Monodromy of multiloop integrals in d dimensions

Roman Lee[1]

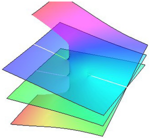
Budker Institute of Nuclear Physics, Novosibirsk

Advances in Quantum Field Theory, August 12, 2025

[1]in collaboration with Andrei Pomeransky

Motivation

- Multiloop integrals are complicated multivalues branching functions of kinematic parameters. As functions of space-time dimensions d, they are meromorphic functions on $\mathbb{C}$.



- Monodromy describes how they change under an analytic continuation along closed paths. Meanwhile, the differential equations describes how they change under an analytic continuation along any, not necessarily closed contour.
- Monodromy captures main properties of differential system with regular singular points[2] and its solutions.

Trivial monodromy group ⇔ rational function
Finite monodromy group ⇔ algebraic function
Discrete monodromy group ⇔ Calabi-Yau periods.

[2]NB: Only differential systems with regular singular points appear in multiloop calculations.

Monodromy of functions I

- Suppose we have algebraic function $y = y(x)$ satisfying a polynomial equation

$$P(y, x) = a_0(x)y^n + a_1(x)y^{n-1} + a_n(x) = 0.$$

In general it has n solutions $y_k(x)$, $(k = 1, \dots n)$. For some values $x \in D = x_1, \dots, x_k$ the distinct roots coalesce. If we move x along some path in $\overline{\mathbb{C}} \backslash D$, the roots also move in the complex plane. For a closed path they can experience permutation. Therefore, we have monodromy representation of the fundamental group $\pi_1(\overline{\mathbb{C}} \backslash D, b)$:

$$\rho : \pi_1(\overline{\mathbb{C}} \backslash D, b) \longrightarrow S_n$$

E.g., let y satisfies

$$y^2 - 2y + x = 0$$

Then $y_{1,2} = 1 \pm \sqrt{1-x}$, $D = \{1, \infty\}$. When going over a closed loop around $x = 1$ the two solutions permute and the monodromy group is S_2.

Monodromy of functions II

- Let $y(x) = \mathrm{Li}_2(x)$. This function has a branch cut $[1, \infty)$. For a loop around $x = 1$ it admits an additive term $2\pi i \ln x$, which has a branching point at $x = 0$. When analytically continued around $x = 0$ it admits a constant term $2i\pi$. So the function space is now 3-dimensional, $\boldsymbol{f} = (\mathrm{Li}_2(x), \ln x, 1)^\mathsf{T}$. The monodromies are

$$\rho(\gamma_0) = \mathcal{M}_0 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 2\pi i \\ 0 & 0 & 1 \end{pmatrix}, \quad \rho(\gamma_1) = \mathcal{M}_1 = \begin{pmatrix} 1 & 2\pi i & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$\boldsymbol{f} \xrightarrow{\gamma_0} \mathcal{M}_0 \boldsymbol{f}, \quad \boldsymbol{f} \xrightarrow{\gamma_1} \mathcal{M}_1 \boldsymbol{f}$$

Monodromy of differential systems

The same notion of monodromies works for the solutions of linear differential systems with entries being rational functions. Consider

$$\partial_x \boldsymbol{f} = M(x)\boldsymbol{f},$$

where M is a rational wrt x matrix. Suppose F is a fundamental matrix of solutions, then upon the analytic continuation along a closed path we have

$$F(x) \to F(x)\mathcal{M}_\gamma,$$

where $\mathcal{M}_\gamma$ is a monodromy matrix depending on the path γ.

Monodromy representation

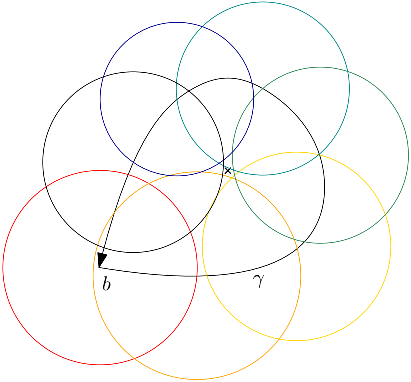
Monodromy matrix depends only on the equivalence class of γ, i.e. we have **monodromy representation of fundamental group**

$$\rho : \pi_1(\overline{\mathbb{C}}\backslash S, b) \longrightarrow GL(n, \mathbb{C})$$
$$\gamma(b, b) \longrightarrow \mathcal{M}_\gamma$$

Monodromy via analytic continuation along the path

We cover our path with pair-wise intersecting disks $D_0, \ldots, D_N$, non of which includes any singular point. In i-th disk we construct fundamental matrix $F_i(x)$ in terms of (ordinary) series expansion. We find constant matching matrices T_i, such that $F_i(x) = F_{i-1}(x) T_i$ for $x \in D_i \cap D_{i-1}$. Then

$$\mathcal{M}_\gamma = T_N \ldots T_1$$



Monodromy via evolution operator I

Fundamental matrix F can be written in terms of path-ordered exponent

$$F_b(x) = U[\gamma(x,b)] \equiv \mathrm{Pexp}\left[\int_{\gamma(b,x)} dx M(x)\right]$$

where by $\gamma(b,x)$ we denoted the path starting and ending at b and x, respectively. This solution is normalized to identity matrix at $x = b$.[3] Then the monodromies, corresponding to a loop $\gamma = \gamma(b,b)$ are

$$\mathcal{M}_\gamma = U[\gamma] = \mathrm{Pexp}\left[\int_\gamma dx M(x)\right]$$

As Pexp can rarely be calculated in closed form, there is no rigorous way to calculate monodromy representation for a given differential system. Finding monodromy is difficult!

Monodromy via evolution operator II

Suppose for convenience that we have differential system in normalized Fuchsian form:

$$\partial_x \boldsymbol{f} = \sum_{a \in S \setminus \infty} \frac{M_a}{x - x_a} \boldsymbol{f},$$

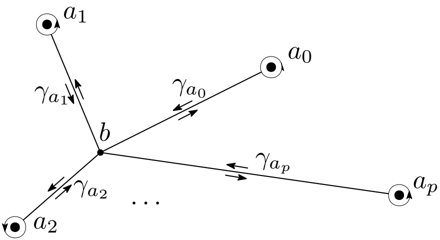
where $S = a_0, \ldots, a_p \subset \overline{\mathbb{C}}$ and the matrices M_a are free from resonance eigenvalues. The matrices $\mathcal{M}_a^0 = \exp(2\pi i M_a)$ are called *local monodromies*. Then the monodromy matrix corresponding to an elementary loop around point a is similar to $\mathcal{M}_a^0$:

$$\mathcal{M}_a = C_a^{-1} \mathcal{M}_a^0 C_a,$$

So the problem is to find *connection matrices* C_a.

Monodromy via evolution operator III

Let us consider the elementary loops as paths going along the straight line segment, loops around a singular point along a small circle, and returns back along the same line. Then the three parts of this contour exactly correspond to the product $\mathcal{M}_a = C_a^{-1}\mathcal{M}_a^0 C_a$.



Here $\mathcal{M}_a^0 = \exp(2\pi i M_a)$ are local monodromies defined above. The connection matrix are defined as

$$C_a = U(\underline{a}, b) \equiv \lim_{x \to a} (x-a)^{-M_a} \operatorname{Pexp}\left[\int_b^x dx M(x)\right]$$

[3]Strictly speaking, this defines single-valued function of x in a small vicinity of regular point b provided that we take path $\gamma(b, x)$ also belonging to this vicinity.

Differential systems with given monodromy I

So, a differential system uniquely (up to common similarity) define monodromy representation. To what extent the monodromy representation define the differential system?
This is closely related to the celebrated 21st Hilbert problem formulated in 1900

Hilbert's 21st problem

Prove that for any monodromy representation there exists a linear differential system with regular points of Fuchsian type.

The positive solution was given by Plemelj in 1908, however in late 1960s the Plemelj's prove was criticized in relation to underlined part. In 1992 Bolibrukh has found a counter-example, so the 21st problem has, in general, negative solution. But if we omit the underlined property, Plemelj's solution was correct.

Differential systems with given monodromy II

But to what extent the differential system is unique?
Let us consider two systems

$$\partial_x F_1 = M_1(x) F_1, \qquad \partial_x F_2 = M_2(x) F_2$$

with the same monodromy representation. Consider now the analitic continuation of the ratio $F_1(x)F_2^{-1}(x)$ over a closed loop. We have

$$F_1(x)F_2^{-1}(x) \xrightarrow{\gamma} F_1(x)\mathcal{M}_\gamma(F_2(x)\mathcal{M}_\gamma)^{-1} = F_1(x)F_2^{-1}(x)$$

So this ratio has trivial monodromy, thus being a rational function.

Differential system form monodromy

Monodromy representation defines differential system uniquely up to linear transformations of functions, with rational in x coefficients.

Monodromy from differential system

As we already mentioned, representing the monodromy via path-ordered exponent does not help. Note that for multiloop calculation we have additional complication: dependence on d.

As usual, if we don't have a rigorous approach, we may try to guess the correct answer. What we need:

- An ansatz for functional dependence on d.
- An ansats for suitable basis in which the monodromy matrices have the suggested form.
- An efficient way to obtain high-precision numerical results for the monodromy matrices for specified numerical value d.
- A way to recognize a function from its numerical value in a single point.

An ansatz for functional dependence on d I

Note that, in addition to differential system [Kotikov, 1991, Remiddi, 1997]

$$\partial_x \boldsymbol{j}(x,d) = M(x,d)\boldsymbol{j}(x,d)$$

there is always dimensional recurrence relations [Tarasov, 1996]

$$\boldsymbol{j}(x,d+2) = L(x,d)\boldsymbol{j}(x,d),$$

where $L(x,d)$ is a rational matrix (wrt d and x). The compatibility condition of these two equations has the form

$$\partial_x L(x,d) = M(x,d+2)L(x,d) - L(x,d)M(x,d)$$

can be explicitly check for each specific example. Then there exists a fundamental matrix of solutions $F(x,d)$ which satisfies both equations. In particular,

$$F(x,d+2) = L(x,d)F(x,d)$$

An ansatz for functional dependence on d II

Let us see how this equation evolves upon analytical continuation around a closed loop γ:

$$F(x, d+2)\rho(d+2) = L(x, d)F(x, d)\rho(d)$$

The we conclude that $\rho(d+2) = \rho(d)$, i.e., ρ is a periodic function of d (with period 2). Periodic functions of d appear also in the DRA method [RL, 2010], where the variable $z = e^{i\pi d}$ was introduced. Any function of z is periodic wrt d.

Anzats for d dependence

There is a monodromy representation in $\mathrm{M}_n(\mathbb{Q}(z))$, i.e., the representation with matrices whose entries are rational functions of $z = \exp(i\pi d)$.

An ansatz for suitable basis. I

We assume now that there is a basis in which monodromies are rational functions of z.
Suppose that there is a non-degenerate eigenvalue $\lambda = z^n$ of one of the monodromy generators. Note that the eigenvalues monodromy generators can be found from local monodromies and have the form z^k with $k \in \mathbb{Z}$. Then we can construct an appropriate basis using the eigenvector v corresponding to this eigenvalue. Namely, we act on this eigenvector by various products of monodromy generators until we find the basis. By construction, this basis will consists of vectors of rational functions in z. Then the transformation to this basic will retain rational form of the monodromy matrices.

Numerical monodromies from generalized power series

Let the system be Fuchsian at $x = a$, i.e.

$$M(x) = \frac{M_a}{x-a} + O(1)$$

Then in the vicinity of a there is a solution in the form

$$U(x, \underline{a}) = \left[\sum_{n=0}^{\infty} H_n^a \cdot (x-a)^n\right](x-a)^{M_a},$$

where H_n^a are some constant matrices.

Finite order recurrence for H_n^a

In Ref. [RL, Smirnov, and Smirnov, 2018] an efficient algorithm based on finite recurrence between the coefficients of these series was introduced. This approach allows us to calculate a lot of expansion terms in linear time, as opposed to the conventional quadratic time. For $O(10^3)$ terms that make a lot of difference!

Guessing rational function from its value in a single point.

Suppose $f(z)$ belongs to $\mathbb{Q}(z)$, i.e., can be represented as a ratio $P(z)/Q(z)$, where $P(z)$ and $Q(z)$ are polynomials with integer coefficients. Can we guess its precise term from its numerical value at one chosen point?

Yes, we can:

- Pick transcendental point $z = z_0$, e.g. $z = \pi$.
- Evaluate $f(z_0)$ with high precision.
- Using PSLQ find integer relation between numbers

$$1, z_0, z_0^2, \ldots, z_0^N, f(z_0), z_0 f(z_0) \ldots, z_0^N f(z_0),$$

  where N is sufficiently large integer number.
- Express $f(z_0)$ from this relation, replace $z_0 \to z$.

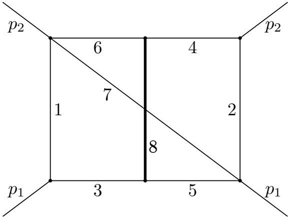
Monodromy from differential equations: heuristic algorithm

1. Pick a numeric d such that $z = \exp(i\pi d)$ is transcendental. E.g., take $d = 12/\pi$.
2. Find sufficiently many terms of generalized power series expansions at each singular point. If necessary, add expansions around some regular points, so that the union of convergence disks is connected.
3. Using these expansions, construct high-precision numeric matrices of monodromy generators along the lines described in the previous section.
4. Pick a vector v_1 corresponding to a non-degenerate eigenvalue $\propto z^k$ ($k \in \mathbb{Z}$) of one of the monodromy generators. Acting on this vector by various elements of monodromy group obtain a basis.
5. Transform the monodromy generators to this basis by applying the similarity transformation $\mathcal{M}_a \to C^{-1}\mathcal{M}_a C$ with C consisting of these vectors as columns and try to recognize their matrix elements as rational functions of z using PSLQ.

Example: maximal cut of three-loop forward box I

$$\partial_x j = \left[\frac{M_0}{x} + \frac{M_{a_+}}{x - a_+} + \frac{M_{a_-}}{x - a_-}\right] j$$

where $a_\pm = \frac{1}{2}(-11 \pm 5\sqrt{5})$. The numerical values of $a_\pm$ differ by two orders of magnitude, so we had to calculate series expansion not only at $x = \{0, a_+, a_-\}$, but in a number of intermediate points: $x = \{-37/9, -29/19, -5/9, -1/5, -1/13\}$. The number of terms in each expansion was 10^3, which gave us about 250 digit precision result for monodromies.



Elliptic sector from Higgs@N^{3}LO [Mistlberger, 2018]

Example: maximal cut of three-loop forward box II

Monodromy matrices

$$\mathcal{M}_0 = \begin{pmatrix} z & (z-1)z & (z-1)z & 1-z \\ 0 & \frac{(z+1)\left(z^2+1\right)}{z^2} & \frac{(z+1)\left(z^2+1\right)}{z^2} & -\frac{z^3+z^2+1}{z^3} \\ 0 & -1 & -1 & \frac{1}{z} \\ 0 & 1 & 0 & 0 \end{pmatrix},$$

$$\mathcal{M}_{a_+} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 1 & z^4 & z^2\left(z^3+z^2+z+2\right) & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix},$$

$$\mathcal{M}_{a_-} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ z^2+1 & -\frac{2z^3+z^2+z+1}{z} & z^4 & \frac{z^5+z^3+2z^2+1}{z} \\ 0 & 0 & 0 & 1 \end{pmatrix},$$

$$\mathcal{M}_\infty = \begin{pmatrix} \frac{1}{z} & 0 & z-1 & 0 \\ 0 & 0 & 0 & 1 \\ \frac{z^3-z^2+z+1}{z^5} & -\frac{(z+1)\left(z^2-z+2\right)}{z^5} & \frac{z^3-z^2+2z+1}{z^3} & \frac{(z+1)\left(2z^2-z+1\right)}{z^5} \\ -\frac{1}{z^2} & \frac{1}{z^2} & -1 & 0 \end{pmatrix}.$$

Conclusion

- Monodromies capture the most nontrivial information about differential systems.
- We have introduced a heuristic approach to calculate the monodromies exactly in the parameter d.
- We have checked, in particular, that found monodromies bilinear constraints dictated by twisted Riemann bilinear relations.
- Surprisingly, in all considered cases, the monodromies can be repesented as matrices whose entries belong to $\mathbb{Z}[z, 1/z]$, i.e., are Laurent polynomials of z!

Conclusion

- Monodromies capture the most nontrivial information about differential systems.
- We have introduced a heuristic approach to calculate the monodromies exactly in the parameter d.
- We have checked, in particular, that found monodromies bilinear constraints dictated by twisted Riemann bilinear relations.
- Surprisingly, in all considered cases, the monodromies can be repesented as matrices whose entries belong to $\mathbb{Z}[z, 1/z]$, i.e., are Laurent polynomials of z!

Thank you!

References

A. V. Kotikov. Differential equation method: The Calculation of N point Feynman diagrams. *Phys. Lett.*, B267:123–127, 1991. doi: 10.1016/0370-2693(91)90536-Y. [Erratum: Phys. Lett.B295,409(1992)].

Bernhard Mistlberger. Higgs boson production at hadron colliders at N^3LO in QCD. *JHEP*, 05:028, 2018. doi: 10.1007/JHEP05(2018)028.

Ettore Remiddi. Differential equations for Feynman graph amplitudes. *Nuovo Cim.*, A110:1435–1452, 1997.

RL. Space-time dimensionality d as complex variable: Calculating loop integrals using dimensional recurrence relation and analytical properties with respect to d. *Nuclear Physics B*, 830:474, 2010. ISSN 0550-3213. doi: DOI:10.1016/j.nuclphysb.2009.12.025. URL http://www.sciencedirect.com/science/article/B6TVC-4Y34PW6-2/2/bd2b4965b69dc349aa8f5f9040fc5d30.

RL, Alexander V. Smirnov, and Vladimir A. Smirnov. Solving differential equations for Feynman integrals by expansions near singular points. *JHEP*, 03:008, 2018. doi: 10.1007/JHEP03(2018)008.

O. V. Tarasov. Connection between feynman integrals having different values of the space-time dimension. *Phys. Rev. D*, 54:6479, 1996. doi: 10.1103/PhysRevD.54.6479.