Dirac Singleton as a Relativistc Field Beyond SM

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Dirac Singleton

Singleton $S = Di + Rac$ was discovered by P.A.M. Dirac,

A Remarkable representation of the 3 + 2 de Sitter group

J. Math. Phys. 4 (1963), 901-909

Dirac observed that the wave equation

$$\Box\phi(x) + \frac{5}{4}\Lambda\phi(x) = 0$$

admits two types of solutions:

Class S slowly decrease at infinity and

Class B fastly decrease at infinity

S is a free conformal scalar field at the boundary of AdS_4.

Scalar singleton is sometimes denoted Rac

There is also its spinor companion Di

Each forms a unitary representation of the conformal algebra $o(3, 2)$

Flato-Fronsdal Thm

A fundamental fact on singletons is the Flato-Fronsdal Thm

$$S \bigotimes S = \sum_{s=0}^{\infty} \phi^{d=4}_{s,m=0}(x) \ldots , \qquad \mathbf{1978}$$

$3d$ conserved currents are holographically dual to $4d$ massless fields

Holography suggests that fields in AdS_{d+1}

are dual to conformal operators on the d-dimensional boundary

Main question of this talk: what is dual to the singleton conformal field?

The answer will be unusual and may have some far going consequences

An infinite-dimensional Lorentz group IRREP in $d+1$ dimensions.

Related fact: at $d=3$ S cannot be localised at a point in the $3d$ space.

From the $4d$ perspective it is nowhere (everywhere).

New issues:

- Lorentz covariant field equations and action for singleton in $(A)dS_4$

Plan

- **Unfolding and holography**

- **Massless scalar field in d and singleton in** $d+1$

- **Speculations on physical applications**

Unfolded Dynamics

First-order form of differential equations

$$\dot{q}^i(t) = \varphi^i(q(t)) \qquad \text{initial values:} \quad q^i(t_0)$$

Unfolded dynamics: multidimensional generalization

$$\frac{\partial}{\partial t} \to \mathrm{d}\,, \qquad q^i(t) \to W^\Omega(\theta, x) = \theta^{\underline{n}_1} \dots \theta^{\underline{n}_p} W^\Omega_{\underline{n}_1 \dots \underline{n}_p}(x)$$

$$\mathrm{d}\mathbf{W}^\Omega(\mathbf{x}) = \mathbf{G}^\Omega(\mathbf{W}(\mathbf{x}))\,, \qquad \mathrm{d} = \theta^{\underline{n}} \partial_{\underline{n}} \qquad \mathbf{MV} \quad \mathbf{1988}$$

$G^\Omega(W)$: **function of "supercoordinates"** W^Ω

$$G^\Omega(W) = \sum_{n=1}^{\infty} f^\Omega{}_{\Phi_1 \dots \Phi_n} W^{\Phi_1} \dots W^{\Phi_n}$$

Covariant first-order differential equations

$d > 1$: **Compatibility conditions**

$$\mathrm{d}^2 = 0 \quad \to \quad G^\Phi(W) \frac{\partial G^\Omega(W)}{\partial W^\Phi} = 0$$

Invariant functionals

The system is invariant under the gauge transformation

$$\delta W^{\Omega} = d\varepsilon^{\Omega} + \varepsilon^{\Lambda} \frac{\partial G^{\Omega}(W)}{\partial W^{\Lambda}} \,,$$

where the gauge parameter $\varepsilon^{\Omega}(x)$ is a $(p_{\Omega} - 1)$-form if W^{Ω} was a p_{Ω}-form

The unfolded equations is useful to write in the Hamiltonian-like form

$$\mathsf{d}F(W) = Q(F(W)) \,, \qquad \forall F(W) \,,$$

where Q is homological vector field in the space of W^{Ω}

$$Q = G^{\Omega} \frac{\partial}{\partial W^{\Omega}} \,, \qquad Q^2 = 0$$

Invariant functionals

$$S = \int_{\Sigma^p} L_p \,, \qquad L_p \in H^p(Q) \;: \quad QL_p = 0 \,, \quad L_p \neq QM_{p-1} \,.$$

Vacuum and Dynamical Fields

A particular example of an unfolded system: Maurer-Cartan equations for a Lie algebra h with a basis $\{T_\alpha\}$

$$\mathrm{d}\omega + \omega\omega = 0\,, \qquad \omega = \omega^\alpha T_\alpha\,, \qquad \omega\omega := \frac{1}{2}\omega^\alpha\omega^\beta[T_\alpha, T_\beta]\,.$$

The zero-curvature equations describe background geometry in a coordinate independent way. Minkowski for h being the Poincareè algebra

$$\omega(x) = e^n(x)P_n + \frac{1}{2}\omega^{nm}(x)M_{nm}\,,$$

If the set W^α contains some p-forms $\mathcal{C}^i$ and G^i are linear in ω and $\mathcal{C}$,

$$G^i = -\omega^\alpha(T_\alpha)^i{}_j\mathcal{C}^j\,,$$

$(T_\alpha)^i{}_j$ form some representation T in an $h-$module V of $\mathcal{C}^i$. Unfolded equation: covariant constancy condition $D_\omega\mathcal{C} = 0$ with $D_\omega \equiv \mathrm{d} + \omega$ in V.

For different Lie algebras h one can describe a different background like, e.g., AdS_d for $h = o(d-1,2)$ or conformally flat for $h = o(d,2)$.

Properties

Unfolded formulation has a number of remarkable properties:

Universaility

Coordinate independence due to the exterior algebra formalism

DOF: zero-forms $C^I(x_0) \in \{W^\Omega(x_0)\}$ at any $x = x_0$, that realize an infinite-dimensional module dual to the space of single-particle states analogous to the phase space in the Hamiltonian approach.

Key fact: unfolded equation makes sense in space-time of any dimension

$$\mathrm{d}W^\Omega(x) = G^\Omega(W(x))\,, \quad x \to X = (x,z)\,, \quad \mathrm{d}_x \to \mathrm{d}_X = \mathrm{d}_x + \mathrm{d}_z\,, \quad \mathrm{d}_z = dz^u \frac{\partial}{\partial z^u}$$

X-dependence is reconstructed in terms of fields $W^\Omega(X_0) = W^\Omega(x_0, z_0)$ at any X_0. To take $W^\Omega(x_0, z_0)$ in space M_X with coordinates X_0 is the same as to take $W^\Omega(x_0)$ in the space $M_x \in M_X$ with coordinates x.

Conformal Setup

Conformal algebra

$$[D\,,P_a] = -P_a\,, \qquad [D\,,K^b] = K^b\,, \qquad [D\,,L_{ab}] = 0\,,$$

$$[P_a\,,K_b] = 2L_{ab} - 2\eta_{ab}D\,,$$

Let M^d be a d–dimensional conformally flat space-time with local coordinates $\mathbf{x}$ and some $o(d,2)$ flat connection

$$\mathsf{d}_{\mathbf{x}} w_{\mathbf{x}}(\mathbf{x}) + w_{\mathbf{x}}(\mathbf{x}) w_{\mathbf{x}}(\mathbf{x}) = 0$$

Flat connection corresponding to Cartesian coordinates is $w_{\mathbf{x}}(\mathbf{x}) = d\mathbf{x}^a P_a\,.$

The dilatation generator D induces standard $\mathbb{Z}$ grading on $o(d,2)$

$$[D\,,T_A] = \Delta(T_A)T_A\,,$$

where $\Delta(T_A)$ is conformal dimension of T_A,

$$\Delta(L) = 0\,, \qquad \Delta(D) = 0\,, \qquad \Delta(K) = 1\,, \qquad \Delta(P) = -1\,.$$

Holography

Introduce an additional coordinate z and differential dz so that $x = (\mathbf{x}, z)$ be local coordinates of AdS_{d+1}. A conformally flat foliated connection in (a local chart of) AdS_{d+1}:

$$W_{\mathbf{x}}^A(x) T_A = z^{\Delta(T_A)} w_{\mathbf{x}}^A(\mathbf{x}) T_A \,, \qquad W_z(x) D = -z^{-1} dz D \,.$$

Analogously, unfolded equations

$$\mathcal{D}_{\mathbf{x}} C_i(\mathbf{x}) = 0 \,, \qquad \mathcal{D}_{\mathbf{x}} := \mathsf{d}_{\mathbf{x}} + W_{\mathbf{x}}^A T_A \,, \qquad \mathsf{d}_{\mathbf{x}} := d\mathbf{x}^n \frac{\partial}{\partial \mathbf{x}^n}$$

in M^d for a set of fields $C_i(\mathbf{x})$ of conformal weights Δ_i extend to the fields and equations

$$C_i(x) = z^{\Delta_i} C_i(\mathbf{x}) \,, \qquad \mathcal{D}_x C_i(x) = 0 \,, \qquad \mathcal{D}_x := \mathsf{d}_x + W_x^A T_A \,.$$

Important comment: if a system was off-shell in M^d this is not so in the extended $d+1$-dimensional space: the dependence on the additional coordinate z is reconstructed in terms of that on $\mathbf{x}$.

Generators of AdS_{d+1}

To identify the $d+1$-dimensional space with (a local chart of) AdS_{d+1} it suffices to redefine $o(d,2)$ generators as

$$P_n = \Big((P_a{+}\lambda^2 K_a)\,, 2\lambda D\Big)\,, \qquad M_{nm} = \Big(L_{ab}, \frac{1}{2\lambda}(P_a{-}\lambda^2 K_a)\delta^d_n, -\frac{1}{2\lambda}(P_b{-}\lambda^2 K_b)\delta^d_m\Big)$$

with $a,b = (0\,,\ldots,d-1)$ $n,m = (0\,,\ldots,d)$, interpreting P_n and M_{nm} as AdS_{d+1} translation (transvection) and Lorentz generators, respectively.

AdS_{d+1} connection is

$$W = h^n P_n + \frac{1}{2}\omega^{nm} M_{nm}$$

In particular

$$e^a = h^a + 2\lambda\omega^{ad}\,.$$

λ is related to the cosmological constant $\Lambda = -\lambda^2$

real and pure imaginary in AdS_{d+1} and dS_{d+1} respectively.

Conformal Scalar within Unfolded Formalism

Singleton $|Rac\rangle$ is a massless conformal scalar field in any d. Its unfolded formulation is described in terms of a zero-form $C(y|\mathbf{x})$, that depends on the space-time coordinates $\mathbf{x}^n$ and auxiliary variables y^n $(n = 0, \dots d-1)$:

$$\mathsf{d}_{\mathbf{x}} C(y|\mathbf{x}) + d\mathbf{x}^n \frac{\partial}{\partial y^n} C(y|\mathbf{x}) = 0\,, \qquad \mathsf{d}_{\mathbf{x}} := d\mathbf{x}^n \frac{\partial}{\partial \mathbf{x}^n}$$

This equation relates the coefficients $C_{a_1 \dots a_n}(\mathbf{x})$ of the expansion

$$C(y|\mathbf{x}) = \sum_{n=0}^{\infty} \frac{1}{n!} C_{a_1 \dots a_n}(\mathbf{x}) y^{a_1} \dots y^{a_n}$$

to higher derivatives in $\mathbf{x}^a$,

$$C_{a_1 \dots a_n}(\mathbf{x}) = (-1)^n \partial_{a_1} \dots \partial_{a_n} C(\mathbf{x})\,, \qquad \partial_a := \frac{\partial}{\partial \mathbf{x}^a}\,,$$

$C(\mathbf{x})$ is the ground component of $C(y|\mathbf{x})$

$$C(\mathbf{x}) := C(0|\mathbf{x})$$

The system is off-shell: no differential conditions on $C(\mathbf{x})$.

To put the system on shell of a massless field it suffices to impose the constraint

$$\Box_y C(y|\mathbf{x}) = 0\,, \qquad \Box_y := \eta^{ab}\frac{\partial^2}{\partial y^a \partial y^b}\,.$$

The system is equivalent to

$$\mathcal{D}_{\mathbf{x}} C(y|\mathbf{x}) = 0\,,$$

$$\mathcal{D}_{\mathbf{x}} := \mathsf{d}_{\mathbf{x}} + e^a P_a + f_a K^a + \frac{1}{2}\omega^{ab} L_{ab} + bD$$

with particular connection with $f = \omega = b = 0$, $e^a = d\mathbf{x}^a$

In terms of y^a, conformal generators are

$$P_a = \frac{\partial}{\partial y^a}\,, \qquad L_{ab} = y_a \frac{\partial}{\partial y^b} - y_b \frac{\partial}{\partial y^a}\,, \qquad D = y^a \frac{\partial}{\partial y^a} + \Delta\,,$$

$$K_a = y^2 \frac{\partial}{\partial y^a} - 2 y_a y^b \frac{\partial}{\partial y^b} - 2\Delta y_a\,,$$

Conformal weight Δ is a number.

Representation Theory Interpretation

If $C(Y|\mathbf{x})$ obeys the constraint $\Box_y C = 0$ then it is obeyed for $T_A C(Y|\mathbf{x})$ for all generators T_A provided that conformal weight is canonical

$$\Delta = \frac{d}{2} - 1$$

for a massless scalar in d dimensions. That $\Box_y C = 0$ respects the unfolded equation, implies conformal invariance of the massless KG equation.

There are two components of $C(y)$ annihilated by the special conformal generators K^a. One is the vacuum (lowest weight) component $C(\mathbf{x})$. Another is the singular vector, associated with the trace component

$$C'(\mathbf{x}) := C^a{}_a(\mathbf{x}) = \Box_x C(\mathbf{x})\,.$$

To prove conformal invariance of a functional built from $C(\mathbf{x})$ and $C'(\mathbf{x})$ it suffices to check its invariance under the action of P_a, L_{ab} and D.

Conformally Invariant Lagrangian in d Dimensions

Dipole Lagrangian in $d+1$ dimensions that is a $d+1$-form with huge gauge symmetry Flato and Fronsdal 1987 ... A.Starinets 1999

Alternative proposal of this talk: the conformally invariant Lagrangian for a scalar field is a closed d-form

$$L = \frac{1}{2}\epsilon_{a_1\dots a_d} e^{a_1}(\mathbf{x})\dots e^{a_d}(\mathbf{x}) C(\mathbf{x}) C'(\mathbf{x}) \,, \qquad C'(\mathbf{x}) = \Box_{\mathbf{x}} C(\mathbf{x})\,.$$

Indeed, L is Q-closed since

special conformal gauge field f_a is absent in de^a, $\mathcal{D}(C)$ and $\mathcal{D}(C')$,

Lorentz connection cancels by Lorentz invariance,

b cancels since L has proper scaling dimension,

contribution of e^b cancels by antisymmetrization over $d+1$ indices a.

The fields $C(y|\mathbf{x})$ still obey the unfolded equations, that are off-shell which means that they just express higher components $C_{a_1\dots a_n}(\mathbf{x})$ via derivatives of $C(\mathbf{x})$ imposing no differential conditions on the latter.

AdS_{d+1} Extension

To extend the d-dimensional singleton to a $d+1$-dimensional space with the same $o(d,2)$ symmetry consider unfolded equations in $d+1$ dimensions

$$D_x C(x) = 0\,, \qquad D_x C_\alpha(x) = 0\,, \qquad D_x := \mathsf{d}_x + W$$

$$W^A_{\mathbf{x}}(x) T_A = z^{\Delta(T_A)} w^A_{\mathbf{x}}(\mathbf{x}) T_A\,, \qquad W_z(x) D = -z^{-1} dz D\,.$$

The $(A)dS_{d+1}$ invariant Lagrangian has the form

$$L = \frac{1}{2}\epsilon_{a_1 \dots a_d} e^{a_1}(x) \dots e^{a_d}(x) C(x) C'(x)\,, \qquad e^a = h^a + 2\lambda\omega^{ad}$$

Now L is a closed d-form in $(A)dS_{d+1}$ invariant up to exact forms (*i.e.*, total derivatives) under the symmetries that leave invariant the background connections, *i.e.*, $(A)dS_{d+1}$ symmetries.

Unusual Features

The $(A)dS_{d+1}$ unfolded equations are no longer off-shell reconstructing the dependence on z. This is why the seemingly non-invariant form of L in view of $e^a = h^a + 2\lambda\omega^{ad}$ still respects Lorentz invariance in AdS_{d+1}: d+1-dimensional Lorentz transformations act on the singleton nonlocally relating fields $\phi(\mathbf{x}, z)$ at different z, acting in the infinite-dimensional module.

Being a local field in d dimensions, from the d+1-dimensional perspective singleton is nowhere (equivalently, everywhere).

λ in the $(A)dS$ connection is pure imaginary in the dS case. Naively, the Lagrangian is not Hermitian in dS. This problem can be resolved by introducing doublets of mutually conjugated fields $C^{\pm}$ associated with $\lambda = \pm i\lambda'$ at real λ'. This allows one to consider singletons as fields in the dark energy dS regime. The modes associated with the evolution along z are either increasing or decreasing that is not too surprising in the expansion regime.

Observe or not observe?

The field equations are three-dimensional rather than four-dimensional. Direct scattering is unlikely observable: good news for SM. However Flato-Fronsdal Thm:

$$S \bigotimes S = \sum_{s=0}^{\infty} \phi_{s,m=0}(x) = graviton + neutral \quad massless \quad scalar + \ldots$$

implying that bilinears of singletons contain graviton, that may have direct dark matter type consequences via additional induced gravity. That S admits a Lorentz covariant formulation allows one to introduce interactions with gravity via usual covariantization of derivatives

Also singletons may be related to another long standing problem of baryon asymmetry: fit the Sakharov conditions necessary for baryon asymmetry: positive cosmological constant may provide a non-equilibrium regime. Moreover, singletons endowed with appropriate inner structure may induce violation of the baryon number conservation. The presence of complex coefficients in the Lagrangian may induce CP violation.

Conclusion

New type of relativistic matter in presence of dark energy

Interesting to explore in the context of long-standing problems including dark matter and even baryon asymmetry induced by the appropriately charged singletons.

Singleton is described as a field in AdS_4 with auxiliary variables y^+_α, y^-_β

$$|\phi(y^+|x)\rangle = \phi(y^+|x)|0\rangle$$

$$[y_\alpha\,, y^+_\beta] = \varepsilon_{\alpha\beta}\,, \qquad y_-|0\rangle = 0\,.$$

Field equations are

$$D|\phi\rangle = 0\,, \qquad \phi(y^+|x) = \sum_{n=0}^{\infty} \frac{1}{n!}\phi_{\alpha_1\ldots\alpha_n}(x) y^{+\alpha_1}\ldots y^{+\alpha_n}$$

where

$$D = \mathsf{d}_x + \frac{i}{z} d\mathbf{x}^{\alpha\beta} y^-_\alpha y^-_\beta - \frac{dz}{2z} y^-_\alpha y^{+\alpha}\,, \qquad \mathsf{d}_x := dx^{\alpha\dot\beta}\frac{\partial}{\partial x^{\alpha\dot\beta}}$$

$$x^{\alpha\dot\alpha} = (\mathbf{x}^{\alpha\dot\alpha}, -\frac{i}{2}\epsilon^{\alpha\dot\alpha} z^{-1})\,,$$

AdS_4 connection in Poincaré coordinates

$$e^{\alpha\dot\alpha} = \frac{1}{2z} dx^{\alpha\dot\alpha}\,, \qquad \omega^{\alpha\beta} = -\frac{i}{4z} d\mathbf{x}^{\alpha\beta}\,, \qquad \bar\omega^{\dot\alpha\dot\beta} = \frac{i}{4z} d\mathbf{x}^{\dot\alpha\dot\beta}\,.$$