How the Analytic perturbation theory appears in QCD LCSRs approach and how it cures pion form factors at small transfers

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based on **[A&M&P arXiv:2505.18349];[M&P&Stefanis PRD103,096003]**

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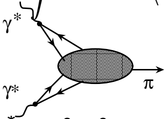
Transition and EM form factors in collinear factorization

$\boldsymbol{\gamma^*(q_1)\,\gamma^*(q_2) \to \pi^0(P)}$, Transition FF $\boldsymbol{F^{\gamma^*\gamma^*\pi}}$:

$$\int d^4z e^{-iq_1\cdot z}\langle\pi^0(P)|T\{j_\mu(z)j_\nu(0)\}|0\rangle = i\epsilon_{\mu\nu\alpha\beta}q_1^\alpha q_2^\beta \cdot \boldsymbol{F}^{\gamma^*\gamma^*\pi}(Q^2,q^2)\,,$$



where $-q_1^2 = Q^2 > 0, \;\; -q_2^2 = q^2 \geq 0$

$$\boldsymbol{F}^{\gamma^*\gamma^*\pi}(Q^2,q^2) = \boldsymbol{T^{(tw2)}(Q^2,q^2,\mu_F^2;x)} \otimes \boldsymbol{\varphi_\pi(x;\mu_F^2)} + \text{twist 4} + \text{ twist 6}$$

[M&P&Stefanis PRD(2016)], [A&M&S PRD(2018), M&P&S PRD(2021), EPJ WC(2022)]

$\boldsymbol{\gamma^*(q)\,\pi^+(P) \to \pi^+(P+q)}$, EM FF $\boldsymbol{F_\pi}$:

$$\left\langle\pi^+(P-q)\left|j_\mu^{em}\right|\pi^+(P)\right\rangle = (2P-q)_\mu \boldsymbol{F_\pi}(q^2)$$

$$\int d^4z e^{-iq\cdot z}\langle\pi^+(P)|T\{j_\mu(z)j_{5\nu}(0)\}|0\rangle = 2P_\mu P_\nu \cdot \boldsymbol{F_\pi}(Q^2,s) + \ldots$$

where $-q^2 = Q^2 > 0, \;\; s = -(P-q)^2 \geq 0$

$$\boldsymbol{F_\pi}(Q^2,s) = \boldsymbol{H^{(tw2)}(Q^2,s,\mu_F^2;x)} \otimes \boldsymbol{\varphi_\pi(x;\mu_F^2)} + \text{twist 4} + \text{twist 6}$$

We focus on the **radiative corrections** to $\boldsymbol{H}$, $(\boldsymbol{T})$ and their **summation**.

OUTLINE

1. **Intro**: Electromagnetic **(EM)** and Transition **(T)** form factors **(FF)** in Light Cone Sum Rules **(LCSR)** approach

2. **RG summations** of the initial factorized **FF**s in **twist 2**.

3. **Dispersive form for pion FFs + RG** generates in **LCSR** a "New" perturbation theory - the known **fractional APT**. Behavior of **FAPT**coupling constants.

4. Twist 4 (6) in **Light cone SR** with **FAPT**

   <u>**LCSR** with **FAPT**, results of data processing:</u>

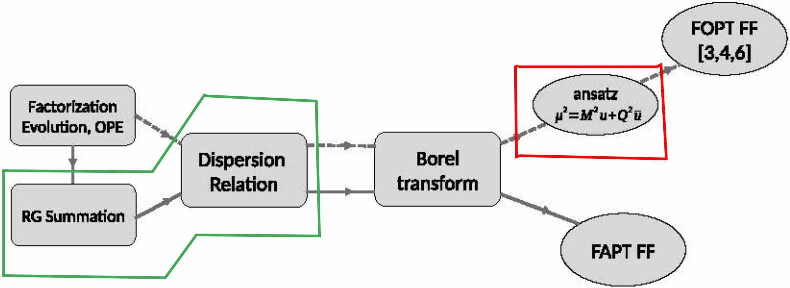
5. Determination of **twist-2 pion DA** and scales of **twist-4,6** explanation for **pion-photon TFF down to 0.35 GeV^2**

6. Predictions for **pion EM FF down to 0.35 GeV^2** **[A&M&P2025]**

7. **Conclusions**

STORE

Review: APT $\Rightarrow$ **Generalized "Fractional" APT**= **FAPT**, coupling behavior.



Factorization
Evolution, OPE
RG Summation
Dispersion
Relation
Borel
transform
ansatz
$\mu^2=M^2u+Q^2\bar{u}$
FOPT FF
[3,4,6]
FAPT FF

The general structure of pion form factors in fixed orders of pQCD

Hard process at $Q^2, s \gg m_\rho^2 \Rightarrow$ **collinear factorization**

$$(F^{\gamma^*\gamma^*\pi})F_\pi(Q^2,s) = \left(H_{\rm LO} + a_s(\mu_F^2)H_{\rm NLO} + a_s^2(\mu_F^2)H_{\rm NNLO} + \ldots\right) \otimes \varphi_\pi$$

$H_{\rm NLO}$ – **[Braun&Khodjamirian&Maul2000, Bijnens&Khod.2002,...]**,

But, we need another, **Detailed representation** for H_l:

$$H_{\rm LO} = a_s^0(\mu_F^2)\, H_0(y) \otimes \mathbb{1} \equiv x/\left(Q^2\bar{x} + xs\right)$$

$$\boxed{a_s H_{\rm NLO}} = a_s^1(\mu_F^2)\, H_0(y) \otimes \left[\mathcal{H}^{(1)} + \underline{L\, V_0}\right](y,x)\,,$$

$$a_s^2 H_{\rm NNLO} = a_s^2(\mu_F^2)\, H_0(y) \otimes \left[\mathcal{H}^{(2)} - \underline{L\,\mathcal{H}^{(1)}\beta_0} + \underline{L\,\mathcal{H}^{(1)} \otimes V_0} - \underline{\frac{L^2}{2}\beta_0 V_0} + \underline{\frac{L^2}{2} V_0 \otimes V_0} + \underline{\underline{L\, V_1}}\right](y,x)\,,$$

$L = L(y) = \ln\left[(Q^2\bar{y} + sy)/\mu_F^2\right]$, this presentation for $F^{\gamma^*\gamma^*\pi}$ in **[M&P&S 2016]**.

Plain terms $H_0(y), \mathcal{H}^{(1)}(y,x), \mathcal{H}^{(2)}(y,x)$ – corrections to parton subprocess;
Underlined terms contribute to **ERBL-factor**, V_0 - kernel or to $\bar{a}_s(y)$;
underlined term – two loops **ERBL**, V_1 - kernel.

Pion form factors in pQCD with **RG improvement 1.**

Collecting all of the "underlined" terms of RG-evolution into

$a_s(\mu^2) \to \bar{a}_s(y) \equiv \bar{a}_s\left(s\bar{y} + Q^2 y\right)$ and **ERBL–factor** **[AMS2018,TFF]**.

$$F_\pi(Q^2,s) = H_0(y) \underset{y}{\otimes} \left\{ \left[1 + \bar{a}_s(y)\mathcal{H}^{(1)}(y,x) + \ldots\right] \underset{x}{\otimes} \exp\left[-\int_{a_s(\mu^2)}^{\bar{a}_s(y)} d\alpha \frac{V(\alpha;x,z)}{\beta(\alpha)}\right] \right\} \underset{z}{\otimes} \varphi_\pi^{(2)}(z,\mu^2),$$

For **unit vectors** of Gegenbauer basis $\{\psi_n \equiv 6x(1-x)C_n^{3/2}(2x-1)\}$:

$$\varphi_\pi^{(2)}(x,\mu^2) = \psi_0(x) + \sum_{n=2,4,\ldots}^{\infty} b_n(\mu^2)\,\psi_n(x), \quad V_0(x,z) \underset{z}{\otimes} \psi_n(z) = -\gamma_n\,\psi_n(x).$$

$$F(Q^2,s) = F_0^{\text{RG}}(Q^2,s) + \sum_{n=2,4,\ldots}^{\infty} b_n(\mu^2) F_n^{\text{RG}}(Q^2,s)$$

One loop resumed result in leading Logs, $\nu_n = \gamma_n/2\beta_0$

$$F_n^{\text{RG}}(Q^2,s) = H_0(y) \underset{y}{\otimes} \left\{ \left[\mathbb{1} + \bar{a}_s(y)\mathcal{H}^{(1)}(y,x)\right] \left(\frac{\bar{a}_s(y)}{a_s(\mu^2)}\right)^{\nu_n} \right\} \underset{x}{\otimes} \psi_n(x)$$

Finally, All the Logs $= L^n$ are accumulated just in $\bar{a}_s^{\nu}(y)$

Pion form factors in pQCD with **RG improvement 2.**

Collecting all of the <u>terms</u> of RG-evolution into

$a_s(\mu^2) \to \bar{a}_s(y) \equiv \bar{a}_s\left(s\bar{y} + Q^2 y\right)$ and **ERBL–factor** **[M&P&S2021-2,TFF]**

$$F_n^{(\mathrm{tw2})}(Q^2, s) = H_0(y) \underset{y}{\otimes} \left\{ \left[1 + \bar{a}_s(y)\mathcal{H}^{(1)}(y,x) + \bar{a}_s^2(y)\mathcal{H}^{(2)}(y,x) + \ldots\right] \underset{x}{\otimes} \exp\left[-\int_{a_s(\mu^2)}^{\bar{a}_s(y)} d\alpha \, \frac{V(\alpha; x, z)}{\beta(\alpha)}\right]\right\} \underset{z}{\otimes} \psi_n(z)\,,$$

One loop resumed result, $\nu_n = \gamma_n/2\beta_0$, gives the simplest expression:

$$F_n^{\mathrm{RG}}(Q^2, s) = \frac{1}{a_s(\mu^2)^{\nu_n}} H_0(Q^2, s; y) \underset{y}{\otimes} \left\{ \bar{a}_s^{\nu_n}(y)\mathbb{1} + \bar{a}_s^{1+\nu_n}(y)\mathcal{H}^{(1)}(y,x)\right\} \underset{x}{\otimes} \psi_n(x)$$

But, the $\bar{a}_s(y) \equiv \bar{a}_s\left(s\bar{y} + Q^2 y\right)$ **is inapplicable** within factorization,
Resum formula **fall out** from the PT applicability domain at $s = 0, y \ll 1$.

$$\gamma^*(q)\,\pi^0(P) \to \pi^0$$

Dispersive form for pion EM FF $+$ RG

==

a “New” perturbation theory –

known Fractional APT.

Properties of FAPT couplings.

Dispersive form of **LCSR** leads to fractional APT 1.

$$\left[F(Q^2,s)\right]_{\text{an}} = \int_0^\infty \frac{\rho_F(Q^2,\sigma)}{\sigma+s-i\epsilon}\,d\sigma,\;\; \rho_F(Q^2,\sigma) = \frac{\textbf{Im}}{\pi}\left[F(Q^2,-\sigma)\right]$$

Naturally appear the known **FAPT** couplings $\mathcal{A}_\nu(Q^2)$-Euclead., $\mathfrak{A}_\nu(s)$-Minkowsk.
+ a generalized one – $\mathcal{I}_\nu(s,Q^2)$ **[A&M&S2018]**,
$a_s^\nu(Q^2) \Rightarrow \{\mathcal{A}_\nu(Q^2),\mathfrak{A}_\nu(s)\}$**[Bakulev&M&Stefanis2005-7]**

$\rho_n(\sigma) = \frac{\textbf{Im}}{\pi}\left[\bar{a}_s^n(-\sigma)\right]\beta_0$, for 1 loop run, $L = \ln(Q^2/\Lambda^2)$, $L_s = \ln(s/\Lambda^2)$:
[Shirkov&Solovtsov1997(∼ 600c)-2007]

$$\rho_1(\sigma) \overset{1l}{=} \frac{1}{L_\sigma^2+\pi^2}$$

$$\mathcal{A}_1[L] = \int_0^\infty \frac{\rho_1(\sigma)}{\sigma+Q^2}\,d\sigma \overset{1l}{=} \frac{1}{L} - \frac{1}{e^L-1}$$

$$\mathfrak{A}_1[L_s] = \int_s^\infty \frac{\rho_1(\sigma)}{\sigma}\,d\sigma \overset{1l}{=} \frac{1}{\pi}\arccos\frac{L_s}{\sqrt{\pi^2+L_s^2}}$$

Generalization is $\mathcal{I}_n(s,Q^2) = \int_s^\infty \frac{\rho_n(\sigma)}{\sigma+Q^2}\,d\sigma$

Inequality: $a_s^n[L] > \left(\mathcal{A}_n[L],\ \mathfrak{A}_n[L]\right) \xrightarrow{L\to\infty} a_s^n[L]$

Dispersive form of **LCSR** leads to fractional APT 2.

$n = 0, \nu(0)=0;$ **important contribution of zero harmonics**

$$\boldsymbol{F_0}^{\text{FAPT}}(Q^2,s)=H_0(y)\underset{y}{\otimes}\left\{\mathbb{1}+\mathbb{A}_1(\boldsymbol{y})\mathcal{H}^{(1)}(y,x)\right\}\underset{x}{\otimes}(\theta(x\geqslant x_0)\psi_0(x))$$

$n > 0, \nu(n)\neq 0;$

$$\boldsymbol{F_n}^{\text{FAPT}}(Q^2,s)=\frac{1}{a_s^{\nu_n}(\mu^2)}H_0(y)\underset{y}{\otimes}\left\{\mathbb{A}_{\nu_n}(\boldsymbol{y})\mathbb{1}+\mathbb{A}_{1+\nu_n}(\boldsymbol{y})\mathcal{H}^{(1)}(y,x)\right\}\underset{x}{\otimes}(\theta(x\geqslant x_0)\psi_n(x))$$

$\bar{\boldsymbol{a}}_{\boldsymbol{s}}^{\nu}(\boldsymbol{y}) \Leftrightarrow \mathbb{A}_\nu(\boldsymbol{y})$, the same expression as for **RG**-case.

$\{\mathbb{A}_\nu\}$ – **nonpower series** **instead of powers of** $\bar{\boldsymbol{a}}_{\boldsymbol{s}}^{\nu}$,

$\mathbb{A}_\nu(\boldsymbol{y}) = \boldsymbol{\mathcal{A}}_\nu(\boldsymbol{Q}(\boldsymbol{y})) - \boldsymbol{\mathcal{I}}_\nu(\boldsymbol{s}(\boldsymbol{y}),\boldsymbol{Q}(\boldsymbol{y})) + \mathfrak{A}_\nu(\boldsymbol{s}(\boldsymbol{y})) - \boldsymbol{\mathcal{A}}_\nu(\boldsymbol{0})$, here

$\boldsymbol{Q}(\boldsymbol{y}) = s\bar{y} + Q^2 y;\ \boldsymbol{s}(\boldsymbol{y}) = (Q^2+s_0)(y-y_0)\geqslant 0;\ y_0 = x_0 = Q^2/(Q^2+s_0)$

LCSR steps to fractional APT 3.

Why $\mathbb{A}_\nu(y) = \mathcal{A}_\nu\left(Q(y)\right) - \mathcal{I}_\nu\left(s(y), Q(y)\right) + \mathfrak{A}_\nu\left(s(y)\right) - \mathcal{A}_\nu(0)$?

(a) Hadronic model inherent to **LCSR** uses **duality interval** $[0, s_0]$

$$\left[F(Q^2, s)\right]_{\text{an}} = \left[\int_0^\infty \to \int_0^{s_0}\right] \frac{\rho_F(Q^2,\sigma)}{\sigma + s - i\epsilon}\, d\sigma$$

At $s_0 \to \infty$ $\mathbb{A}_\nu(y) \Rightarrow \mathcal{A}_\nu(Q(y)) - \mathcal{A}_\nu(0)$

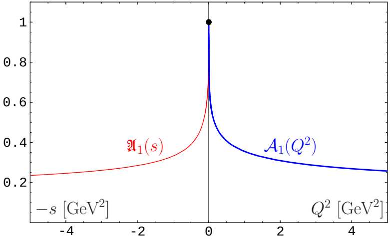
(b) What is $\mathcal{A}_\nu(0)$ here? Requires Physical constrains on **FAPT**, $\mathcal{A}_\nu(0) = 0$

(c) Borel transform $\mathbf{M^2\, B}_{(\mathbf{s}\to\mathbf{M^2})}\left[F(Q^2, s)\right] = F^{\pi}_{\text{LCSR}}(Q^2, M^2 \simeq 1\text{GeV}^2)$

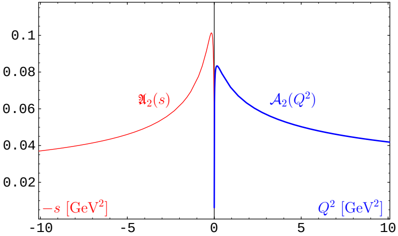
$$F^{\pi}_{\text{LCSR}(n)}(Q^2, M^2) = \frac{1}{a_s^{\nu_n}(\mu_0^2)} \left\{ \left[\mathfrak{A}_{\nu_n}\left(s(u)\right) + \int_0^{s(u)} \rho_{\nu_n}(s)\omega_1(s,u)ds\right] \underset{u}{\otimes} \mathbb{1} + \right.$$
$$\left. \left[\mathfrak{A}_{(\nu_n+1)}(s(u)) + \int_0^{s(u)} \rho_{(\nu_n+1)}(s)\omega_1(s,u)ds\right] \underset{u}{\otimes} \mathcal{H}^{(1)}(u,v) \right\}$$
$$\exp\left(-\frac{Q^2}{M^2}\frac{\bar{u}}{u}\right) \underset{v}{\otimes} \left(\theta(v \geqslant v_0)\psi_n(v)\right);\ v_0 = Q^2/(Q^2 + s_0)\,.$$

Physical constrains on FAPTcharges.

Coupling images: $\mathfrak{A}_1(s)$ & $\mathcal{A}_1(Q^2)$



Square-images: $\mathfrak{A}_2(s)$ & $\mathcal{A}_2(Q^2)$



The behavior of **FAPT** charges in the vicinity of the origin
$Q^2 = 0$ is not appropriate at index $\nu \leqslant 1$. Therefore

$$\mathfrak{A}_1^{(k)}(0), \mathcal{A}_1^{(k)}(0)$$

should be taken equal to **0** at **k=0 (Tw-2), 1 (Tw-4), ...**

To hold the **correspondence with PT asymptotics**
we put **"calibrated FAPT"** conditions:

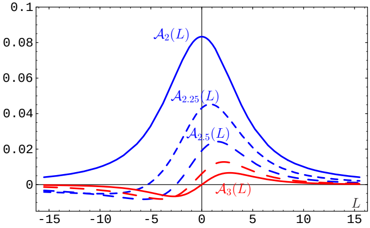
$$\boxed{\mathcal{A}_\nu^{(k)}(0) = \mathfrak{A}_\nu^{(k)}(0) = 0 \text{ for } 0 < \nu \leqslant 1}$$

Variety of coupling models where suggested to fulfill this property
[Ayala et al, 2017-20]

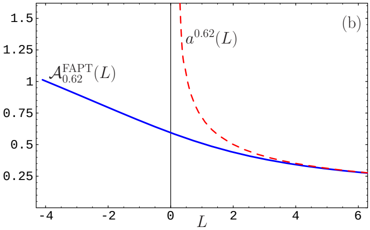
FAPT(Eucl): $\mathcal{A}_\nu[L]$ versus L

$$\mathcal{A}_\nu[L] = \frac{1}{L^\nu} - \frac{\mathrm{Li}_{1-\nu}(e^{-L})}{\Gamma(\nu)}$$

Fractional $\nu \in [2, 3]$:



Comparison with $\bar{a}_s^\nu[L]$:



where $\nu = 0.62 = \gamma_2/2\beta_0$

$$\bar{a}_s^{1+\nu}[L] \gg \mathcal{A}_{1+\nu}[L] \gg \mathcal{A}_{2+\nu}[L] \text{ at } L \sim 1$$

Twist 4,6 in LCSRs

$$H_{(\text{tw4})}(Q^2, s) = \frac{\delta^2_{\text{tw4}}(\mu_0^2)}{(\bar{y}\, Q^2 + y\, s)^2} \left[\frac{a_s(\boldsymbol{\mu^2})}{a_s(\mu_0^2)}\right]^{\boldsymbol{\nu_{t4}}} \otimes (y\varphi_\pi^{(4)}(y)); \boldsymbol{\nu_{t4}} = \gamma_{tw4}/\beta_0$$

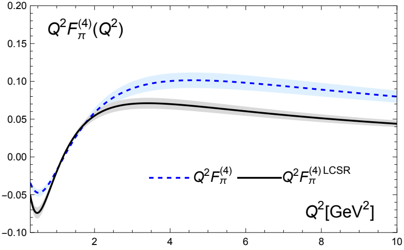
Hypothesis:

$\boldsymbol{\mu^2}$ is formed by the same Logs $\boldsymbol{L} = \ln\left[(\bar{y}Q^2 + ys)/\mu_0^2\right]$, so $\boldsymbol{\mu^2} = \bar{y}Q^2 + ys = \mathbf{Q(y)}$

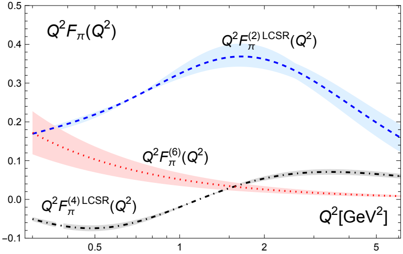
$$H^{\text{LCSR}}_{(\text{tw4})}(Q^2, \sigma) = \frac{\delta^2_{\text{tw4}}(\mu_0^2)}{(\bar{y}Q^2 + ys)^2} \frac{\square(Q(y), s(y))}{a_s^{\nu_{t4}}(\mu_0^2)} \otimes \Big(y\theta(y \geqslant y_0)\varphi_\pi^{(4)}(y)\Big)$$

$$\square(Q(y), s(y)) = \mathcal{A}_{\nu_{t4}}(Q(y)) - \mathcal{I}_{\nu_{t4}}(s(y), Q(y)) + \mathfrak{A}_{\nu_{t4}}(s(y)) + Q(y)\mathcal{I}'_{\nu_{t4}}(s(y), 0)$$

Solid, black line – $F^{(4)\text{LCSR}}_{\text{RG}}$



Comparison of $\boldsymbol{F}^{(2)\text{LCSR}}$, $\boldsymbol{F}^{(4)\text{LCSR}}$, $F^{(6)}$



Dashed, blue – the standard $F^{(4)}$

Light Cone Sum Rules with FAPT,

New prediction for the pion TFF

$$\gamma(q^2 \simeq 0)\gamma^*(Q^2) \to \pi^0$$

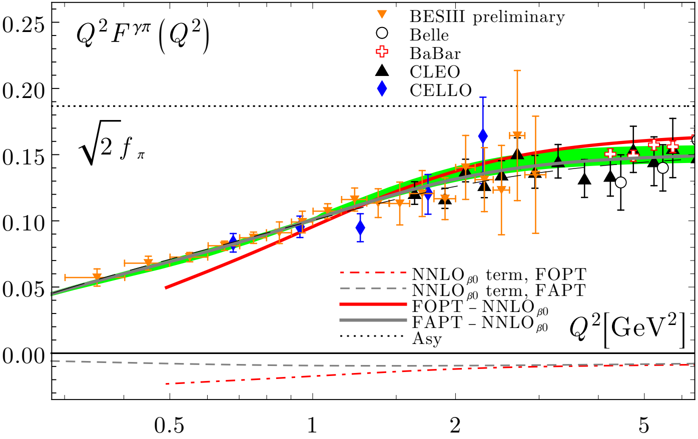
Ayala C. & M.S. & Pimikov A. & Stefanis N.

PRD103(2021) 096003, EPJ Web Conf.258(2022) 0301, PRD98(2018) 096017

Predictions of $\mathbf{TFF_{LCSR}}$ in **FAPT** vs the experimental data

$$F^{\gamma\pi}_{\text{LCSR}}(Q^2) = F^{\gamma\pi}_{\text{LCSR}(0)}(Q^2) + \sum_{n=2,4} b_n(\mu^2)\, F^{\gamma\pi}_{\text{LCSR}(n)}(Q^2) + \text{Tw-4,6}$$

(30% near lowest Q^2_{exp})



Green line&green strip around - **FAPT predictions** for $Q^2 F^{\gamma\pi}_{\text{LCSR}}$, $\chi^2_{pdf} = 0.57$

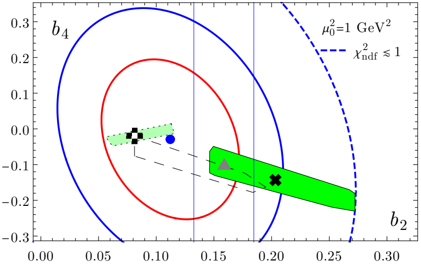
Red line - FOPT prediction at N^{2}LO for $Q^2 F^{\gamma\pi}_{\text{LCSR}}$ – **fall down**

The **fitted parameters** are the scale of Tw-6 $\langle \bar{q} q \rangle^2$ within its error bars and the certain pattern of **pion DA from BMS bunch.**

The processing of the BESIII+ data on $\mathbf{TFF_{LCSR}}$ up to 3.1 GeV^2

We **extract and reconcile** the hadronic characteristics presented in **TFF :**
twist-2 DA, b_2, b_4; the scales of twist-4,6 – $\delta^2_{\text{tw-4}}, \delta^2_{\text{tw-6}}$.

$$F^{\gamma\pi}_{\text{LCSR}}(Q^2) = F^{\gamma\pi}_{\text{LCSR};0}(Q^2) + \sum_{n=2,4} b_n(\mu^2)\, F^{\gamma\pi}_{\text{LCSR}(n)}(Q^2) + \text{Tw-4,6}$$



- best fit DA $\chi^2_{\text{ndf}} = \mathbf{0.38}$
- **red ellipse -** 1σ-region
- **blue solid ellipse -** 2σ-region
- **blue dashed ell.-**$\chi^2_{\text{ndf}} \leq 1$-region
- ✖center of **BMS domain[BMS2001]**
- ✜platykurtic **[Stefanis PLB738,2014]**

twist-2 DA $\in$ **BMS compromise**: ▲ $(b_2(\mu_0^2) = \mathbf{0.159}, b_4(\mu_0^2) = -\mathbf{0.098})$

b_2 lattice(vert. blue lines)**[Bali etal.JHEP08,2019]**

twist-4 : $\delta^2_{\text{tw-4}}(\mu_0^2) = \mathbf{0.19} \pm \mathbf{0.04}$ GeV^2

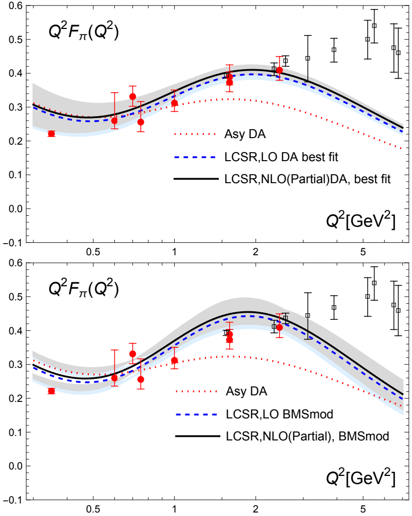
twist-6 : $\delta^2_{\text{tw-6}}(\mu_0^2) = \mathbf{1.61} \times 10^{-4}$ GeV^6

Processing BESIII, CELLO, Cleo data in window $[\mathbf{0.35} \leqslant Q^2 \leqslant \mathbf{3.1}]$ GeV^2,
[M&P&S PRD103(2021)096003]

Predictions of **EM FF$_{\text{LCSR}}$** in **FAPT** vs the experimental data

$$F^{\pi}_{\text{LCSR}}(Q^2) = F^{\pi}_{\text{LCSR(0)}}(Q^2) + \sum_{n=2,4} b_n(\mu^2)\, F^{\pi}_{\text{LCSR}(n)}(Q^2) + \textbf{Tw-4,6}$$

The scales $\delta^2_{\text{tw-4}}(\mu_0^2)$ for **Tw-4**, $\delta^2_{\text{tw-6}}(\mu_0^2)$ for **Tw-6** as well as $b_{2,4}(\mu_0^2 = 1\text{GeV}^2)$ are taken from the previous analysis of pion **TFF**



best fit ● DA (b_2^{bf}=0.112, b_4^{bf}=−0.029)

●JLab experimendata**[2008]** with bars

☐ boxes with bars – lattice res.**[2024]**

— — **dashed LO**

—– **solid NLO**

▲DA (b_2=$0.159^{+0.025}_{-0.027}$, b_4=$-0.098^{+0.05}_{-0.03}$)

– **BMS modified bunch**

• • • Asy DA

CONCLUSIONS

1. The composition of **RG sum** and **LCSRs** for pion Form Factors naturally leads to a generalization of **Fractional APT** that improves perturbative corrections to amplitudes.

2. The applicability of the **FAPT** to exclusive processes demands **new conditions** for the **FAPT** charges, $\mathcal{A}_{\nu}^{(k)}(0) = \mathfrak{A}_{\nu}^{(k)}(0) = 0, \forall \nu, k = 0, 1, \ldots$, as a "feedback"

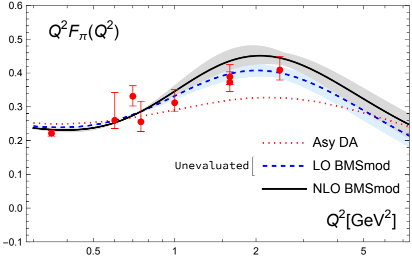
3. **LCSRs** augmented with **RG summation** of radiative corrections gives improved Q^2-behavior of Transition and EM Form Factors and **extends** the domain of **QCD applicability below 1 GeV**^{2}

4. Good description of pion **EM FF** in comparison with experimental data is obtained by using the nonperturbative inputs of the **Tw-2, Tw-4, Tw-6**, extracted in similar analysis of pion **Transition FF**

Predictions of **EM FF** in **FAPT**+ standard NLO vs the experimental data

$$F^{\pi}_{\text{LCSR}}(Q^2) = F^{\pi}_{\text{LCSR}(0)}(Q^2) + \sum_{n=2,4} b_n(\mu^2)\, F^{\pi}_{\text{LCSR}(n)}(Q^2) + \textbf{Tw-4,6}$$

The scales $\delta^2_{\text{tw-4}}(\mu_0^2)$ for **Tw-4**, $\delta^2_{\text{tw-6}}(\mu_0^2)$ for **Tw-6** as well as $b_{2,4}(\mu_0^2 = 1\text{GeV}^2)$ are taken from the previous analysis of pion **TFF**



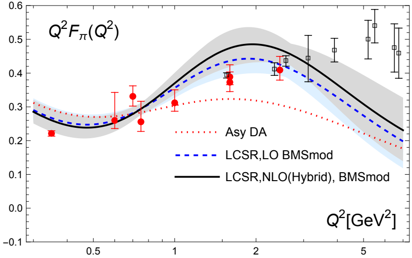
best fit ● DA (b_2^{bf}=0.112, b_4^{bf}=−0.029)

●JLab experimendata**[2008]** with bars

☐ boxes with bars – lattice res.**[2024]**

— — **dashed LO**

—– **solid NLO**



▲DA (b_2=$0.159^{+0.025}_{-0.027}$, b_4=$-0.098^{+0.05}_{-0.03}$)

– **BMS modified bunch**

• • • Asy DA

STORE

Fractional Analytic Perturbation Theory

FAPT couplings $\mathcal{A}_\nu, \mathfrak{A}_\nu$

Dispersive “Källen–Lehmann” representation

Different coupling images in **Euclidean, $\mathcal{A}_n$,** and **Minkowsk., $\mathfrak{A}_n$,** regions $\overline{\alpha}_s^n \to \{\mathcal{A}_n, \mathfrak{A}_n\}$ **[Shirkov&Solovtsov1997(534cit)-07]**–nonpower series

$$\left[f(Q^2)\right]_{\text{an}} = \int_0^\infty \frac{\rho_f(\sigma)}{\sigma + Q^2 - i\epsilon}\, d\sigma, \quad \rho_n(\sigma) = \frac{\text{Im}}{\pi}\left[\overline{a}_s^n(-\sigma)\right]\beta_0$$

For 1 loop run, $L = \ln(Q^2/\Lambda^2),\ L_s = \ln(s/\Lambda^2)$:

$$\rho_1(\sigma) \stackrel{1l}{=} \frac{1}{L_\sigma^2 + \pi^2}$$

$$\mathcal{A}_1[L] = \int_0^\infty \frac{\rho_1(\sigma)}{\sigma + Q^2}\, d\sigma \stackrel{1l}{=} \frac{1}{L} - \frac{1}{e^L - 1}$$

$$\mathfrak{A}_1[L_s] = \int_s^\infty \frac{\rho_1(\sigma)}{\sigma}\, d\sigma \stackrel{1l}{=} \frac{1}{\pi}\arccos\frac{L_s}{\sqrt{\pi^2 + L_s^2}}$$

Inequality:

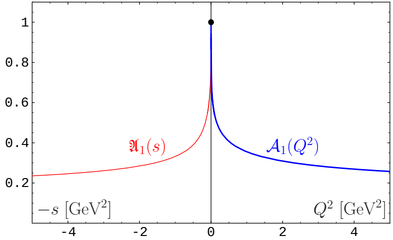
$$a_s^n[L] > (\mathcal{A}_n[L],\ \mathfrak{A}_n[L]) \stackrel{L\to\infty}{\longrightarrow} a_s^n[L]$$

Generalization of $(\mathcal{A}_n,\ \mathfrak{A}_n)$: $\mathcal{I}_n$ **[Ayala&MS&2018]**

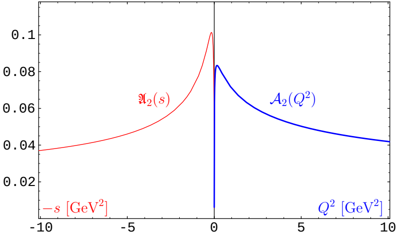
$$\mathcal{I}_n(s, Q^2) = \int_s^\infty \frac{\rho_n(\sigma)}{\sigma + Q^2}\, d\sigma$$

APT: Distorting mirror [Shirkov&Solovtsov1997-2007]

Coupling images: $\mathfrak{A}_1(s)$ & $\mathcal{A}_1(Q^2)$



Square-images: $\mathfrak{A}_2(s)$ & $\mathcal{A}_2(Q^2)$



FAPT(Eucl): Properties of $\mathcal{A}_\nu[L]$ [Bakulev,MS,Stefanis 2005-07]

Euclidean coupling :

$$\mathcal{A}_\nu[L] = \frac{1}{L^\nu} - \frac{e^{-L}\Phi(e^{-L}, 1-\nu, 1)}{\Gamma(\nu)} \equiv \frac{1}{L^\nu} - \frac{\mathrm{Li}_{1-\nu}(e^{-L})}{\Gamma(\nu)}$$

Here $\Phi(z, \nu, 1)$ is **Lerch's** transcendental, Li_ν - PolyLog functions.
They are analytic functions in ν. Properties:
The charge $\mathcal{A}_\nu(Q^2)$ **is Bounded** for $\nu \geqslant 1$,

- $\mathcal{A}_0[L] = 1$;
- $\mathcal{A}_{-m}[L] = L^m$ for $m \in \mathbb{N}$;
- $\mathcal{A}_m[L] = (-1)^m \mathcal{A}_m[-L]$ for $m \geq 2$, $m \in \mathbb{N}$;
- $\mathcal{A}_\nu[\pm\infty] = 0$ for $\nu > 1$;

$\mathcal{A}_\nu[-\infty] = (\infty)^{1-\nu}$ for $\nu < 1$ *i.e.,*
$\mathcal{A}_\nu(Q^2 \to 0)$ becomes Unbounded for $\nu < 1$