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Functorial properties of Schwinger-DeWitt expansion: heat kernel for nonminimal operators

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Outline

Schwinger-DeWitt expansion for heat kernels of minimal operators

Infrared problem and functoriality of the formalism

Higher derivative minimal operators

Extension to nonminimal causal operators

Perturbation theory and subtraction procedure

Examples of Proca and vector field operators in curved spacetime

Heat kernel regularity (nondegenerate vs degenerate principal symbol)

Background (mean) field formalism

Generating functional

$$Z[J] = \int D\varphi \,\exp \frac{1}{\hbar}\Big(- S[\varphi] - \int dx\, \varphi(x) J(x)\Big)$$

Effective action

$$e^{-\Gamma[\phi]/\hbar} = \int D\varphi \,\exp \frac{1}{\hbar}\left(-S[\varphi] + \int dx\, \big(\varphi(x) - \phi(x)\big)\, \frac{\delta \Gamma[\phi]}{\delta\phi(x)}\right)$$

mean field ($\phi(x)$)

quantum field ($\varphi(x)$)

Loop expansion

$$S[\varphi] \to \Gamma[\phi] = S[\phi] + \hbar\, \Gamma_{1-\mathrm{loop}}[\phi] + \hbar^2 \Gamma_{2-\mathrm{loop}}[\phi] + \ldots$$

Inverse propagator
--- action Hessian

$$H(\nabla)\,\delta(x,y) = \frac{\delta^2 S[\phi]}{\delta\phi(x)\,\delta\phi(y)},$$

Vertices

$$S^{(n)}(x_1, \ldots x_n) = \frac{\delta^n S[\phi]}{\delta\phi(x_1)\ldots\delta\phi(x_n)}$$

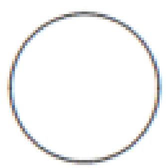
One-loop and two-loop orders

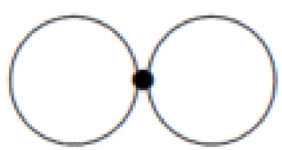
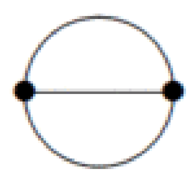
$$\Gamma_{1-\mathrm{loop}} = \frac{1}{2}\,\ln \mathrm{Det}\, F(\nabla) = \frac{1}{2}\,\mathrm{Tr}\,\ln F(\nabla),$$

$$\Gamma_{2-\mathrm{loop}} = \frac{1}{8}\int dx_1 dx_2 dx_3 dx_4\, G(x_1,x_2)\, S^{(4)}(x_1,x_2,x_3,x_4)\, G(x_3,x_4)$$

$$+\frac{1}{12}\int dx_1 dx_2 dx_3\, dy_1 dy_2 dy_3\, S^{(3)}(x_1,x_2,x_3)$$

$$\times\, G(x_1,y_1)\, G(x_2,y_2)\, G(x_3,y_3)\, S^{(3)}(y_1,y_2,y_3)$$

$\Gamma_{1-\mathrm{loop}} = \frac{1}{2}$ 

$\Gamma_{2-\mathrm{loop}} = \frac{1}{8}$  $+\ \frac{1}{12}$ 

Propagators and vertices in external background field !

Heat kernel and proper time method

Proper time method

$$G \equiv -F^{-1}(\nabla) = \int_0^\infty d\tau\, K(\tau),$$

$$\frac{1}{2}\,\mathrm{Tr}\,\ln F(\nabla) = -\frac{1}{2}\int_0^\infty \frac{d\tau}{\tau}\,\mathrm{Tr}\,K(\tau)$$

Heat kernel and its functional trace

$$\hat{K}(\tau|\,x,y) = e^{-\tau\hat{F}(\nabla)}\,\delta(x,y),$$

$$\mathrm{Tr}K(\tau) = \int dx\,\mathrm{tr}\,\hat{K}(\tau|\,x,x)$$

Operator $\hat{F}(\nabla) = F^A_B(\nabla)$ **acting in the space of fields** $\varphi = \varphi^A(x)$

Generic matrix notation $\hat{P} = P^A_B$

Covariant d'Alembertian

Minimal second order operator in curved spacetime

$$\hat{F}(\nabla) = -\Box - \hat{P} + \frac{\hat{1}}{6}R + m^2\hat{1}, \qquad \Box = g^{\mu\nu}\nabla_\mu\nabla_\nu$$

Generalized "curvatures" and covariant derivatives

$$\Re = \hat{P},\ \hat{\mathcal{R}}_{\mu\nu},\ R_{\mu\nu\alpha\beta},$$

$$[\nabla_\mu,\nabla_\nu]\,v^\alpha = R^\alpha_{\ \beta\mu\nu}v^\beta,$$

$$[\nabla_\mu,\nabla_\nu]\,\varphi = \hat{\mathcal{R}}_{\mu\nu}\,\varphi, \qquad [\nabla_\mu,\nabla_\nu]\,\hat{P} = [\hat{\mathcal{R}}_{\mu\nu},\hat{P}]$$

Heat kernel (Schwinger-DeWitt) expansion for minimal second order operators

Synge world function

$$e^{-s\hat{F}(\nabla)}\delta(x,y)=\frac{\Delta^{1/2}(x,y)}{(4\pi s)^{d/2}}\,g^{1/2}(y)\,e^{-\frac{\sigma(x,y)}{2s}}\sum_{n=0}^{\infty}s^n\,\hat{a}_n(x,y)$$

Schwinger-DeWitt (Gilkey-Seely) coefficients and their coincidence limits

$$\hat{a}_0(x,x)=\hat{1},\quad \hat{a}_1(x,x)=\hat{P},$$

$$\hat{a}_2(x,x)=\frac{1}{180}\,(R^2_{\alpha\beta\gamma\delta}-R^2_{\mu\nu}+\Box R)\,\hat{1}+\frac{1}{12}\hat{\boldsymbol{R}}^2_{\mu\nu}+\frac{1}{2}\hat{P}^2+\frac{1}{6}\Box\hat{P},\ldots$$

$$\hat{a}_n(x,x)\propto\overbrace{\nabla\cdots\nabla}^{2p}\,\overbrace{\Re\cdots\Re}^{m},\quad m+p=n$$

One-loop divergences $\quad \Gamma^{\rm div}_{\rm one-loop}=-\frac{1}{32\pi^2\varepsilon}\int d^4x\,g^{1/2}{\rm tr}\,\hat{a}_2(x,x),\quad \varepsilon=2-\frac{d}{2}\to 0$

Applicable only to minimal second order operators of the form $\hat{F}(\nabla)=-\Box+\hat{P}$ **!**

Applications to ***non-minimal and higher-derivative operators by alternative methods bypassing the use of heat kernel.***

For instance --- the method of universal functional traces (I. Jack and H. Osborn (1984), G.A.Vilkovisky & A.B., Phys. Rept. 119 (1985) 1)

$$\mathrm{Tr}\ \ln\Big(\Box^N + P(\nabla)\Big) = N\,\mathrm{Tr}\ \ln\Box + \mathrm{Tr}\ \ln\Big(1 + P(\nabla)\frac{1}{\Box^N}\Big)$$

$$= N\,\mathrm{Tr}\ \ln\Box + \mathrm{Tr}\,P(\nabla)\frac{1}{\Box^N} + \cdots$$

⇓

$$\Gamma^{\mathrm{div}} = \sum_{m,n}\int d^4x\,\mathcal{R}_n^{\mu_1\ldots\mu_m}\nabla_{\mu_1}\ldots\nabla_{\mu_m}\frac{\hat{1}}{\Box^n}\delta(x,y)\Big|_{y=x}^{\mathrm{div}}$$

↑ ***universal functional traces***

$$\nabla\ldots\nabla\frac{\hat{1}}{\Box^n}\,\delta(x,y)\Big|_{y=x}^{\mathrm{div}} = \frac{(-1)^n}{\Gamma(n)}\nabla\ldots\nabla\int_0^\infty d\tau\,\tau^{n-1}\,e^{\tau\Box}\,\hat{\delta}(x,y)\Big|_{y=x}^{\mathrm{div}}$$

↑ *Schwinger-DeWitt expansion*

UV divergences of several universal functional traces with $\Box \to \hat{F}(\nabla)$

$$\frac{\hat{1}}{F(\nabla)}\,\delta(x,y)\Big|_{y=x}^{\rm div} = -\frac{1}{16\pi^2\varepsilon}\,g^{1/2}\hat{P},$$

$$\nabla_\mu\frac{\hat{1}}{F(\nabla)}\,\delta(x,y)\Big|_{y=x}^{\rm div} = -\frac{1}{16\pi^2\varepsilon}\,g^{1/2}\Big(\frac{1}{2}\nabla_\mu\hat{P} - \frac{1}{6}\,\nabla^\nu\hat{\mathcal{R}}_{\nu\mu}\Big),$$

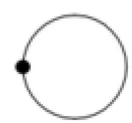
$$\frac{\hat{1}}{F^2(\nabla)}\,\delta(x,y)\Big|_{y=x}^{\rm div} = \frac{1}{16\pi^2\varepsilon}\,g^{1/2}\hat{1},$$

$$\nabla_\mu\frac{\hat{1}}{F^2(\nabla)}\,\delta(x,y)\Big|_{y=x}^{\rm div} = 0,$$

$$\nabla_\mu\nabla_\nu\frac{\hat{1}}{F^2(\nabla)}\,\delta(x,y)\Big|_{y=x}^{\rm div} = \frac{1}{16\pi^2\varepsilon}\,g^{1/2}\Big(\frac{1}{6}\,R_{\mu\nu}\hat{1} - \frac{1}{2}\,g_{\mu\nu}\hat{P} + \frac{1}{2}\,\hat{\mathcal{R}}_{\mu\nu}\Big)$$

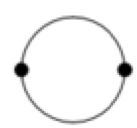
Diagrammatically – tadpoles in external background field:

$$\nabla_x\ldots\nabla_x G(x,y)\Big|_{y=x} =$$



Functional differentiation yields polarization (self-energy) operators:

$$\varPi(x,y) \equiv \frac{\delta}{\delta P(y)}\nabla_x\ldots\nabla_x G(x,y)\Big|_{y=x} = \nabla_x\ldots\nabla_x G(x,y)G(y,x) =$$



Another example of indirect methods ***--- dimensional reduction*** *on a static background with a generic 3-metric in Horava gravity*

$$\mathrm{Tr}_4 \ln\left(-\hat{1}\partial_t^2 + \hat{F}(\nabla)\right) = \int dt\, \mathrm{Tr}_3 \sqrt{\hat{F}(\nabla)}$$

A.O. Barvinsky, A.V. Kurov, S. M. Sibiryakov, Phys.Rev.D 105 (2022) 044009, arXiv:2110.14688

3D functional trace

How to proceed with the square root of the 6-th order differential operator?

$$\mathbb{F} = \sum_{a=0}^{6} \mathcal{R}_{(a)} \sum_{6\geq 2k\geq a} \alpha_{a,k} \nabla_1 ... \nabla_{2k-a} (-\Delta)^{3-k}, \quad \mathcal{R}_{(a)} = O\left(\frac{1}{l^a}\right)$$

Pseudodifferential operator – infinite series in curvature invariants $\mathcal{R}_{(a)}$

$$\sqrt{\mathbb{F}} = \sum_{a=0}^{\infty} \mathcal{R}_{(a)} \sum_{k\geq a/2}^{K_a} \tilde{\alpha}_{a,k} \nabla_1 ... \nabla_{2k-a} \frac{1}{(-\Delta)^{k-3/2}}$$

Infrared problem in massless models

IR divergent at $m \geq d/2$

$$\frac{1}{2}\,\mathrm{Tr}\,\ln F(\nabla) = -\frac{1}{2}\int_0^\infty \frac{d\tau}{\tau}\,\mathrm{Tr}\,K(\tau) = \frac{1}{(4\pi)^{d/2}}\sum_{n=0}^{\infty}\int_0^\infty d\tau\,\tau^{m-\frac{d}{2}-1}\int dx\,g^{1/2}(x)\,a_m(x,x)$$

Evisceration of DeWitt coefficients

$$a_m(x,x) = \sum \nabla^{2k}\,\mathfrak{R}^{m-k} = c_m\,\mathfrak{R}\,(-\Box)^m\,\mathfrak{R} + O[\mathfrak{R}]$$

$$\sum_m \int dx\,g^{1/2}(x)\,a_m(x,x)\tau^m = \int dx\,g^{1/2}(x)\Big[\mathfrak{R}\underbrace{\sum_m c_m(-\tau\Box)^m}_{f(-\tau\Box)}\mathfrak{R} + \dots\Big] = \int dx\,g^{1/2}(x)\Big[\mathfrak{R}\,f(-\tau\Box)\,\mathfrak{R} + \dots\Big]$$

$$\sum_m c_m\int_0^\infty d\tau\,\tau^{m-\frac{d}{2}-1}(-\Box)^m \quad\Longrightarrow\quad \int_0^\infty \frac{d\tau}{\tau^{d/2+1}}\,f(-\tau\Box) < \infty \qquad \text{IR finite}$$

We look for functorial properties of DeWitt expansion---keeping them intact: functoriality.
Avoid proper time integration in the infinite range.

Nonminimal operators

$$\hat{H}(\nabla) = H^A_B(\nabla), \qquad \hat{1} \equiv \delta^A_B$$

$$\hat{H}(\nabla) = \hat{D}^{a_1\ldots a_{2N}} \nabla_{a_1} \ldots \nabla_{a_{2N}} + O\Big(\nabla^{2N-1}\Big), \qquad \hat{D}^{a_1\ldots a_{2N}} \not\propto \hat{1}$$

Examples:

Vector field in curved spacetime

$$H^a_b(\nabla) = -\Box\delta^a_b + \alpha\nabla^a\nabla_b + R^a_b, \quad \alpha \neq 1$$

Proca operator in curved spacetime

$$H^a_b(\nabla) = -\Box\delta^a_b + \nabla^a\nabla_b + R^a_b$$

4D Horava gravity (not causal, but spatial part $D^{kl}_{ij}(\nabla)$ is O(3)-invariant)

$$\hat{H}^{kl}_{ij}(\nabla) = -\delta^{kl}_{ij}\,\partial^2_\tau + D^{kl}_{ij}(\nabla)$$

3D covariant Laplacian

$$D^{kl}_{ij}(\nabla) = -\left[\nu_5 \delta_{ij}{}^{kl} + \frac{4\nu_4(1-\lambda)+\nu_5}{1-3\lambda} g_{ij} g^{kl}\right]\Delta^3 + \left[2\nu_5 - \frac{1}{\sigma}\right]\delta^{(k}_{(i}\nabla_{j)}\nabla^{l)}\Delta^2$$
$$+ \frac{4\nu_4(1-\lambda)+\nu_5}{1-3\lambda} g_{ij}\nabla^{(k}\nabla^{l)}\Delta^2 + \left[4\nu_4 + \nu_5 + \frac{l(1+\xi)}{\sigma}\right]\nabla_{(i}\nabla_{j)} g^{kl}\Delta^2$$
$$- \left[4\nu_4 + 2\nu_5 + \frac{\xi}{\sigma}\right]\nabla_{(i}\nabla_{j)}\nabla^{(k}\nabla^{l)}\Delta + \left[\sim \text{200 lower derivative terms}\right],$$

Minimal higher-derivative operators

A.B., A.Kurov, W.Wachowski,,
arXiv:2408.10990

Generalized exponential functions

$$e^{-\frac{\sigma(x,x')}{2\tau}} \to \sum_\alpha \mathcal{E}_{N,\alpha}\Big(-\frac{\sigma(x,x')}{2\,\tau^{1/N}}\Big)$$

Split form of the 2N-th order minimal operator

$$\hat{F}(\nabla) = (\Box + P)^N + \hat{W}(\nabla), \quad \hat{W}(\nabla) = O\big[\nabla^{2N-1}\big]$$

$$\hat{K}_F(\tau|x,x') = \frac{1}{(4\pi\tau^{1/N})^{d/2}}\Big\{\sum_{m=0}^{\infty}\tau^{\frac{m}{N}}\sum_{k=0}^{\infty}\mathcal{E}_{N,\frac{d}{2}+Nk-m}\Big(-\frac{\sigma}{2\,\tau^{1/N}}\Big)\,\hat{a}_{m,k}(F|x,x') + \sum_{m=1}^{\infty}\tau^{-\frac{m}{N}}\sum_{k\geq\frac{m}{N-1}}^{\infty}\mathcal{E}_{N,\frac{d}{2}+Nk+m}\Big(-\frac{\sigma}{2\,\tau^{1/N}}\Big)\,\hat{a}_{-m,k}(F|x,x')\Big\}$$

Negative powers !

$$\hat{a}_{m,k}(F|x,x') = \sum_l \underbrace{\hat{U}_{m,k,l}(\nabla_x)}\Big[\Delta^{1/2}(x,x')\,\hat{a}_l(-\Box + P|x,x')\Big]$$

special differential operators
(fully algorithmic procedure)

SDW coefficients of $-\Box + P$

Three steps for the construction of $\left\{\hat{U}_k(\nabla)\right\}_n$

1) Multiple nested commutators:

$$\hat{V}_0(\nabla) = \hat{W}, \quad \hat{V}_k(\nabla) = \Big[\dots\Big[\Big[\hat{W}, \underbrace{\hat{H}^M\Big], \hat{H}^M\Big], \dots, \hat{H}^M}_{k}\Big], \quad k > 0$$

2) Local differential operators:

$$\hat{U}_k(\nabla) = \hat{1},$$

$$\hat{U}_k(\nabla) = \sum_{\substack{n, k_1, \cdots k_n \\ n + |\mathbf{k}| = k}} \frac{(-1)^n \, \hat{V}_{k_1}(\nabla) \cdots \hat{V}_{k_n}(\nabla)}{k_1! \cdots k_n! \, (k_1+1)(k_1+k_2+2) \cdots (k_1+\cdots+k_n+n)}, \; k > 0$$

3) n-th order ***SYNGIFICATION*** *of* $\hat{U}_k(\nabla), \quad \hat{U}_k(\nabla) \to \left\{\hat{U}_k(\nabla)\right\}_{\mathbf{n}}$:

Replacement of ***n*** *covariant derivatives in each of their monomials by Synge world function vectors* $\nabla_a \to -\nabla_a \sigma(x, x')/2$

$$\left\{\nabla_{a_1} \dots \nabla_{a_j}\right\}_n = \frac{1}{2^n n!} \frac{\partial^n}{\partial k^n} \Big(\nabla_{a_1} - k\sigma_{a_1}\Big) \dots \Big(\nabla_{a_j} - k\sigma_{a_j}\Big) \Big|_{k=0}$$

Literature on nonminimal operators:

V. P. Gusynin, *Asymptotics of the heat kernel for non-minimal differential operators, Ukr. Math. J. 43, 1432 (1991).*
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Lack of systematic approach, manifest covariance in curved spacetime, and universality

We will consider a systematic method for nonminimal operators of causal theories.

Causal theories

Principal symbol

$$\hat{D}(\nabla) \to \hat{D}(ip) = (-1)^N \hat{D}^{a_1 \ldots a_{2N}} p_{a_1} \ldots p_{a_{2N}}$$

Causality condition

$$\det \hat{D}(ip) = C(p^2)^{NM}, \quad p^2 = g^{ab} p_a p_b$$

Wave equation characteristic surface is a light cone $p^2 = 0$

Eigenvalues of the principal symbol

$$\hat{D}(ip)\varphi_i = \lambda_i (p^2)^N \varphi_i$$

Heat kernel of the operator truncated to principal symbol

$$\hat{K}_H(\tau \,|\, x, y) \simeq \int \frac{d^d p}{(2\pi)^d} e^{ip(x-y)} \times \sum_i e^{-\tau \lambda_i (p^2)^N} \hat{\Pi}_i(n), \quad n_a = \frac{p_a}{\sqrt{p^2}}$$

Projectors onto lambda-eigenspaces

$$\hat{\Pi}_i \hat{\Pi}_k = \delta_{ik} \hat{\Pi}_i, \quad \sum_i \hat{\Pi}_i = \hat{1}$$

Projector structure for spin s theory

$$\hat{\Pi}_i = \big[\, \Pi_i \big]^A_B = \hat{\pi}_i^{a_1 \ldots a_{2s}} \frac{p_{a_1} \ldots p_{a_{2s}}}{(p^2)^s}, \quad A = a_1 a_2 \cdots a_s, \; B = b_1 b_2 \cdots b_s$$

Leading order approximation

Promotion to curved spacetime

$$\delta_{ab} \to g_{ab}(x), \quad p_a \to -i\nabla_a, \quad p^2 \to -\Box = -g^{ab}\nabla_a\nabla_b$$

From momentum space to coordinate space:

$$\hat{K}_H(\tau\,|\,x,y) \simeq \int \frac{d^d p}{(2\pi)^d}\, e^{ip(x-y)} \times \sum_i e^{-\tau\lambda_i (p^2)^N}\, \hat{\Pi}_i(n) \quad \Rightarrow \quad \sum_i \hat{\pi}_i^{a_1 \dots a_{2s}} \nabla_{a_1} \dots \nabla_{a_{2s}} \frac{e^{-\tau\lambda_i(-\Box)^N}}{(-\Box)^s}$$

To generic minimal operator

$$(-\Box)^N \quad \Rightarrow \quad F = \hat{F}(\nabla) = (-\Box + \hat{P})^N = (-\Box)^N + O[\,\mathfrak{R}\,]$$

Quasi-projectors

$$\Pi_i = \hat{\Pi}_i(\nabla) = \hat{\pi}_i^{a_1 \dots a_{2s}} \nabla_{a_1} \dots \nabla_{a_{2s}} \frac{\hat{1}}{[\,F(\nabla)\,]^s}$$

$$\Pi_i \Pi_j = \delta_{ij} + O[\,\mathfrak{R}\,]$$

curvature corrections

Subtraction procedure

The goal --- to avoid IR problems of $\tau \to \infty$

source of IR divergences

$$\frac{\hat{1}}{F^s} = \frac{1}{\Gamma(s)} \int_0^\infty d\tau \, \tau^{s-1} e^{-\tau F}$$

$$K_H(\tau) \simeq \sum_i \Pi_i e^{-\tau\lambda_i F} = 1 + \sum_i \Pi_i \left(e^{-\tau\lambda_i F} - 1\right) = 1 - \sum_i \Pi_i F\lambda_i \int_0^\tau d\tau_1 \, e^{-\tau_1\lambda_i F}$$

Repeat *n* times

$$K_H(\tau) \simeq \sum_{k=0}^{n-1} \frac{(-\tau)^k}{k!} \sum_i \Pi_i (F\lambda_i)^k + (-1)^n \int_0^\tau d^n\tau \sum_i \Pi_i (F\lambda_i)^n \, e^{-\tau_n\lambda_i F}$$

$$\int_0^\tau d^n\tau \, (...) \equiv \int_0^\tau d\tau_1 \int_0^{\tau_1} d\tau_2 ... \int_0^{\tau_{n-1}} d\tau_n \, (...)$$

$$H^k - \sum_i \Pi_i (F\lambda_i)^k = O[\mathfrak{R}]$$

$$\sum_{k=0}^{n-1} \frac{(-\tau)^k}{k!} H^k$$

local differential operator

$$\Pi_i F^n = \hat{\pi}_i^{a_1 \dots a_{2s}} \nabla_{a_1} \dots \nabla_{a_{2s}} \hat{F}^{n-s}(\nabla)$$

local for n>s or n=s

Leading order approximation (not the flat-space one)

$$\mathbb{K}_s(\tau) = \sum_{k=0}^{s} \frac{(-\tau)^k}{k!} H^k + \int_0^\tau d^n\tau \sum_i \Pi_i (-F\lambda_i)^{s+1} e^{-\tau_n \lambda_i F}$$

Perturbation theory

$$\left(\frac{\partial}{\partial \tau} + H\right) \mathbb{K}_s(\tau) = -\mathbb{W}_s(\tau)$$

$$\mathbb{W}_s(\tau) = (-1)^s \int_0^\tau d^s\tau \left\{ \left[H^{s+1} - \sum_i H\, \Pi_i (F\lambda_i)^s \right] + \sum_i \left[H\Pi_i (F\lambda_i)^s - \Pi_i (F\lambda_i)^{s+1} \right] e^{-\tau_s \lambda_i F} \right\}$$

commutators $\sim \mathfrak{R}, \nabla\mathfrak{R}$

$$\mathbb{W}_s(\tau) = O[\mathfrak{R}]$$

$$\Downarrow$$

$$\mathbb{K}(\tau) = \mathbb{K}_s(\tau) + \sum_{n=1}^{\infty} \mathbb{K}_s^{(n)}(\tau)$$

$$\mathbb{K}_s^{(n)}(\tau) = \int_0^\tau d^n\tau\, \mathbb{K}_s(\tau_n)\, \mathbb{W}_s(\tau - \tau_1) \mathbb{W}_s(\tau_1 - \tau_2)\, \mathbb{W}_s(\tau_2 - \tau_3) \dots \mathbb{W}_s(\tau_{n-1} - \tau_n)$$

Typical perturbation term multilinear in operator exponent $e^{-\tau F}$

$$\mathbb{K}_s^{(n)}(\tau) \sim \int d\tau_0 \cdots d\tau_n\, K\, e^{-\tau_0 F}\, W\, e^{-\tau_1 F} \ldots W\, e^{-\tau_n F}$$

"vertices"

⇓

Convolution of kernels

$$\cdots \int d^d x'\, W K_F(\tau\,|\,x,x')\, W K_F(\tau'\,|\,x',x'') \cdots$$

INSTEAD:

Commutation of all operator exponents to the right

$$\mathbb{K}_s^{(n)}(\tau) \sim \int d\tau_0 \cdots d\tau_n\, K\, e^{-\tau_0 F}\, W\, e^{-\tau_1 F} \ldots W\, e^{-\tau_n F}$$

$$\Big[e^{-\tau F}, W\Big] = \sum_{n=1}^{\infty} \frac{(-\tau)^n}{n!} \overbrace{\Big[F,[F,\cdots[F}^{n},W]\cdots]\Big]\, e^{-\tau F}$$

$$\overbrace{\Big[F,[F,\cdots[F}^{n},W]\cdots]\Big] = O\Big[\nabla^n \mathfrak{R} + \mathfrak{R}^n\Big]$$

⇓

Linear map from minimal to nonminimal

$$e^{-\tau H(\nabla)} = \int d\tau'\, \mathfrak{B}(\tau,\tau',\mathfrak{R}\,|\,\nabla)\, e^{-\tau' F(\nabla)}$$

differential operator

Substitute Schwinger-DeWitt expansion

Proca model operator (without mass term)

$$H^a_b(\nabla) = -\Box\delta^a_b + \nabla^a\nabla_b + R^a_b$$

Choice of minimal operator *F*

$$F^a_b(\nabla) = -\Box\delta^a_b + R^a_b$$

scalar field $\Box$

$$\nabla_a F^a_b(\nabla) = -\Box\nabla_b, \quad F^a_b(\nabla)\nabla^b = -\nabla^a\Box$$

$$\nabla_a \frac{\delta^a_b}{F(\nabla)} = -\frac{1}{\Box}\nabla_b, \quad \frac{\delta^a_b}{F(\nabla)}\nabla^b = -\nabla^a\frac{1}{\Box}$$

remarkable properties

degenerate symbol

Projectors and eigenvalues of the principal symbol

$$\Pi_1 = \delta^a_b - \nabla^a\frac{1}{\Box}\nabla_b, \;\; \lambda_1 = 1, \qquad \Pi_2 = \nabla^a\frac{1}{\Box}\nabla_b, \;\; \lambda_2 = 0$$

Leading order = exact answer

$$\mathbb{K}_1(\tau) = [\,e^{-\tau F}\,]^a_b + \nabla^a\frac{1-e^{\tau\Box}}{\Box}\nabla_b = [\,e^{-\tau F}\,]^a_b + \nabla^a\int_0^\tau d\tau_1\, e^{\tau_1\Box}\nabla_b$$

$$\mathbb{W}_1 = 0$$

How important is a good choice of minimal *F* !

Smoothness of heat kernel

$$K_H(\tau\,|\,x,x') \sim \int_0^{\tau} d\tau'\, e^{-\frac{\sigma(x,x')}{2\lambda_i \tau'}}\, \tau'^{\gamma-1} \propto \left[\frac{\sigma(x,x')}{2\lambda_i}\right]^{\gamma}$$

singular for relevant and marginal EFT operators with $\gamma = m + 1 - d/2 \leq 0$

UV source of the problem

$$\mathfrak{B}(\tau,\tau',\mathfrak{R}\,|\,\nabla) \sim \nabla\nabla\cdots\nabla\delta(x,x') + \frac{\#}{\left[\sigma(x,x')\right]^{\#}}$$

singular at $x' \to x$ *(contradicts boundedness of well-defined heat kernel)*

⇩

Better subtraction scheme?

Modified subtraction scheme and perturbation theory

replace the subtracted term

$$1 \Rightarrow e^{-\beta\tau F}, \quad \beta > 0$$

$$e^{-\beta\tau F} + \left(e^{-\tau\lambda_i F} - e^{-\beta\tau F}\right) = e^{-\beta\tau F} - \int_{\beta\tau}^{\lambda_i\tau} d\tau_1\, e^{-\tau_1 F}$$

$$\mathbb{K}_s(\tau) \Rightarrow \mathbb{K}_s^{(\beta)}(\tau) = \sum_{k=0}^{s} \frac{\tau^k}{k!}(\beta F - H)^k\, e^{-\beta\tau F} + \sum_i \int_{\beta\tau}^{\lambda_i\tau} d^{s+1}\tau\, \Pi_i(-F)^{s+1}\, e^{-\tau_{s+1}F}$$

$\neq 0$

every term contains operator exponent

effects only perturbation terms

$$\frac{\partial}{\partial\beta}\mathbb{K}_s^{(\beta)}(\tau) = O[\mathfrak{R}]$$

$$\mathbb{K}_s^{(\beta)}(\tau),\ \mathbb{W}_s^{(\beta)}(\tau),\ \mathbb{K}_s^{(n,\beta)}(\tau) \Rightarrow \mathbb{K}_s^{(\beta)}(\tau),\ \mathbb{W}_s^{(\beta)}(\tau),\ \mathbb{K}_s^{(n,\beta)}(\tau)$$

Nonmimimal vector field operator with a nondegenerate symbol

$$H^a_b(\nabla) = -\Box\delta^a_b + \alpha\nabla^a\nabla_b + R^a_b, \qquad \alpha \neq 1$$

Projectors and eigenvalues

$$\Pi_1 = \delta^a_b - \nabla^a\frac{1}{\Box}\nabla_b,\;\; \lambda_1 = 1, \qquad \Pi_2 = \nabla^a\frac{1}{\Box}\nabla_b,\;\; \lambda_2 = 1-\alpha$$

Leading order is exact, choice of $\beta = 1$ (simplifies---annihilates the contribution of Π_1)

$$\Big[K_H\Big]^a_{\;b'}(\tau\,|\,x,x') = \Big[K_F\Big]^a_{\;b'}(\tau\,|\,x,x') - \nabla^a\nabla_{b'}\int\limits_\tau^{(1-\alpha)\tau} d\tau_1\, K_{-\Box}(\tau\,|\,x,x')$$

minimal part

nonminimal part

$$\int\limits_\tau^{(1-\alpha)\tau} d\tau_1\, K_{-\Box}(\tau\,|\,x,x') = \frac{\Delta^{1/2}(x,x')}{(4\pi)^{d/2}}\, g^{1/2}(x') \sum_{m=0}^{\infty} I\Big(\tau, m-\tfrac{d}{2}+1, \sigma(x,x')\Big)\, a_m(-\Box\,|\,x,x')$$

$$I\Big(\tau,\gamma,\sigma\Big) = \Big(\frac{\sigma}{2}\Big)^\gamma\Big[\Gamma\Big(-\gamma, \frac{\sigma}{2\tau(1-\alpha)}\Big) - \Gamma\Big(-\gamma, \frac{\sigma}{2\tau}\Big)\Big]$$

Regularity $$I\Big(\tau,\gamma,\sigma\Big)=\tau^{\gamma}\sum_{n=0}^{\infty}\Big(-\frac{\sigma}{2\tau}\Big)^{n}\frac{1-(1-\alpha)^{\gamma-n}}{n!(n-\gamma)},\quad \sigma\to 0$$

Coincidence limit of the full heat kernel

$$\big[K_H\big]^a_{\;b}(\tau\,|\,x,x)=\frac{g^{1/2}(x)}{(4\pi\tau)^{d/2}}\sum_{m=0}^{\infty}\tau^m\Big\{\big[a_m(F\,|\,x,x)\big]^a_{\;b}$$
$$+\Big[\frac{1-(1-\alpha)^{m-\frac{d}{2}}}{m-\frac{d}{2}}\delta^a_b+\frac{1-(1-\alpha)^{m-\frac{d}{2}+1}}{m-\frac{d}{2}+1}\tau\Big(\frac{1}{6}R^a_b+\underbrace{\nabla^a\nabla_{b'}}_{\text{derivatives}}\Big)\Big]a_m(-\Box\,|\,x,x')\Big|_{x'=x}\Big\}$$

Degenerate principal symbol case (Proca), $\alpha=1$

Heat kernel is ***singular:***

$$I\Big(\tau,\gamma,\sigma\Big)\Big|_{\alpha=1}=-\Big(\frac{\sigma}{2}\Big)^{\gamma}\Gamma\Big(-\gamma,\frac{\sigma}{2\tau}\Big)\to\infty,\ \ \sigma\to 0,\ \ \gamma=m+1-\frac{d}{2}<0$$

for UV (relevant and marginal terms)

Conclusions

Systematic approach to heat kernel of nonminimal causal operators:

principal symbol eigenvalues + auxiliary minimal operator F

$\Downarrow$

LO heat kernel + subtraction scheme + perturbation theory
+ commutator algebra of operator exponents

$\Downarrow$

direct use of Schwinger-DeWitt expansion coefficients

Unfinished issue: when F *is higher-derivative* $F=(-\Box)^N+\cdots\,,\quad N>2$

$$K_F(\tau\,|\,x,x')=\sum_{m=1}^{\infty}\frac{(\cdots)}{\tau^m}+\cdots\,,\qquad (\cdots)\Big|_{x'=x}=0$$

$K_H(\tau\,|\,x,x')\sim\nabla\cdots\nabla K_F(\tau\,|\,x,x')$, *is* $K_F(\tau\,|\,x,x)$ *regular at* $\tau\to 0$ **?**

Non-causal operators, generalized causality **?**