Critical non-Abelian Vortex String and $\mathcal{N} = 2$ supersymmetric Liouville Theory

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Introduction

Seiberg and Witten 1994 : Confinement in the monopole vacuum of $\mathcal{N}=2$ supersymmetric QCD
Abelian confinement
In the search for a non-Abelian confinement
Non-Abelian vortex strings
were found in $\mathcal{N}=2$ U(N) QCD
Hanany, Tong 2003
Auzzi, Bolognesi, Evslin, Konishi, Yung 2003
Shifman Yung 2004
Hanany Tong 2004
Non-Abelian string : Orientational zero modes
Rotation of color flux inside SU(N).
Non-Abelian vortex string is BPS and preserves $\mathcal{N}=(2,2)$ supersymmetry on its world sheet.

$\mathcal{N} = 2$ SQCD:

In monopole vacua Abelian strings confine quarks

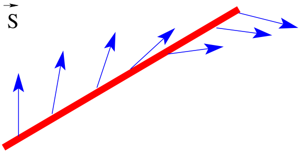
In quark vacua non-Abelian strings confine monopoles

In both cases an "observer" will see colorless hadrons

= stringy states

Next problem:

How to quantize confining solitonic string outside critical dimension???



Shifman and Yung, 2015: Non-Abelian vortex in $\mathcal{N} = 2$ supersymmetric QCD can behave as a critical superstring

Idea:

Non-Abelian string has more moduli then Abrikosov-Nielsen-Olesen string.

It has translational + orientaional moduli

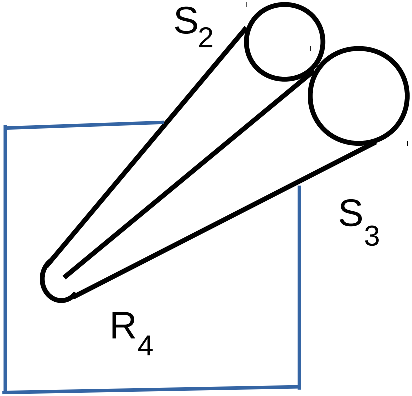
We can fulfill the criticality condition: In $\mathcal{N} = 2$ QCD with $U(N = 2)$ gauge group and $N_f = 4$ quark flavors.

- The solitonic non-Abelian vortex have six orientational and size moduli, which, together with four translational moduli, form a ten-dimensional space.
- For $N_f = 2N$ 2D world sheet theory is conformal.

For $N = 2$ and $N_f = 4$ the target space of the 2D sigma model on the string world sheet is

$$R^4 \times Y_6,$$

where Y_6 is a non-compact Calabi-Yau manifold studied by Candelas, Witten and Vafa, namely conifold.



We studied states of closed type IIA string propagating on $R^4 \times Y_6$ and interpreted them as hadrons in 4D $\mathcal{N} = 2$ QCD.

Conifold $==$ non-compact CY.

Looking for states with normalizable wave function over Y_6
String states localized near the conifold singularity. They are 4D SQCD states

Solitonic string-gauge duality:

4D $\mathcal{N} = 2$ SQCD at weak coupling is in the Higgs phase and can be described in terms of (s)quarks and Higgsed gauge bosons, while at strong coupling hadrons of this theory can be understood as closed string states formed by the non-Abelian vortex string.

To find the spectrum of string states we used Little String Theory approach

Ghoshal, Vafa, 1995; Giveon Kutasov 1999

Critical string on a conifold is equivalent to non-critical string on

$$\mathcal{R}^4 \times \mathcal{R}_\phi \times S^1,$$

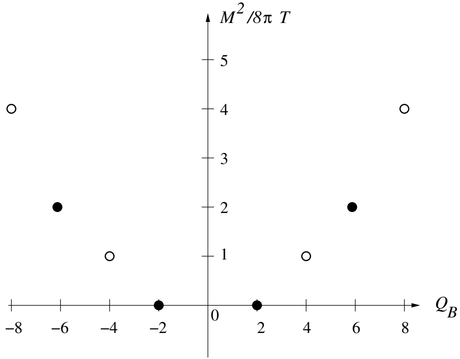
$\mathcal{R}_\phi$ is a real line associated with the Liouville field ϕ and the theory has a linear in ϕ dilaton, such that string coupling is given by

$$g_s = e^{-\frac{Q}{2}\phi} .$$

$\mathcal{N} = 2$ supersymmetric Liouville theory

Shifman and Yung, 2017

spectrum of low lying string states = hadrons of $\mathcal{N} = 2$ QCD



The importance of physical results obtained for the spectrum requires to put the above equivalence on a firmer ground.

$\mathbb{WCP}(N, N)$ models

World sheet sigma models on non-Abelian strings in $\mathcal{N} = 2$ SQCD with $N_f = 2N$ are $\mathbb{WCP}(N, N)$ models. Can be understood as Higgs branches of U(1) gauge theory, $e_0 \to \infty$ (*Witten, 1993*). Conformal in the zero mass limit.

$$S = \int d^2x \left\{ \left|\nabla_\alpha n^i\right|^2 + \left|\tilde{\nabla}_\alpha \rho^j\right|^2 - \frac{1}{4e_0^2} F^2_{\alpha\beta} + \frac{1}{e_0^2} \, |\partial_\alpha \sigma|^2 \right.$$
$$+ \frac{1}{2e_0^2} D^2 - \left|\sqrt{2}\sigma + m_i\right|^2 \left|n^i\right|^2 + \left|\sqrt{2}\sigma + m_j\right|^2 \left|\rho^j\right|^2$$
$$\left. + D \left(\left|n^i\right|^2 - \left|\rho^j\right|^2 - \mathrm{Re}\,\beta \right) - \frac{\vartheta}{2\pi} F_{01} \right\},$$

where $i = 1, ..., N$, $j = (N+1), ..., 2N$ and the complex scalar fields n^i and ρ^j have charges $\mathtt{Q} = +1$ and $\mathtt{Q} = -1$

$$\nabla_\alpha = \partial_\alpha - iA_\alpha \,, \qquad \tilde{\nabla}_\alpha = \partial_\alpha + iA_\alpha \,,$$

Inverse coupling constant β is introduced as 2D FI term (twisted superpotential)

$$\Sigma = \sigma + \sqrt{2}\theta_R\bar{\lambda}_L - \sqrt{2}\bar{\theta}_L\lambda_R + \sqrt{2}\theta_R\bar{\theta}_L(D - iF_{01})$$

$$-\frac{\beta}{2}\int d^2\tilde{\theta}\sqrt{2}\,\Sigma = -\frac{\beta}{2}\,(D - iF_{01}), \qquad \beta = \mathrm{Re}\,\beta + i\,\frac{\vartheta}{2\pi}$$

Twisted masses m_i and m_j coincide with quark masses of $2N$ flavors in 4D SQCD.

Dimension of the Higgs branch at $m_i = m_j = 0$

$$\mathrm{dim}_R \mathcal{H} = 4N - 1 - 1 = 2\,(2N - 1)$$

The model is conformal and $\mathcal{N} = (2, 2)$ supersymmetric $\Rightarrow$ target space is Ricci-flat and Kähler $\Rightarrow$ Calabi-Yau
Central charge

$$\hat{c}_{CY} \equiv \frac{c_{CY}}{3} = \mathrm{dim}_C \mathcal{H} = 2N - 1$$

For $N = 2$ $\mathrm{dim}_R \mathcal{H} = 6$ – conifold 6+4=10 – critical non-Abelian string
We consider all N, moreover use large N approximation as a first step.

Coulomb branch

D'Adda, Davis, DiVeccia, Salamonson, 1983; Witten, 1993 ...

Critical points of the exact twisted superpotential for Σ are given by the vacuum equation

$$\prod_{i=1}^{N}\left(\sqrt{2}\,\sigma + m_i\right) = e^{-2\pi\beta} \prod_{j=N+1}^{2N}\left(\sqrt{2}\,\sigma + m_j\right)$$

In the limit $m_i = m_j = 0$

$$\sigma^N = e^{-2\pi\beta}\,\sigma^N, \qquad \sigma = 0 \quad \text{for} \quad \beta \neq 0$$

For $\beta = 0$ σ is arbitrary – Coulomb branch

We show that the Coulomb branch is described by $\mathcal{N} = 2$ Liouville theory

Conifold, $N = 2$

Massless 4D states

Deformations preserving Ricci-flat metric on Y_6 are $h^{(1,1)}$ Kahler form deformations and $h^{(1,2)}$ deformations of the complex structure.
$h^{(p,q)}$ are Hodge numbers – numbers of harmonic (p, q) forms on Y_6
Kahler form deformations = variations of 2D coupling β
D-term condition in $\mathbb{WCP}(2, 2)$ model

$$|n^i|^2 - |\rho^j|^2 = \beta, \qquad i = 1, 2, \qquad j = 1, 2$$

Resolved conifold
β - non-normalizable mode (quadratically divergent in the IR)

Complex structure deformations

Construct U(1) gauge invariant "mesonic " variables"

$$w^{ij} = n^i \rho^j .$$

$$\det w^{ij} = 0$$

Take $\beta = 0$

Complex structure deformation ⇒ Deformed conifold

$$\det w^{ij} = b$$

Modulus b has logarithmically normalizable metric ⇒ physical massless state (baryon) in 4D
Deformed conifold = Coulomb branch of $\mathbb{WCP}(2,2)$ model

$\mathcal{N} = 2$ Liouville theory

Target space: $R \times S^1_Q$. Linear dilaton $\Phi(\phi) = -\frac{Q}{2}\phi$

$$T = -\frac{1}{2}\left[(\partial_z\phi)^2 + Q\,\partial_z^2\phi + (\partial_z Y)^2\right], \qquad Y \sim Y + 2\pi Q$$

Central charge

$$c_L = 3 + 3Q^2, \qquad \hat{c}_L \equiv \frac{c_L}{3} = 1 + Q^2.$$

Liouville interaction

$$L_{int} = \mu \int d^2\theta\, e^{-\frac{\phi + iY}{Q}}$$

This superpotential is a marginal deformation

$$\Delta_L\left(e^{-\frac{\phi+iY}{Q}}\right) = \frac{1}{2}\left(-\frac{1}{Q^2} + 1 + \frac{1}{Q^2}\right) = \frac{1}{2}, \qquad \Delta = \left(\frac{1}{2}, \frac{1}{2}\right).$$

Primary operators

Dixon, Peskin, Lykken, 1989; Mukhi, Vafa, 1993; Evans, Gaberdiel, Perry, 1998

The spectrum of primary operators was computed exactly using mirror description: $SL(2,R)/U(1)$ WZNW model at level

$$k = 2/Q^2$$

$$V_{j,m} \approx \exp\left(Q\,j\phi + iQ\,mY\right), \qquad \phi \to \infty$$

$$V_{j;m} = g_s \Psi_{j;m}(\phi, Y) = e^{-\frac{Q\phi}{2}}\, \Psi_{j;m}(\phi, Y), \qquad \Psi_{j;m} \sim e^{Q(j+\frac{1}{2})\phi + iQ\,mY}$$

Normalizable states have $j \leq -\frac{1}{2}$

Conformal dimension

$$\Delta_{j,m} = \frac{Q^2}{2}\left\{m^2 - j(j+1)\right\} = \frac{1}{k}\left\{m^2 - j(j+1)\right\}$$

We have (for $k = 1$)

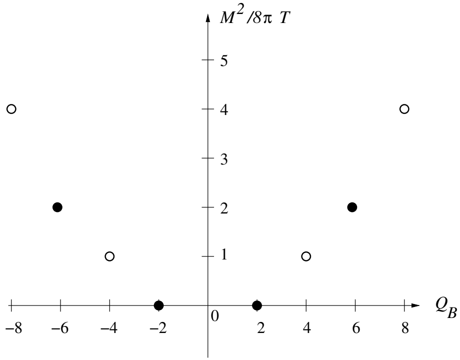
- Discrete series

$$j = -\frac{1}{2}, -1, -\frac{3}{2}, \ldots, \qquad m = \pm\{j, j-1, j-2, \ldots\}$$

- Principal continues representations
  $j = -\frac{1}{2} + is$, $m =$ integer or half-integer

The discrete representations include the normalizable and borderline $j = -\frac{1}{2}$ states localized near the tip of the cigar.

Spectrum of low lying string states = hadrons of $\mathcal{N} = 2$ QCD

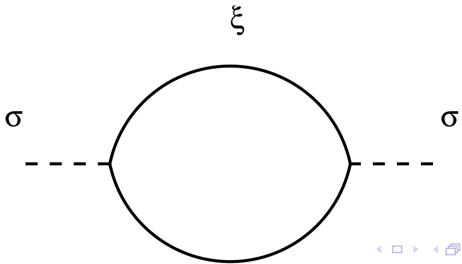


Large N calculation

Take $\mathbb{WCP}(N,N)$ at large N and $\beta = 0$. Fields n and ρ become "massive" at $\sigma \neq 0$ and can be integrated out. *Witten, 1979* for CP$(N-1)$ model. Similar calculation for our model.

$$S_{\rm eff}^{\rm kin} = \int d^2x \left\{ -\frac{1}{4e^2} F_{\alpha\beta}^2 + \frac{1}{e^2}|\partial_\alpha \sigma|^2 + \frac{1}{2e^2} D^2 \right\}$$

$$\frac{1}{e^2} = \left(\frac{1}{e_0^2} + \frac{2N}{4\pi}\frac{1}{2|\sigma|^2} \right)\Bigg|_{e_0^2 \to \infty} = \frac{2N}{4\pi}\frac{1}{2|\sigma|^2} \qquad \sum_1^{2N} \mathtt{Q}^2 = 2N$$



Effective action for the σ kinetic term is

$$S_{\rm eff}^{\sigma} = \frac{2N}{4\pi}\int d^2x\,\frac{1}{2}\,\frac{|\partial_\alpha\sigma|^2}{|\sigma|^2}$$

Change of variables

$$\sigma = \gamma\, e^{-\frac{\phi+iY}{Q}}$$

gives

$$S_{\rm eff}^{\sigma} = \frac{1}{4\pi}\int d^2x\,\left\{\frac{1}{2}(\partial_\alpha\phi)^2 + \frac{1}{2}(\partial_\alpha Y)^2\right\}$$

Here Y is a compact variable, $Y + 2\pi Q \sim Y$, where

$$Q|_{N\to\infty} \approx \sqrt{2N}$$

To get background charge we need to introduce world sheet metric

$$\int d^2x\sqrt{h}\left\{h^{\alpha\beta}\left(\partial_\alpha\bar{n}_i\partial_\beta n^i + \partial_\alpha\bar{\rho}_j\partial_\beta\rho^j\right) + 2\,|\sigma|^2\left(\left|n^i\right|^2 + \left|\rho^j\right|^2\right)\right\}$$

This gives for σ kinetic term

$$S_{\rm eff}^{\sigma} = \frac{1}{4\pi}\int d^2x\sqrt{h}\,\frac{Q^2}{2}\,h^{\alpha\beta}\partial_\alpha \log\left(\sigma(h)^{1/4}\right)\partial_\beta \log\left(\overline{\sigma}(h)^{1/4}\right)$$

$$S_{\rm eff}^{\sigma} = \frac{1}{4\pi}\int d^2x\sqrt{h}\,\left\{\frac{1}{2}\,h^{\alpha\beta}(\partial_\alpha\phi\partial_\beta\phi + \partial_\alpha Y\partial_\beta Y) - \frac{Q}{2}\phi\, R^{(2)}\right\},$$

where 2D Ricci scalar $R^{(2)} = -\frac{1}{\sqrt{h}}\partial_\alpha^2 \log\sqrt{h}$ in conformal gauge.

The Liouville (twisted) superpotential comes from the 2D FI term of $\mathbb{WCP}(N,N)$ model (we keep μ fixed at $\beta\to 0$):

$$L_{\rm FI} = -\frac{\beta}{\sqrt{2}}\int d^2\tilde{\theta}\,\Sigma = \mu\int d^2\tilde{\theta}\,e^{-\frac{\phi+iY}{Q}}, \qquad \mu = -\frac{\gamma\beta}{\sqrt{2}}$$

This FI term was a marginal deformation of original $\mathbb{WCP}(N,N)$ WCP model so we expect that this twisted superpotential is also a marginal deformation of $\mathcal{N}=2$ Liouville theory.

σ is superpartner of A_α, hence

$$\Delta(\sigma) = \left(\frac{1}{2}, \frac{1}{2}\right)$$

Exact equivalence

Relax large N condition.

The form of the action after integrating out fields n and ρ is fixed on dimensional grounds and by supersymmetry. We need only to find $Q(N) \approx \sqrt{2N}$. Require that two central charges should be the same.

$$\hat{c}_{CY} = 2N - 1 = 1 + Q^2 = \hat{c}_L$$

gives

$$Q = \sqrt{2(N-1)}, \qquad k = \frac{1}{N-1}$$

For the conifold case $N = 2$

Coulomb branch of $\mathbb{WCP}(2,2) \Longleftrightarrow$ deformed conifold

Therefore marginal deformations on the Liouville side and on the conifold side should be identified

$$\mu \sim b$$

Mass deformation

Now using the $\mathcal{N}=2$ Liouville theory approach we make a step towards broadening the class of 4D $\mathcal{N}=2$ SQCDs where the solitonic string-gauge duality can be applied.

We introduce quark masses in $\mathcal{N}=2$ SQCD and changing values of mass parameters interpolate between SQCDs with different gauge groups and numbers of quark flavors.

2D $\mathbb{WCP}(N,N)$ world sheet model:
Take $N=2K$

$$m_i = (0, ..., 0, M, ..., M), \qquad m_i = m_{i+N} \\ \leftarrow K \rightarrow \leftarrow K \rightarrow \tag{1}$$

Fields ρ^j has the same masses as fields n^i, $j = i + N$, while half of n^i fields acquire masses M.

Starting point: $M \to \infty$

Half of n and ρ fields decouple. We have $\mathbb{WCP}(K,K)$ model

Final point: $M \to 0$

We have $\mathbb{WCP}(N,N)$ model $N = 2K$.

4D $\mathcal{N} = 2$ SQCD:
For $K = 2$ and $N = 4$ we interpolate between SQCD with U(2) gauge group and $N_f = 4$ quark flavors and SQCD with U(4) gauge group and $N_f = 8$ flavors in 4D.

Now consider $\mathbb{WCP}(N,N)$ model with nonzero twisted masses starting with the large N approximation.

$$S_{\text{eff}}^{\sigma} = \frac{1}{4\pi}\int d^2x \sum_{A=1}^{2N} \frac{|\partial_\alpha \sigma|^2}{|\sqrt{2}\sigma + m_A|^2}$$
$$= \frac{1}{4\pi}\int d^2x \, \frac{1}{2}\, \frac{|\partial_\alpha \sigma|^2}{|\sigma|^2} \sum_{A=1}^{2N} \frac{1}{\left|1 + \frac{m_A}{\sqrt{2}\sigma}\right|^2} \tag{2}$$

Take $\sigma = \gamma\, e^{-\frac{\phi + iY}{Q}}$, $N = 2K$ and $m_i = (0, ..., 0, M, ..., M)$. We get

$$S_{\text{eff}} = \frac{1}{4\pi}\int d^2x \, g_{cl}(\phi, Y) \left(\frac{1}{2}\,(\partial_\alpha \phi)^2 + \frac{1}{2}\,(\partial_\alpha Y)^2\right)$$

$$g_{cl}(\phi, Y) = 1 + \frac{1}{\left|1 + \frac{M}{\sqrt{2}\gamma}\, e^{\frac{\phi + iY}{Q}}\right|^2}, \qquad Q^2 \approx 2K$$

We use this just as initial conditions, namely

$$g_{cl}(\phi, Y) \approx 1 + \frac{2\gamma^2}{|M|^2}\, e^{-\frac{2\phi}{Q}}$$

for the metric warp factor and

$$\Phi \approx -\frac{Q}{2}\,\phi$$

for the dilaton.
True metric and dilaton will be found by solving the gravity equations of motion

Relax large K approximation

$$Q = \sqrt{2(K-1)}, \qquad \text{for} \;\; K = 2 \qquad Q = \sqrt{2}$$

Gravity equations

The bosonic part of the action of the type-II supergravity in the string frame is given by

$$S = \frac{1}{2\kappa^2} \int d^D x \sqrt{-G}\, e^{-2\Phi} \left\{ R + 4G^{MN} \partial_M \Phi \partial_N \Phi_N + \cdots \right\}$$

Einstein's equations:

$$R_{MN} + 2D_M D_N \Phi = 0$$

Dilaton equation:

$$R = 4G^{MN} \partial_M \Phi \partial_N \Phi - 4G^{MN} D_M D_N \Phi + p,$$

where $p = \frac{D-10}{2}$.
Minkowski 4D $\times$ deformed Liouville theory. $D = 6,\ p = -2$
Ansatz for the internal metric:

$$ds^2_{\rm int} = g(\phi, Y) \left\{ d^2\phi + d^2 Y \right\}$$

Solutions to gravity equations

Solution for for the dilaton:

$$\Phi(\phi) = -\frac{Q}{2}\,\phi + \frac{1}{2}\,\ln g$$

and for the metric warp factor:

$$g(\phi) = \frac{1}{1 - \frac{1}{A}\,e^{-Q\phi}} = \frac{1}{1 - e^{-Q(\phi-\phi_0)}},$$

where A is a constant and $\phi_0 = -\frac{1}{Q}\ln A$.
We see that these solutions satisfy initial conditions for the mass-deformed metric and dilaton with

$$A = \frac{M^2}{2\gamma^2}\,, \qquad \phi_0 = -\frac{1}{Q}\ln\left(\frac{M^2}{2\gamma^2}\right)$$

only if $Q = \sqrt{2}$

The metric warp factor develop a naked singularity at $\phi = \phi_0$

$$g|_{\phi\to\phi_0} \approx \frac{1}{Q(\phi - \phi_0)}$$

where the curvature is singular. Thus, the geometry is defined only at $\phi > \phi_0$.

Turns out that the Liouville superpotential (Liouville wall) is not modified and is still a marginal deformation of the theory.

Liouville wall prevents field ϕ from penetrating to the region of large negative values.

$$\phi_{\rm wall} \sim -\, Q \ln \frac{1}{|b|}$$

- At $\phi_0 \ll \phi_{\rm wall}$ string theory describe hadrons of slightly deformed $\mathcal{N} = 2$ SQCD with U(2) gauge group and $N_f = 4$ quark flavors
- At $\phi_0 \gg \phi_{\rm wall}$ string theory describe hadrons of $\mathcal{N} = 2$ SQCD with U(4) gauge group and $N_f = 8$ quark flavors

Conclusions

- We demonstrate that the Coulomb branch of $\mathbb{WCP}(N, N)$ models can be effectively described by $\mathcal{N} = 2$ Liouville theory. We show this in large N approximation first and then extrapolate this to arbitrary N requiring that the central charges of both theories coincide. This gives $Q = \sqrt{2(N-1)}$.
- For $N = 2$ case we identify the coefficient μ in front of the Liouville superpotential with the parameter of the conifold complex structure deformation b. The Higgs and the Coulomb branches of $\mathbb{WCP}(2, 2)$ model are geometrically distinct and correspond to resolved and deformed conifold respectively.

- We show that non-Abelian critical string supported in mass-deformed $\mathcal{N}=2$ SQCD interpolating between theory with U(2) gauge group and $N_f = 4$ quarks and theory with U(4) gauge group and $N_f = 8$ quarks is associated with mass deformation of $\mathcal{N}=2$ Liouville world sheet theory.
- To find the true string vacuum we solve the effective gravity equation of motion.