ОИЯИ
ОБЪЕДИНЁННЫЙ ИНСТИТУТ ЯДЕРНЫХ ИССЛЕДОВАНИЙ

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Non-Renormalizable Interactions Beyond the Leading Order

В ожидании_е.

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The questions:

- Are renormalizable theories just low-energy efficient theories, or does the principle of renormalizability limit the class of permissible interactions?
- What to do with gravity which is not renormalizable?

Non-renormalizable theories are not accepted due to:

- UV divergences are not under control - infinite number of new types of divergences
- The amplitudes increase with energy (in PT) and violate unitarity

However:

- Quantum field theory is formulated for all types of interactions independently on renormalizability
- R-operation equally works for NR theories and leads to local counter terms resulting in finite amplitudes

☑ We attempt to address these issues in NR theories and to show that one can work with NR interactions beyond tree level

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JHEP 06 (2022) 141, arXiv:2112.03091 [hep-th]
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In collaboration with

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Renormalization

Bogolyubov-Parasiuk Theorem: In any local quantum field theory to get the UV finite S-matrix one has to introduce local counter terms to the Lagrangian in each order of perturbation theory - R-operation

$$\mathcal{L} \Rightarrow \mathcal{L} + \Delta\mathcal{L}$$

BPHZ R-operation $$RG = (1-K)R'G$$

In renormalizable case this is equivalent to the operation of multiplication by a renormalization constant Z

$$Z = 1 - \sum_i KR'G_i$$

In non-renormalizable case the BP theorem is still valid and the counter terms are also local (at maximum are polynomial over momenta)

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- Multiplication operation is replaced by acting of an operator $Z \to \hat{Z}$

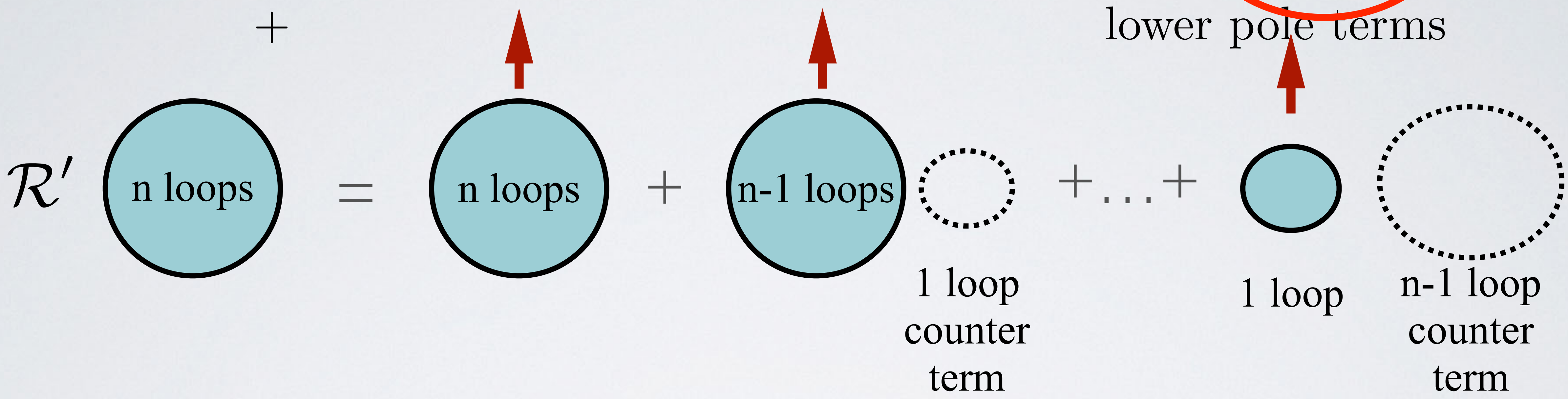
$\hat{Z}$ is a function (polynomial) of momenta (s,t,u for the 4-point case) and/or the fields

BPHZ R-operation

Locality:

$$\mathcal{R}'G_n = \frac{A_n^{(n)}(\mu^2)^{n\epsilon}}{\epsilon^n} + \frac{A_{n-1}^{(n)}(\mu^2)^{(n-1)\epsilon}}{\epsilon^n} + \ldots + \frac{A_1^{(n)}(\mu^2)^{\epsilon}}{\epsilon^n}$$

$+$ lower pole terms



$A_k^{(n)}(\mu^2)^{k\epsilon}$ terms appear after subtraction of (n-k) loop counter terms

Statement: $R'G_n$ is local, i.e. terms like $\log^k \mu^2/\epsilon^m$ should cancel for any k and m

- Due to locality all higher order divergences are related to the lower ones

The Local Counter Terms

Consequence:

Leading divergences: $$A_n^{(n)} = (-1)^{n+1}\frac{A_1^{(n)}}{n}$$ Coefficients of $1/\epsilon^n$

The leading divergences are governed by 1 loop diagrams!

SubLeading divergences: $$B_n^{(n)} = \left(\frac{2}{n(n-1)}B_2^{(n)} + \frac{2}{n}B_1^{(n)}\right)$$ Coefficients of $1/\epsilon^{n-1}$

The sub leading divergences are governed by 2 loop diagrams!

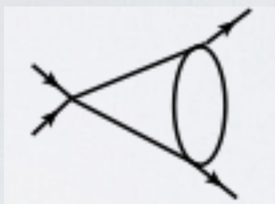
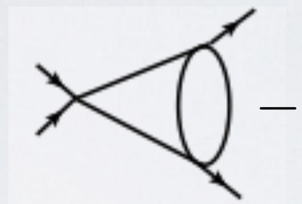
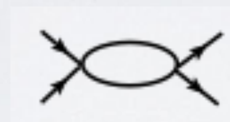
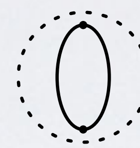
- These properties allow one to write down the recurrence relations connecting the subsequent orders of the counterterms and to evaluate them algebraically without calculating the diagrams. This can be done in renormalizable and non-renormalizable theories. The difference is a more complicated structure of these relations in NR case.

- These recurrence relations can be promoted to the RG equations for the scattering amplitudes, effective potential, etc which sum up the leading divergences (logarithms) and to find out the high energy/field behaviour

Two loop example

ϕ^4_4

 $= \left(\frac{A_2^{(2)}}{\epsilon^2} + \frac{A_1^{(2)}}{\epsilon} \right) (\frac{\mu^2}{s})^{2\epsilon}$

$\mathcal{R}'$  $=$  $-$   $= \left(\frac{A_2^{(2)}}{\epsilon^2} + \frac{A_1^{(2)}}{\epsilon} \right) (\frac{\mu^2}{s})^{2\epsilon} - \frac{A_1^{(1)}}{\epsilon} (\frac{\mu^2}{s})^{\epsilon} \frac{A_1^{(1)}}{\epsilon}$

$$= \frac{A_2^{(2)}}{\epsilon^2} - \frac{(A_1^{(1)})^2}{\epsilon^2} + 2\frac{A_2^{(2)}}{\epsilon} \log(\mu^2/s) - \frac{(A_1^{(1)})^2}{\epsilon} \log(\mu^2/s) = -\frac{1}{2}\frac{(A_1^{(1)})^2}{\epsilon^2} + \ldots$$

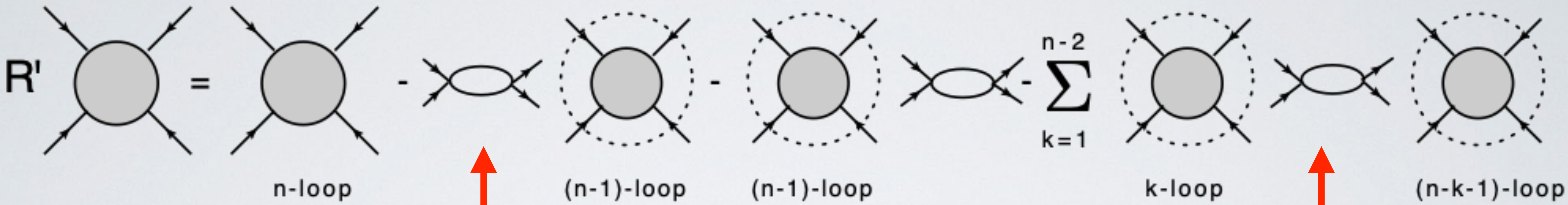
non-local terms to be cancelled

Leading divergence is given by the one-loop term $A_2^{(2)} = \frac{1}{2}(A_1^{(1)})^2$

- These statements are universal and are valid in non-renormalizable theories as well.
- The only difference is that the counter term $A_1^{(1)}$ depends on kinematics and has to be integrated through the remaining one-loop graph.
- As a result $A_2^{(2)}$ is not the square of $A_1^{(1)}$ anymore but is the integrated square .
- This last statement is the general feature of any QFT irrespective of renormalizability

Leading divergences in Scattering Amplitudes

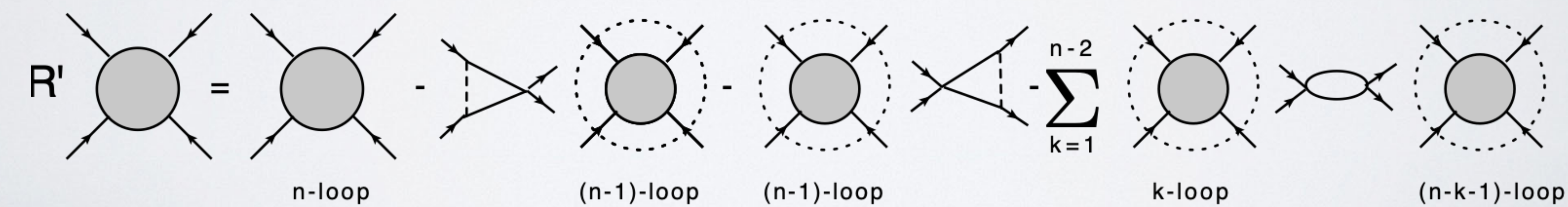
Quartic vertices



- ··· terms with higher loop remaining diagrams

$A_n^{(n)}$ $A_1^{(n)}$ $A_1^{(n)}$

Cubic vertices



- ··· terms with higher loop remaining diagrams

The Recurrence Relation for the Scattering Amplitude

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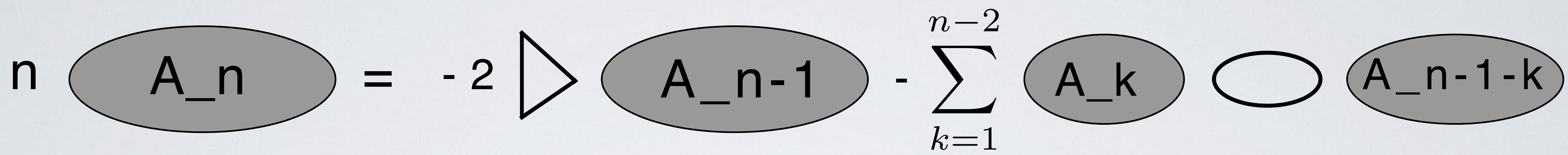

- This is the general recurrence relation that reflects the locality of the counter terms in any theory
- In renormalizable theories A_n is a constant and this relation is reduced to the algebraic one
- In non-renormalizable theories A_n depends on kinematics and one has to integrate through the one loop diagrams

Taking the sum $\sum_n A_n(-z)^n = A(z)$ one can transform the recurrence relation into integro-diff equation

$$\frac{d}{dz}A(z) = b_0\{-1 - 2\int_\triangle A(z) - \int_\bigcirc A^2(z)\}$$

$$\frac{d}{dz} = \frac{d}{d\log\mu^2}$$

This is the generalized RG equation valid in any (even non-renormalizable) theory!

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SYM_D

D=6 N=2

S-channel $S_n(s,t)$ **T-channel** $T_n(s,t)$ $T_n(s,t) = S_n(t,s)$

Exact all-loop recurrence relation $S_3 = -s/3,\ T_3 = -t/3$

$$nS_n(s,t) = -2s \int_0^1 dx \int_0^x dy\ (S_{n-1}(s,t') + T_{n-1}(s,t'))$$

$$n \geq 4$$

$$t' = t(x-y) - sy$$

D=8 N=1

S-channel $S_n(s,t)$ **T-channel** $T_n(s,t)$ $T_n(s,t) = S_n(t,s)$

Exact all-loop recurrence relation $S_1 = \frac{1}{12},\ T_1 = \frac{1}{12}$

$$nS_n(s,t) = -2s^2 \int_0^1 dx \int_0^x dy\ y(1-x)\ (S_{n-1}(s,t') + T_{n-1}(s,t'))|_{t'=tx+yu}$$

$$+ \quad s^4 \int_0^1 dx\ x^2(1-x)^2 \sum_{k=1}^{n-2} \sum_{p=0}^{2k-2} \frac{1}{p!(p+2)!} \frac{d^p}{dt'^p}(S_k(s,t') + T_k(s,t')) \times$$

$$\times \frac{d^p}{dt'^p}(S_{n-1-k}(s,t') + T_{n-1-k}(s,t'))|_{t'=-sx}\ (tsx(1-x))^p$$

RG Equation

SYM_D

D=6 N=2

$$\Sigma(s,t,z) = z^{-2}\sum_{n=3}^{\infty}(-z)^n S_n(s,t)$$

$$\frac{d}{dz}\Sigma(s,t,z) = s - \frac{2}{z}\Sigma(s,t,z) + 2s\int_0^1 dx \int_0^x dy\ (\Sigma(s,t',z)+\Sigma(t',s,z))|_{t'=xt+yu}$$

Linear equation

D=8 N=1

$$\Sigma(s,t,z) = \sum_{n=1}^{\infty}(-z)^n S_n(s,t)$$

$$\frac{d}{dz}\Sigma(s,t,z) = -\frac{1}{12} + 2s^2\int_0^1 dx \int_0^x dy\ y(1-x)\ (\Sigma(s,t',z)+\Sigma(t',s,z))|_{t'=tx+yu}$$
$$-s^4\int_0^1 dx\ x^2(1-x)^2\sum_{p=0}^{\infty}\frac{1}{p!(p+2)!}\left(\frac{d^p}{dt'^p}(\Sigma(s,t',z)+\Sigma(t',s,z))|_{t'=-sx}\right)^2\ (tsx(1-x))^p.$$

Non-linear equation

Solution of RG Equations - Genaral Case

$$\frac{d}{dz}A(z) = b_0\{-1 - 2\int_\Delta A(z) - \int_\bigcirc A^2(z)\}$$

In the r.h.s. one has a second degree polynomial:

- Two real roots - solution is an exponent (decreasing or increasing depending on a theory and kinematics) SYM_6
- Degenerate real root - solution with a pole at low (Asymptotic Freedom) or high (Zero Charge) energies depending on a kinematics ϕ^4_D
- Two complex roots - solution with infinite number of periodic poles in both directions SYM_8

Solution for the four-fermion theory in D=4 dimensions see talk by A.Borlakov

Effective Potential in Scalar Theory

Generating functional for Green functions

$$Z(J) = \int \mathcal{D}\phi \ \exp\left(i \int d^4x \ \mathcal{L}(\phi, d\phi) + J\phi \right)$$

$$W(J) = -i \log Z(J)$$ IPI generating functional

Effective action

$$\Gamma(\phi) = W(J) - \int d^4x J(x)\phi(x)$$ Legendre transformation

$$e^{i\Gamma(\Phi)} = \int \mathcal{D}\hat{\Phi} \ e^{i(S[\Phi+\hat{\Phi}]-\hat{\Phi}\Gamma'[\Phi])}$$

Shifted Classical action

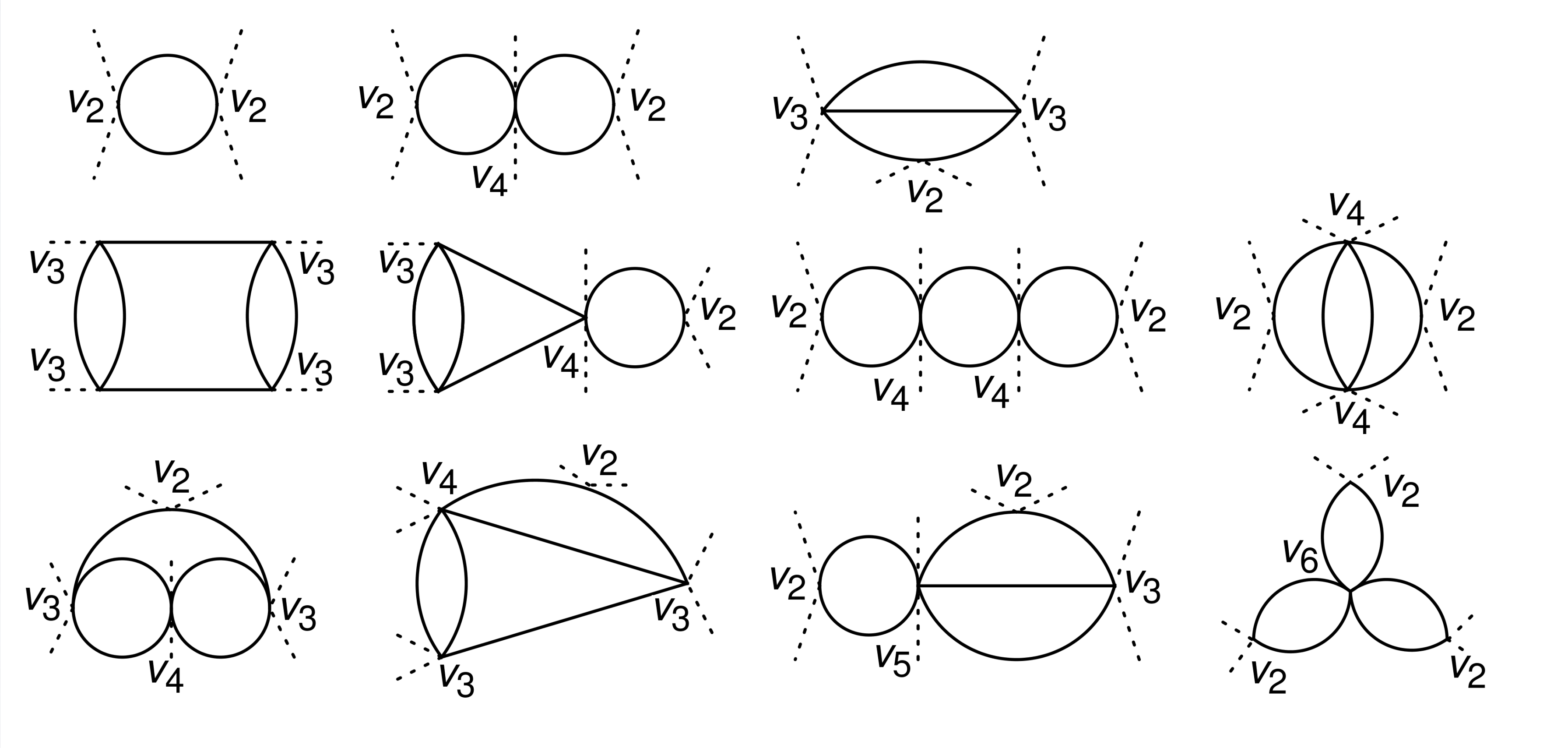
$$S[\Phi + \hat{\Phi}] = S[\Phi] + \hat{\Phi}S'[\Phi] + \frac{1}{2}\hat{\Phi}^2 S''[\Phi] + \frac{1}{3!}\hat{\Phi}^3 S'''[\Phi] + \ldots$$

Classical external field | Field dependent mass | Interaction vertex

Effective Potential in Scalar Theory

V_{eff} Is the sum of all vacuum 1PI diagrams

$$\mathcal{L} = \frac{1}{2}(\partial_\mu \phi)^2 - gV_0(\phi)$$



$$v_2(\phi) \equiv \frac{d^2 V_0(\phi)}{d\phi^2}$$

$$v_n \equiv d^n V_0 / d\phi^n$$

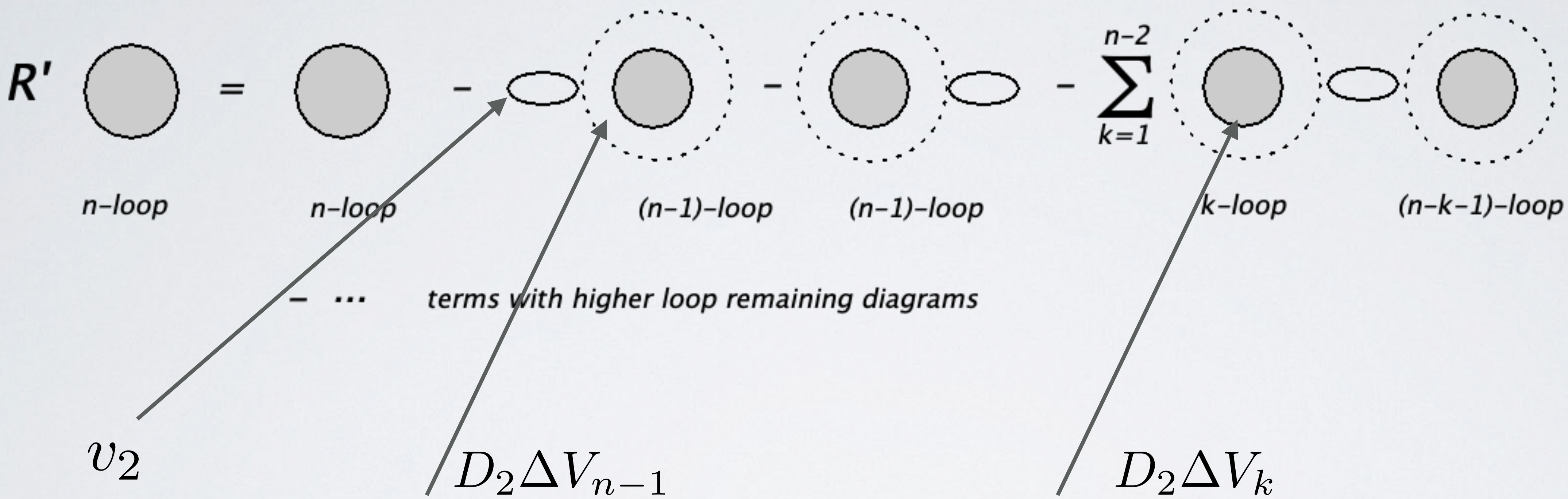
Shown are UV divergent vacuum diagrams in arbitrary scalar theory up to three loops

$$V_{eff} = g \sum_{n=0}^{\infty} (-g)^n V_n.$$

Recurrence relations for the leading poles

Kazakov, Iakhibbaev, Tolkachev 22

Action of R'-operation on divergent diagram



$$n\Delta V_n = \frac{1}{2}v_2 D_2\Delta V_{n-1} + \frac{1}{4}\sum_{k=1}^{n-2} D_2\Delta V_k D_2\Delta V_{n-1-k}, \quad n \geq 2 \quad \Delta V_1 = \frac{1}{4}v_2^2$$

$$n\Delta V_n = \frac{1}{4}\sum_{k=0}^{n-1} D_2\Delta V_k D_2\Delta V_{n-1-k}, \quad n \geq 1, \quad \Delta V_0 = V_0$$

RG pole equation for arbitrary potential

$$\Sigma(z,\phi) = \sum_{n=0}^{\infty}(-z)^n \Delta V_n(\phi) \qquad z = \frac{g}{\epsilon}$$

Kazakov, Iakhibbaev, Tolkachev 22

RG pole equation

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$$\frac{d\Sigma}{dz} = -\frac{1}{4}(D_2\Sigma)^2 \qquad \Sigma(0,\phi) = V_0(\phi)$$

This a non-linear partial differential equation!

Effective potential

$$V_{eff}(g,\phi) = g\Sigma(z,\phi)|_{z\to -\frac{g}{16\pi^2}\log g v_2/\mu^2}. \qquad v_2(\phi) \equiv \frac{d^2V_0(\phi)}{d\phi^2}$$

RG Equations in Subleading Order Scattering Amplitude

$$\frac{dS_(s,t,z)}{dz} = s(S_1 + T_1) \otimes (S_1 + T_1)$$

Coefficients of $1/\epsilon^n$

$$\frac{d^2 S_2(s,t,z)}{dz^2} = s\frac{d}{dz}(S_1 + T_1) \otimes (S_2 + T_2)$$
$$+s^2(S_1 + T_1) \otimes (S_2 + T_2) \otimes (S_1 + T_1)$$
$$+s(S_1 + T_1) \otimes \otimes (S_1 + T_1)$$

Coefficients of

$1/\epsilon^{n-1}$

- Equation for the subleading order function as well as for all the subsequent orders is always linear!
- This seems to be in contradiction with the usual RG equation but is not!

RG Equations in Subleading Order

Effective potential

Σ_1 - leading order

Σ_2 - subleading order

$$D_n \equiv \frac{\partial^n}{\partial \phi^n}$$

$$\frac{d\Sigma_1(z,\phi)}{dz} = -\frac{1}{4}\left(D_2\Sigma_1(z,\phi)\right)^2$$

$$\begin{aligned}\frac{d\Sigma_2(z,\phi)}{dz} = \; & D_2\Sigma_1(z,\phi)D_4\Sigma_2(z,\phi)D_2\Sigma_1(z,\phi) \\ + \; & D_2\Sigma_1(z,\phi)D_3\Sigma_2(z,\phi)D_3\Sigma_1(z,\phi) \\ + \; & D_2\Sigma_1(z,\phi)D_4\Sigma_1(z,\phi)D_2\Sigma_2(z,\phi)\end{aligned}$$

- Equation for the subleading order function as well as for all the subsequent orders is always linear!
- This seems to be in contradiction with the usual RG equation which is non-linear, however, the proper form of the usual RG equation is also linear!!

RG Equations in subleading orders see talk by D.Tolkachev

ϕ^4_6

Structure of UV divergences in NR Theories (Local Counter Terms)

Loop Expansion (non-renormalizable case)

UV divergences within dim reg

$$\Delta \mathcal{L}_1 \sim \lambda^2 (s+t+u)\Phi^4 (\frac{1}{\epsilon} + c_{11})$$

Momentum space

$$\Delta \mathcal{L}_1 \sim \lambda^2 \partial^2 \Phi^2 \Phi^2 (\frac{1}{\epsilon} + c_{11}),$$

Coordinate space

Four-point function →

$$\begin{array}{lllllll}
\Delta\mathcal{L} & = & \lambda^2\partial^2\Phi^2\Phi^2 + \lambda^3[\partial^4\Phi^2\Phi^2 & + & \partial^2\Phi^2\partial^2\Phi^2)] + & \lambda^4[...] + & \lambda^5[...] \\
 & & (\frac{1}{\epsilon}+c_{11}) \qquad (\frac{1}{\epsilon^2}+\frac{1}{\epsilon}+c_{12}) & & (\frac{1}{\epsilon^2}+\frac{1}{\epsilon}+c_{13}) & & \\
 & & \searrow \qquad\qquad \searrow & & & & \searrow \\
 & & \lambda^3\Phi^6 \qquad + \qquad \lambda^4[\partial^2\Phi^4\Phi^2 & + & \partial^2\Phi^2\Phi^4] & & \\
 & & (\frac{1}{\epsilon}+c_{21}) \qquad (\frac{1}{\epsilon^2}+\frac{1}{\epsilon}+c_{22}) & & (\frac{1}{\epsilon^2}+\frac{1}{\epsilon}+c_{23}) & & \\
 & & & & \searrow & & \searrow \\
 & & & & \lambda^5\Phi^8 & & \\
 & & & & (\frac{1}{\epsilon^2}+\frac{1}{\epsilon}+c_{32}), & &
\end{array}$$

Effective potential ↘

How to fix the (infinite) arbitrariness?

- In <u>renormalizable</u> models: fix the four-point function at one point - 1 constraint

$$\Gamma_4(s_0, t_0, u_0, g_0) = g_0 \longleftarrow \quad \text{fixed}$$

This allows to fix the coupling and to calculate the four-point amplitude at arbitrary kinematics and to calculate the other amplitudes.

$$\longrightarrow \quad \Gamma_4(s, t, u, g_0), \Gamma_6, \Gamma_8, \ldots$$

- In <u>non-renormalizable</u> models: fix the four-point function at some interval of s, t and u (for an analytical function this means to fix the whole function)

$$\Gamma_4(s, t, u, g) \longleftarrow \quad \text{fixed} \qquad \text{infinite \# of constraints}$$

This fixes the infinite # of arbitrary coefficients (one dimensional, the first raw), however, the whole massive of coefficients is two dimensional. To fix the other coefficients in multi-point amplitudes one considers the contributions of these amplitudes to the four-point function and figures out their values from there. Then one can calculate

$$\longrightarrow \quad \Gamma_6, \Gamma_8, \ldots$$

Resume

- **The UV divergences in non-renormalizable theories are local and can be removed by local counter terms like in renormalizable ones**

- **The main difference is that the renormalization constant Z depends on kinematics and acts like an operator rather than simple multiplication**

- **Based on locality of the counter terms due to the Bogoliubov-Parasiuk theorem one can construct the recurrence relations that define all loop divergences starting from one loop**

- **The recurrence relations can be converted into the generalized RG equations just like in renormalizable theories**

- **The RG equations allow one to sum up the leading (subleading, etc) divergences in all loops and define the high-energy/field behaviour**

- **The arbitrariness of subtractions can be reduced to ONE amplitude as a function of kinematical variables. Then the other amplitudes are calculated unambiguously.**