}\right)^n \; , \quad \beta_0 = \frac{11}{3} N_c - \frac{2}{3} n_f \; , \tag{9}$$

$$\int_0^{\mu^2} \frac{d\lambda^2}{\lambda^2} \left(\frac{\alpha_s(\lambda^2)}{\pi}\right)^n = \left(\frac{\alpha_s(\mu^2)}{\pi}\right)^n \left[-\frac{1}{n\epsilon} + \left(\frac{\alpha_s(\mu^2)}{4\pi}\right) \frac{\beta_0}{(n+1)\epsilon^2} \right.$$
$$+ \left(\frac{\alpha_s(\mu^2)}{4\pi}\right)^2 \left(\frac{1}{(n+2)\epsilon^2}\right) \left(\beta_1 - \frac{\beta_0^2}{\epsilon}\right)$$
$$\left. + \left(\frac{\alpha_s(\mu^2)}{4\pi}\right)^3 \left(\frac{1}{(n+3)\epsilon^2}\right) \left(\beta_2 - \frac{2\beta_0\beta_1}{\epsilon} + \frac{\beta_0^3}{\epsilon^2}\right) \right] , \tag{10}$$

$$\int_0^{\mu^2} \frac{d\lambda^2}{\lambda^2} \left(\frac{\alpha_s(\lambda^2)}{\pi}\right)^n \ln\left(\frac{\mu^2}{\lambda^2}\right) = \left(\frac{\alpha_s(\mu^2)}{\pi}\right)^n \left[\frac{1}{(n\epsilon)^2} - \left(\frac{\alpha_s(\mu^2)}{4\pi}\right)\right.$$

$$\times \frac{\beta_0}{(n+1)\epsilon^3}\frac{2n+1}{n(n+1)} + \left(\frac{\alpha_s(\mu^2)}{4\pi}\right)^2 \frac{1}{(n+2)\epsilon^4}\left(\beta_0^2(\frac{1}{n} + \frac{1}{n+1} + \frac{1}{n+2})\right.$$

$$\left.-\epsilon\beta_1(\frac{1}{n} + \frac{1}{n+2})\right) + \left(\frac{\alpha_s(\mu^2)}{4\pi}\right)^3 \frac{1}{(n+3)\epsilon^5}\left(-\beta_0(\beta_0^2 - \epsilon\beta_1)\right.$$

$$\left.\left.\left(\frac{1}{n} + \frac{1}{(n+1)} + \frac{1}{(n+2)} + \frac{1}{(n+3)}\right) - (\beta_2\epsilon^2 - \beta_0\beta_1\epsilon)\left(\frac{1}{n} + \frac{1}{(n+3)}\right)\right)\right] \quad (11)$$

The common feature of these formulas is the representation of $\mathcal{Z}$ as an exponential with an index containing $\sum_{i<j} \ln(-(p_i + p_j)^2)$ (all momenta are considered incoming) over emitting particles i, j.
Such a representation contradicts the absence of in overlapping channels.
Since the exponent in the infrared singular factor $\mathcal{Z}$ contains the sum $\sum_{i<j} \ln(-(p_i + p_j)^2)$ over all channels, then when expanding the exponential we obtain products of powers of $\ln(-(p_i + p_j)^2$ over all channels, including overlapping ones, i.e. terms having discontinuities in overlapping channels, which contradicts the hypothesis of the absence of such discontinuities.
It contradicts also the Steinmann relations in their usual interpretation as justification of this hypothesis.
But there is a question: is this interpretation correct?

As it was said already, the Steinmann relations are used to justify the statement of absence of simultaneous discontinuities of multiparticle amplitudes in energy invariants of overlapping channels.

But originally the Steinmann relations have few common with this statement.

The Steinmann's papers were devoted to investigation of connection between two approaches to axiomatic quantum field theory: Wightman's one and the LSZ (Lehmann-Symanzik-Zimmerman) approach. In the first one, the consequences of the basic postulates of the theory for the system of the vacuum averages

$$W(x_0, ...x_{n-1}) =< A(x_0)...A(x_n)) >_0 \qquad (12)$$

of the products of field operators at arbitrary points in space-time were studied and it was shown that the system of all W uniquely defines the theory.

Unfortunately, the concept of S-matrix cannot be incorporated into this formalism. The approach of Lehman, Simanczyk, and Zimmerman is being made to construct the theory as a theory of the S-matrix. In this approach, the vacuum averages

$$r(x; x_1, x_2, ...x_n) =< R(x; x_1, x_2, ...x_n) >_0 \tag{13}$$

from retarded products of field operators were studied. For $n+1$ operators this product is defined as

$$n = 0 : \quad R(x) = A(x),$$

$$n \geq 1 : R(x; x_1, x_2, ...x_n) = (-i)^n \sum_{\mathcal{P}(x_1, x_2, ...x_n)} \theta(x - x_1)\theta(x_1 - x_2)....$$
$$\theta(x_{n-1} - x_n)[...[A(x), A(x_1)]...A(x_n)] \tag{14}$$

The connection with the S-matrix is established using the so-called asymptotic condition, a statement about the behavior of the field in the limit of $t \to \infty$. From this condition, it is possible to derive a complex system of equations

$$r(x;y,x_1,x_2,...x_n) - r(y;x,x_1,x_2,...x_n) = \sum_{i_1,...i_k} \sum_{k=0 \sum_{l=0}^{\infty}}^{n}$$

$$\frac{(-i)^l}{(n-k)!l!} \int du_1...du_l dv_1...dv_l \hat{K}_{u_l}...\hat{K}_{u_l} r(x;x_{i_1}...x_{i_k}u_1...u_l)$$

$$\Delta^+(u_1-v_1)...\Delta^+(u_l-v_l)\hat{K}_{v_1}...\hat{K}_{v_l} r(y;x_{i_{k+1}}...x_{i_n}v_1...v_l) - (x \leftrightarrow y), \quad (15)$$

where

$$\hat{K}_z = -\partial_z^2 - m^2, \;\; i\Delta^+(x-y) = \int \frac{d^3k e^{-ik(x-y)}}{(2\omega_k)(2\pi)^3}$$

The system of functions (14) as a whole also define the theory, as well as the system .

The functions $r(x; x_1, x_2, ...x_n)$ (13) can be expressed algebraically in terms of $W(x; x_1, x_2, ...x_n)$ by definition. For given $W(x; x_1, x_2, ...x_n)$ the resulting expression satisfies all required properties, except for (15, which follows from the asymptotic conditions.
On the contrary, if $\{r_n\}$ is a system of functions with the correct properties, including (15), which follows from the asymptotic conditions then the corresponding field operator, and hence the Wightman functions, can be calculated, that is, the equations are solvable with respect to $\{W_n\}$. The solvability conditions were investigated in this paper The Steinmann's papers. Actually, the problem which was considered in these papers is: under what conditions is the connection between $\{r_n\}$ and $\{W_n\}$ solvable with respect to $\{W_n\}$? Are the properties of $\{r_n\}$ without (15) sufficient? If not, what properties are needed?

The Steinmann's papers are devoted mainly to investigation of properties of the retarded commutators $\{r_n\}$ resulting from the following postulates:

1. The vectors of states form a Hilbert space with a positively defined metric.
2. The theory of invariants with respect to an inhomogeneous Lorentz proper group.
3. The theory is local, i.e. $[A(x), A(y)] = 0$ if $x - y$ is spatially similar.
4. There are no negative energy states. There is only one state Ω (vacuum) with energy 0.

Note that the postulate 4. actually forbids massless particles.

There is an additional restriction:

The Lehman asymptotic condition or similar requirements are not assumed.

Rejection of the asymptotic condition actually means rejection of consideration of the S-matrix.

The Steinmann relations were obtained for the retarded commutators $\{r_n\}$, and not for matrix elements of the S matrix. Therefore, they can not be considered as justification of the the absence of simultaneous discontinuities in overlapping channels.

Actually, if such discontinuities are infrared singular, they can not be prohibited at all in axiomatic quantum field theories It was indicated by Steinmann himself. In particular, he wrote in O. Steinmann, The Infrared Problem in Electron Scattering, Acta Phys. Austriaca Suppl. **11** (1973), 167-198

"The axiomatic way of defining S also does not work. In the known proofs of asymptotic conditions it is assumed that the particles under consideration belong to isolated one-particle hyperboloids in the energy-momentum spectrum of the relevant superselection sector. This is not the case for electrons."

Thus, firstly,
usage of the Steinmann relations as justification of the statement of the absence of simultaneous discontinuities of multiparticle amplitudes in overlapping channels is not correct.
and secondly,
the statement of the absence of simultaneous discontinuities in overlapping channels contradicts the factorization of infrared singularities.
Nevertheless, this statement is used under the name Steinmann relations in quantum chromodynamics, and in supersymmetric theories, and even in cosmology:
P. Benincasa, A. J. McLeod and C. Vergu, Steinmann Relations and the Wavefunction of the Universe, Phys. Rev. D **102** (2020), 125004.
In fact, what is meant here is not the Steinmann relations, but the statement that there are no simultaneous discontinuities in overlapping channels.

The statement of the absence of simultaneous discontinuities in overlapping channels arose during the creation of the Regge theory of multiparticle processes.

To create this theory, knowledge of the analytical properties of many-particle amplitudes was required. In the absence of any reliably established properties of these amplitudes, various models were used: the ladder model

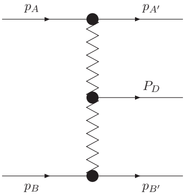
I. T. Drummond, P. V. Landshoff, and W. J. Zakrzewski, Signature in production amplitudes, Phys. Lett. B, 28:676–678, 1969,

the hybrid Gribov model,

I. T. Drummond, P. V. Landshoff, and W. J. Zakrzewski, The two-reggeon/particle coupling, Nucl. Phys. B, 11:383–405, 1969,

I. T. Drummond, Multi-reggeon behavior of production amplitudes, Phys. Rev., 176:2003–2013, 1968.,

the dual resonance model
J. H. Weis, Factorization of multi-Regge amplitudes, Phys. Rev. D, 4:1777–1787, 1971,
Carleton E. DeTar and J. H. Weis, Analytic structure of the triple-regge vertex, Phys. Rev. D, 4:3141–3161, 1971.
It was recognized that the reggeon-reggeon-particle vertex $V^{D}_{R_1 R_2}(q_1, q_2)$ in the direct generalization of the regge pole contribution to the elastic amplitude (1) for the case of the process $AB \to A'DB'$



$$\mathcal{A}_{AB}^{A'DB'} = \Gamma_{AA'}(t_1) s_1^{\alpha_1} \xi_{\alpha_1} V_{R_1 R_2}^{D}(q_1, q_2) s_2^{\alpha_2} \xi_{\alpha_2} \Gamma_{BB'}(t_2), \qquad (16)$$

in the Multi-Regge kinematics

$$s \gg s_i \gg |t_i|, \quad i = 1,2, \quad s = (p_A + p_B)^2, \quad s_1 = (p'_A + p_D)^2,$$
$$s_2 = (p'_B + p_D)^2, \quad t_1 = (p_A - p'_A)^2, t_2 = (p_B - p'_B)^2) \qquad (17)$$

has a complicated analytical structure. Finally, Regge theory for multiparticle amplitudes was built on the system of postulates. R. C. Brower, Carleton E. DeTar, and J. H. Weis, Regge Theory for Multiparticle Amplitudes, Phys. Rept., 14:257, 1974.
One of the most important postulated properties of many-particle amplitudes is the absence of simultaneous discontinuities in the squares of the invariant masses of overlapping channels
(Recall that two channels are said to overlap when they have common particles but are not subchannels of each other).

This property allows one to write a multi-Regge representation for many-particle amplitudes in a form that explicitly shows all their analytical properties. Contribution of Reggeons with trajectories $\alpha_i(t)$ and signatures τ_i to the amplitude of the process $AB \to A'CB'$
is presented in the form

$$\mathcal{A}_{AB}^{A'DB'} = \Gamma_{AA'}(t_1)\left[s^{\alpha_2}\xi_{\alpha_2}s_1^{\alpha_1-\alpha_2}\xi_{\alpha_1\alpha_2}\right]V_R(t_1,t_2,\kappa)$$

$$\times\left[s^{\alpha_1}\xi_{\alpha_1}s_2^{\alpha_2-\alpha_1}\xi_{\alpha_2\alpha_1}\right]V_L(t_1,t_2,\kappa)\Big]\Gamma_{BB'}(t_2), \qquad (18)$$

where $\kappa = \frac{s_1 s_2}{s}, \quad \xi_{\alpha_1\alpha_2} = \frac{e^{-i\pi(\alpha_1-\alpha_2)}+\tau_1\tau_2}{\sin(\pi(\alpha_1-\alpha_2))}$.
The advantage of this representation is that for $t_i < 0$ the functions V_L and V_R are real, which allows us to uniquely separate the amplitude into real and imaginary parts. Similar representations exist for the production of a larger number of particles (with the number of vertices increasing with the number of particles).

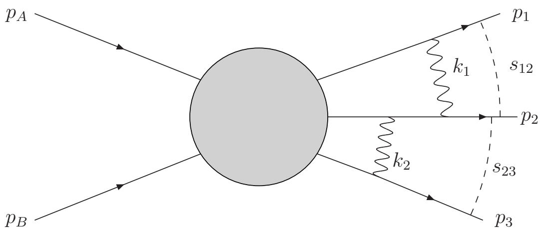
The existence of simultaneous discontinuities in the energy invariants of overlapping channels can be verified both in the presence of infrared singularities and in their absence by considering the two-loop radiative correction in QED to the process with three charged particles in the final state (as, for example, in the Bethe-Heitler process). We will consider only diagrams with insertions of photon vertices into the external lines of the Born approximation diagrams, neglecting the photon momenta in the internal lines. Note that according to the Landau criterion
[Landau:1959fi] L. D. Landau, On analytic properties of vertex parts in quantum field theory, Nucl. Phys., 13(1):181–192, 1959.
the singularities of these diagrams are contained among the singularities of the total amplitude, since the latter include the singularities of diagrams in which some of the lines are missing, i.e. the vertices they connect merge.

Let us denote the momenta of the final particles as $p_1,\ p_2,\ p_3$. For simplicity, we will assume that the masses of the particles are equal. Let

$$s_{ij} = (p_i + p_j)^2, \ \ S = (p_1 + p_2 + p_3)^2 = s_{12} + s_{13} + s_{23} - 3m^2. \quad (19)$$

Note that in the physical region (i.e. in the region where the momenta of all particles have physical meaning) the sign of s_{ij} coincides with the sign of $(p_i p_j)$, so the sign of the product $s_{12}s_{13}s_{23}$ is always positive. In this case, according to the accepted terminology, any two of these channels are overlapping. One of these channels and the s-channel are non-overlapping, since it contains all 3 particles.
Let us investigate the presence of simultaneous discontinuities in the channels s_{12} and s_{23}. Such discontinuities can only be given by diagrams in which one of the photons connects the lines of particles with momenta p_1 and p_2, and the other with momenta p_3 and p_2.

One of this diagram is presented here.



The contribution of such a diagram is proportional to the Born amplitude multiplied by the factor

$$I \equiv I(s_{12}, s_{23}, s_{13}) = \int \frac{d^4k_1}{(2\pi)^4 i} \frac{1}{d_{10}d_{11}d_{12}} J_3(\tilde{s}_{23}; \tilde{m}^2), \qquad (20)$$

where

$$\tilde{s}_{23} = (p_2 + p_3 - k_1)^2, \; \tilde{m}^2 = (p_2 - k_1)^2;$$

$$J_3(\tilde{s}_{23}; \tilde{m}^2, m^2) = \int \frac{d^4 k_2}{(2\pi)^4 i} \frac{1}{d_{20} d_{21} d_{22}}, \tag{21}$$

$$d_{10} = (k_1^2 - \lambda^2 + i0), \;\; d_{11} = ((k_1 + p_1)^2 - m^2 + i0),$$
$$d_{12} = ((k_1 - p_2)^2 - m^2 + i0), \;\; d_{20} = (k_2^2 - \lambda^2 + i0),$$
$$d_{21} = ((k_2 + p_3)^2 - m^2 + i0), \;\; d_{22} = ((k_1 + k_1 - p_2)^2 - m^2 + i0). \tag{22}$$

Let us consider the discontinuities of I with respect to the invariant s_{12}. There are only two such discontinuities: a two-particle one, which occurs due to the simultaneous vanishing of d_{11} and d_{12}, and a three-particle discontinuity, which occurs due to the simultaneous vanishing of d_{11}, d_{22}, and d_{20}. The second of them has no singularities, since the vanishing of any of the remaining denominators would contradict the law of conservation of energy-momentum. For a

$$\Delta_{s_{12}} I = \int \frac{d^4k}{(2\pi)^4 i} \frac{(2\pi i)^2 \delta((p_1+k)^2 - m^2 \delta((p_2-k)^2 - m^2))}{k^2 - \lambda^2} \times J_3(\tilde{s}_{23}; \tilde{m}^2). \quad (23)$$

We will use the Sudakov parametrization (light-cone variables). Introducing the light-cone vectors l_1 and l_2 such that

$$p_1 = l_1 + \frac{m^2}{\tilde{s}} l_2, \;\; p_2 = l_2 + \frac{m^2}{\tilde{s}} l_1, \;\; l_1^2 = 0, \;\; l_2^2 = 0, \;\; (l_1 + l_2)^2 = \tilde{s},$$

$$s_{12} = \tilde{s}(1 + \frac{m^2}{\tilde{s}})^2, \;\; \tilde{s} = s_{12} \frac{(1+v_{12})^2}{4}, \;\; v_{12} = \sqrt{1 - \frac{4m^2}{s}}, \quad (24)$$

representing k as

$$k = -\beta l_1 + \alpha l_2 + k_\perp, \;\; (k_\perp l_1) = (k_\perp l_2) = 0, \;\; k_\perp^2 \equiv -\vec{k}^2 \leq 0, \quad (25)$$

so that

$$k^2 = -\tilde{s}\alpha\beta - \vec{k}^2, \;\; (p_1 + k)^2 - m^2 = \tilde{s}\alpha(1-\beta) - m^2\beta - \vec{k}^2,$$

$$(p_2 - k)^2 - m^2 = \tilde{s}\beta(1-\alpha) - m^2\alpha - \vec{k}^2, \qquad (26)$$

and using $d^4k = \frac{\tilde{s}}{4} d\alpha d\beta d\vec{k}_\perp^2 d\phi$, where ϕ is the azimuth angle of $\vec{k}_\perp$, we obtain, passing to the integration variable $z = \beta/(1 - m^2/\tilde{s})$,

$$\Delta_{s_{12}} I = -\frac{2i}{(4\pi)^2\sqrt{s_{12}(s_{12}-4m^2)}} \int_0^1 \frac{dz}{z + \frac{\lambda^2}{s_{12}-4m^2}} \int_0^{2\pi} d\phi J_3(\tilde{s}_{23}; m^2), \qquad (27)$$

$$J_3(\tilde{s}_{23}; m^2) = -\frac{1}{(4\pi)^2}\int_0^1 dx \int_0^1 \frac{y dy}{(y^2\tilde{p}_x^2 + \lambda^2(1-y))}, \qquad (28)$$

where $\tilde{p}_x = x(p_2 - k) - (1-x)p_3$, and

$$(p_2 - k)^2 = m^2, \;\; k = -z(p_1 - p_2) + k_\perp, \;\; \vec{k}_\perp^2 = (s_{12} - 4m^2)z(1-z). \qquad (29)$$

Here it should be said that $\tilde{s}_{23} = (p_3 + p_2 - k)^2$ we must calculate in the physical region, since we use Sudakov's parametrization.
We obtain

$$\tilde{p}_x^2 = m^2 - x(1-x)(s_{13}z + s_{23}(1-z) + 2(\vec{p}_{3\perp}\vec{k}_\perp)). \qquad (30)$$

The most singular contribution to $\Delta_{s_{12}} I$ comes from $z = 0$ to $J((p_3 + p_2 - k)^2)$, so that for this contribution we have

$$\Delta_{s_{12}}^{sing} I = -\frac{4\pi i}{(4\pi)^2\sqrt{s_{12}(s_{12} - 4m^2)}} \int_0^1 \frac{dz}{z + \frac{\lambda^2}{s_{12}-4m^2}} J_3(s_{23}), . \qquad (31)$$

The presence of a discontinuity $\Delta_{s_{23}} J_3(s_{23})$ makes obvious the existence of a double discontinuity in overlapping channels.

The singular part of the double discontinuity is

$$\Delta_{s_{23}}\Delta_{s_{12}}^{sing} I = \Delta_{s_{12}} J_3(s_{12}) \Delta_{s_{23}} J_3(s_{23}) \qquad (32)$$

according to infrared factorization.
It is worth noting that the double discontinuity (by s_{12} and s_{23}) may be not only from the contribution I under consideration, but also from the contribution I' corresponding to the diagram in which the vertices of the interaction of photons with a particle with momentum p_2 change places. But the calculation of the double discontinuity depends on the order in which the discontinuities are calculated. As already mentioned, $\Delta_{s_{23}} I$ has no singularities; similarly $\Delta_{s_{12}} I'$. Therefore (32) gives the full discontinuity.

Thus, infrared singularity destroys the hypothesis of the absence of simultaneous discontinuities in overlapping channels.
The representation (27) can be used also for analysis of nonsigular contributions to double discontinuities. Using the Feynman parametrization

$$\frac{1}{d_{20}d_{21}d_{22}} = \int_0^1 dx \int_0^1 \frac{2ydy}{\left[(1-y)d_{20} + y\Big(xd_{22} + (1-x)d_{21}\Big)\right]^3} \tag{33}$$

and performing in (20) integration over d^4k_2, we obtain

$$J_3(\tilde{s}_{23}; m^2) = -\frac{1}{(4\pi)^2}\int_0^1 dx \int_0^1 \frac{ydy}{(y^2\tilde{p}_x^2 + \lambda^2(1-y))}, \tag{34}$$

where $\tilde{p}_x = x(p_2 - k) - (1 - x)p_3$,

$$(p_2-k)^2 = m^2, \quad k == -z(p_1-p_2)+k_\perp, \quad \mathbf{k}_\perp^2 = (s_{12}-4m^2)z(1-z), \tag{35}$$

so that

$$\tilde{p}_x^2 = -x(1 - x)(p_3 + p_2 - k)^2 + m^2,$$
$$(p_3 + p_2 - k)^2 = s_{13}z + s_{23}(1 - z) + 2(\mathbf{p}_{3\perp}\mathbf{k}_\perp). \tag{36}$$

Performing in (27) integration over ϕ, one has

$$\Delta_{s_{12}} I = \frac{i}{(4\pi)^3\sqrt{s_{12}(s_{12} - 4m^2)}} \int_0^1 \frac{dz}{z + \frac{\lambda^2}{s_{12}-4m^2}} \int_0^1 dx \int_0^1 \frac{ydy}{D}, \tag{37}$$

where

$$D = \sqrt{\left(\lambda^2(1 - y) + y^2\left(m^2 - x(1 - x)\left(s_{23}(1 - z) + s_{13}z\right)\right)\right)^2 - B^2}, \tag{38}$$

$$B^2 - 4(y^2x(1-x))^2\mathbf{p}_{3\perp}^2\mathbf{k}_\perp^2$$
$$= 4(y^2x(1-x))^2 z(1-z)\left[s_{23}s_{13} - \frac{m^2}{s_{12}}(s_{23}+s_{13}+s_{12}-4m^2)^2\right]. \tag{39}$$

Let us consider the simplest case of a finite "photon mass" and zero electron mass. In this case, the denominator in (37) is

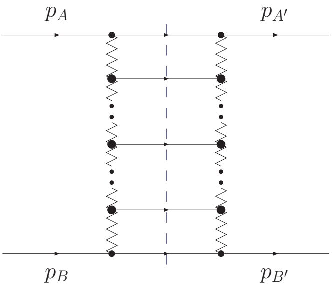
$$D_{m=0}) = \sqrt{t^2(s_{12}z - s_{23}(1-z))^2 + l^2 - 2lt(s_{12}z + s_{23}(1-z))}, \tag{40}$$

where $t = y^2x(1-x), l = \lambda^2(1-y)$. At negative s_{13} and s_{23}, as well as when s_{13} and s_{23} have different signs, the expression under the root is positive in the entire range of change of the integration variables x, y, z. At positive s_{13} and s_{23} it negative at $b_+ > s_{23} > b_-$,

$$b_{\pm} = \frac{1}{1-z}\left[\sqrt{b_0} \pm \sqrt{s_{13}z}\right]^2, \qquad (41)$$

where $b_0 = l/t$. Its negativity means that there is a discontinuity in s_{23}. At $z = 0$ and fixed x, y, the discontinuity is at a single point (delta-shaped singularity). When x, y change in the area of integration, this point runs through the entire real positive semiaxis. At $z \neq 0$, the points $b_{\pm}$, as well as the difference $b_{+} - b_{-}$, also run through all values on the real semiaxis with a change in x, y, which indicates the presence of a discontinuity at $s_{23} > 0$.

Violation of the hypothesis of the absence of simultaneous discontinuities in overlapping channels turns out to be important for further development of the BFKL approach.
The BFKL approach is founded on the gluon Reggeization.
In the dispersive method, used for the derivation of the BFKL equation, the unitarity relations are used for the calculation of imaginary parts of elastic amplitudes. Regge form of multiparticle amplitudes is used in unitarity relations.
In the unitarity relations, multiple production amplitudes in the multi-Regge kinematics (MRK) must be taken into account.
MRK is the kinematics where all particles have limited transverse momenta (with respect to momenta of colliding particle) and are combined into jets with limited invariant mass of each jet and large (increasing with s) invariant mass of any pair of jets.
The muli-Regge form was used for these amplitudes.



The s-channel discontinuity.

The fallacy of the hypothesis of the absence of simultaneous discontinuities in overlapping channels, and hence of the multi-Regge form of multiparticle amplitudes based on this hypothesis, may cast doubt on the derivation of the BFKL equation.

However, these doubts are unfounded both in the LLA and in the NLLA since in these approximations only the real part of the amplitudes included in the unitarity relations was used in deriving the BFKL equation.
It is quite clear in the LLA, where imaginary parts of the multiparticle amplitudes are neglected.
This is also true in the NLLA.
The reason is that in this approximation one of two amplitudes in the unitarity relations can lose $\ln s$, while the second one must be taken in the LLA. The LLA amplitudes are real, so that only real parts of the NLLA amplitudes are important in the unitarity relations.

Unfortunately, it is not so in the NNLLA. In this approximation two powers of $\ln s$ can be lost compared with the LLA in the product of two amplitudes in the unitarity relations. It can be done losing one $\ln s$ in each of the amplitudes, so that their analytical properties become important.

- Factorization of infrared singularities is incompatible with the statement of absence of simultaneous discontinuities in energy invariants of overlapping channels of multiparticle amplitudes.
- Steinmann relations are relations between vacuum expectation values of retarded commutators of field operators derived in axiomatic quantum field theory. They have no relations to S-matrix and perturbation theory.
- They can not be used for justification of the statement of absence of simultaneous discontinuities.
- This statement arose during creation of the Regge theory of multiparticle processes.
- Simultaneous discontinuities exist in the absence of infrared singularities as well.
- Existence of of simultaneous discontinuities means that the commonly accepted Regge form of the multiparticle amplitudes is not valid in QCD.