Conformal four-point ladder integrals in diverse dimensions and polylogarithms

A. P. Isaev[1]

[1]Bogoliubov Laboratory of Theoretical Physics,
JINR, Dubna, Russia;
Faculty of Physics, Moscow State University

In collaboration with ***S.Derkachov and L.Shumilov***
(Steklov MI, St.Petersburg, Hamburg University)

International Conference "Advances in Quantum Field Theory (AQFT'25)"
11-15 August 2025, BLTP JINR, Dubna

План

1. Evaluating Feynman diagrams in dimensions $D > 4$ is an important aspect of dimensional regularization and higher-dimensional field theories (e.g., in string inspired theories, SUSY, Kaluza-Klein models and ...).
Relatively recently: D.I. Kazakov, L.V. Bork, R.M. Iakhibbaev, M.V. Kompaniets, D.M. Tolkachev, D.E. Vlasenko, ... (nonrenormalizable theories, dual conformal symmetry, ...)
K.B. Alkalaev, S. Mandrygin (2025), ...

2. The relations of (conformal) Feynman integrals in different dimensions D by means of special operators acting on the variables of the external legs, masses, etc, were considered by many authors S.E. Derkachov, J. Honkonen, Y.M. Pis'mak (1990); O. Tarasov (1996); J. Drummond, J. Henn, J. Plefka (2009); R. Lee (2010); M.F. Paulos, M. Spradlin, A. Volovich (2012); F. Loebbert, S. Stawinski (2024); A.C. Petkou, a.o. (2024),...

3. For certain F.D., analytical results are expressed in terms of **multiple (nested) zeta-values and (multiple) polylogarithms** – subjects for investigations in mathematics.

Feynman diagrams (graphs) visualize perturbative integrals. We associate each vertex of the graph with the point in $\mathbb{R}^D$, while the lines (edges) of the graph (with index α) are propagators of massless particles

$$x \overset{\alpha}{\text{———}} y \quad = \quad 1/(x-y)^{2\alpha}$$

where $\boxed{(x-y)^{2\alpha} := \Big(\sum_{i=1}^{D}(x_i - y_i)\,(x_i - y_i)\Big)^{\alpha}}$, $\alpha \in \mathbb{C}$, $x, y \in \mathbb{R}^D$.

We have 2 types of vertices: the boldface vertices $\bullet$ denote the integration over $\mathbb{R}^D$ and not boldface vertices (not integrated).

These Feynman diagrams are called F.D. in the configuration space.

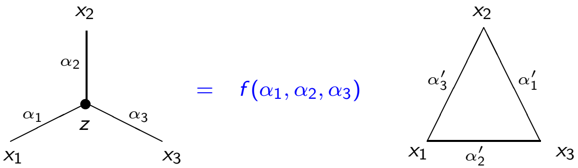
Star-triangle relation (STR):

$$\int \frac{d^D z}{(x_1 - z)^{2\alpha_1}\,(x_2 - z)^{2\alpha_2}\,(x_3 - z)^{2\alpha_3}} = \frac{f(\alpha_1,\alpha_2,\alpha_3)}{(x_1 - x_2)^{2\alpha_3'}\,(x_2 - x_3)^{2\alpha_1'}\,(x_1 - x_3)^{2\alpha_2'}} \, ,$$

where $\boxed{\alpha_1 + \alpha_2 + \alpha_3 = D}$ – conformal condition and

$$f(\alpha_1,\alpha_2,\alpha_3) = (2\pi)^{2D}\, a(\alpha_1)\, a(\alpha_2)\, a(\alpha_3),$$

$$a(\alpha) = \frac{\pi^{-D/2}\,\Gamma(\alpha')}{2^{2\alpha}\,\Gamma(\alpha)} \, , \qquad \alpha' := D/2 - \alpha \, .$$



E.S. Fradkin and M.Y. Palchik, Phys. Rept. 44 (1978) 249;
A.B. Zamolodchikov, Phys.Lett. B 97 (1980) 63;
A.N.Vasilev, Y.M.Pismak, Y.R.Khonkonen, Theor.Math.Phys. 47 (1981) 465.;
D.I.Kazakov, Phys.Lett.B 133 (1983) 406;

A.B. Zamolodchikov, "Fishing-net" Diagrams as a Completely Integrable System, Phys.Lett. B 97 (1980) 63-66. *INSPIRE 112 citations*

The star-triangle relation can be visualized on $\mathbb{R}^2$ as the Yang-Baxter Equation (faces are painted in a checkerboard pattern).

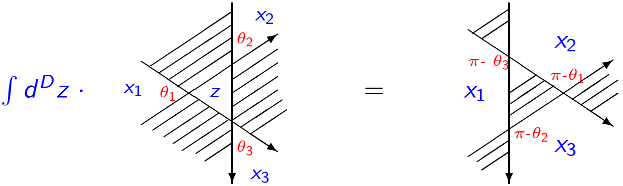
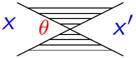


Fig. **1**

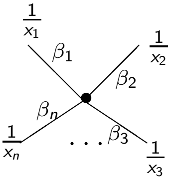
where $\theta_1 + \theta_2 + \theta_3 = \pi$ and



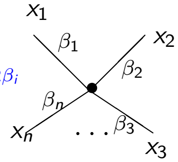
$$= \frac{\Gamma(\alpha(\theta))}{\pi^{\alpha(\theta)}} \frac{1}{(x - x')^{2\alpha(\theta)}}, \qquad \alpha(\theta) := \frac{D}{2\pi}(\pi - \theta).$$

A.Zamolodchikov calculated "fishing-net" planar Feynman diagrams as partition functions of certain integrable statistical model.

Remark, K. Symanzik (1972). The n-point conformal vertex, is the vertex for which the sum of line indices is $\boxed{\sum_{i=1}^{n} \beta_i = D}$. It is transformed under $x_i \to \frac{1}{x_i}$ as



$$= \prod_{i=1}^{n} x_i^{2\beta_i}$$



$$\boxed{\frac{1}{x_i} := \frac{x_{i\mu}}{(x_i)^2}}$$

An integral $I(x_1, ..., x_n; \vec{\beta})$, which is depicted as a Feynman graph with n external lines having indices $\vec{\beta} = (\beta_1, ..., \beta_n)$, and all conformal internal boldface vertices is transformed as

$$I(\frac{1}{x_1}, ..., \frac{1}{x_n}; \vec{\beta})) = \prod_{i=1}^{n} x_i^{2\beta_i} I(x_1, ..., x_n; \vec{\beta}))$$

Then, the conformal invariant function can be chosen as

$$f(u_1, u_2, ...) = I(x_1, ..., x_n; \vec{\beta}) \prod_{i=1}^{n} \frac{(x_{i,i+1})^{\beta_i} (x_{i,i+2})^{\beta_i}}{(x_{i+1,i+2})^{\beta_i}} , \qquad (n + i = i) ,$$

where $u_1, u_2, ...$ are conformal invariant cross-ratios.

Operator formalism for massless diagrams.

Let $\{\hat{q}^\mu_a, \ \hat{p}^\nu_b\}$ $(a, b = 1, ..., n)$ be generators of the D-dimensional Heisenberg algebras $\hat{\mathrm{H}}_a$ (a=1,...,n)

$$[\hat{q}^\mu_a, \ \hat{q}^\nu_b] = 0 = [\hat{p}^\mu_a, \ \hat{p}^\nu_b] \ , \qquad [\hat{q}^\mu_a, \ \hat{p}^\nu_b] = i \, \delta^{\mu\nu} \, \delta_{ab} \qquad (\mu, \nu = 1, ..., D) \ .$$

We introduce states $|x_a\rangle \in V_a$ which diagonalize coordinates $\hat{q}^\mu_a$:

$$\hat{q}^\mu_a |x_a\rangle = x^\mu_a |x_a\rangle \ .$$

These states form the basis in the representation space V_a of subalgebra $\hat{\mathrm{H}}_a$. We also introduce the dual states $\langle x_a|$ such that the orthogonality and completeness conditions are valid

$$\langle x_a | x'_a \rangle = \delta^D(x_a - x'_a), \qquad \boxed{\int d^D x_a \ |x_a\rangle\langle x_a| = I_a},$$

where I_a is the unit operator in V_a and there are no summations over indices a. So, we have the algebra $\hat{\mathrm{H}}^{(n)} = \otimes_{a=1}^n \hat{\mathrm{H}}_a$ which acts in the space $V_1 \otimes \cdots \otimes V_n$ with basis vectors $|x_1\rangle \otimes \cdots |x_n\rangle$.

Further we use operators $(\hat{q}_a)^{2\alpha} = (\sum_\mu \hat{q}_a^\mu \hat{q}_a^\mu)^\alpha$ and $(\hat{p}_a)^{2\beta} = (\sum_\mu \hat{p}_a^\mu \hat{p}_a^\mu)^\beta$ with non-integer α and β. These operators are understood as integral operators with kernels; e.g. for $\mathrm{H}^{(1)}$: $\langle x|\,(\hat{q})^{-2\alpha}\,|y\rangle = (x)^{-2\alpha}\,\delta^D(x-y)$ and

$$\langle x|\frac{1}{(\hat{p})^{2\beta}}|y\rangle = \int \frac{d^D k}{(2\pi)^D}\,\frac{e^{ik(x-y)}}{(k)^{2\beta}} = \frac{a(\beta)}{(x-y)^{2\beta'}}\;,$$

$$a(\beta) := \frac{2^{-2\beta}}{\pi^{D/2}}\,\frac{\Gamma(\beta')}{\Gamma(\beta)}\;, \qquad \beta' := D/2-\beta\;.$$

Note that STR is

$$\int d^D z\, \langle x|\hat{p}^{2\alpha}|z\rangle\, \hat{z}^{2(\alpha+\beta)}\, \langle z|\hat{p}^{2\beta}|y\rangle = \hat{x}^{2\beta}\,\langle x|\hat{p}^{2(\alpha+\beta)}|y\rangle \hat{y}^{2\alpha} \quad \Longleftrightarrow$$
$$\langle x|\hat{p}^{2\alpha}\,\hat{q}^{2(\alpha+\beta)}\,\hat{p}^{2\beta}|y\rangle = \langle x|\hat{q}^{2\beta}\,\hat{p}^{2(\alpha+\beta)}\,\hat{q}^{2\alpha}|y\rangle$$

and STR in operator approach takes a remarkable form [API, NPB 662(2003)461]:

$$\boxed{\hat{p}^{2\alpha}\,\hat{q}^{2(\alpha+\beta)}\,\hat{p}^{2\beta} = \hat{q}^{2\beta}\,\hat{p}^{2(\alpha+\beta)}\,\hat{q}^{2\alpha}} \;\Rightarrow\; \hat{q}^{2\alpha}\,\boxed{\cdots}\,\hat{p}^{2\alpha} \;\Rightarrow$$

$$\hat{q}^{2\alpha}\,\hat{p}^{2\alpha}\cdot\hat{q}^{2\gamma}\,\hat{p}^{2\gamma} = \hat{q}^{2\gamma}\,\hat{p}^{2\gamma}\cdot\hat{q}^{2\alpha}\,\hat{p}^{2\alpha}\big|_{\gamma=\alpha+\beta} \;\Rightarrow\; \mathcal{H}_\alpha\,\mathcal{H}_\gamma = \mathcal{H}_\gamma\,\mathcal{H}_\alpha\,,$$

which is commutativity condition for operators $\mathcal{H}_\alpha = \hat{q}^{2\alpha}\,\hat{p}^{2\alpha} \quad (\forall\alpha)$.

The spectral theory of $\mathcal{H}_\beta$ follows from the theory of infinite dim. irreps of $\mathfrak{sl}_2$:
$[\hat{q}^2, \ \hat{p}^2] = 4H \ , \quad H\,\hat{q}^2 = \hat{q}^2\,(H+2) \ , \quad H\,\hat{p}^2 = \hat{p}^2\,(H-2)$, where
$H := i\,(\hat{q}\,\hat{p}) + D/2$ – Cartan element and quadratic Casimir is:

$$C_{(2)} := (\hat{p}^2\hat{q}^2 + 2H + H^2) = (\mathcal{H}_1 + (H+1)^2 - 1) \ .$$

Proposition 1. *Let* $|\psi_{j,\nu}\rangle$ *be common eigenvectors of operators* $\mathcal{H}_\beta = \hat{p}^{2\beta}\,\hat{q}^{2\beta}$ *:*

$$H_\beta\,|\psi_{j,\nu}\rangle = \tau_{j,\nu}(\beta)\,|\psi_{j,\nu}\rangle \ , \qquad (1)$$

where $\tau_{j,\nu}(\beta)$ *are corresponding eigenvalues. We numerate* $|\psi_{j,\nu}\rangle$ *by two numbers* $\nu, j \in \mathbb{R}$ *which are fixed by eigenvalues of* $H = -H^\dagger$ *and* $C_{(2)}$*:*

$$H\,|\psi_{j,\nu}\rangle = -2i\nu|\psi_{\nu,j}\rangle \ , \quad C_{(2)}\,|\psi_{j,\nu}\rangle = 4j(j-1)\,|\psi_{j,\nu}\rangle \ .$$

Then, the eigenvalues $\tau_{j,\nu}(\beta)$ *in (1) are*

$$\tau_{j,\nu}(\beta) = 4^\beta\,\frac{\Gamma(j+\beta-i\nu)\,\Gamma(j+i\nu)}{\Gamma(j-\beta+i\nu)\,\Gamma(j-i\nu)} \ , \qquad \boxed{j := \frac{D}{4} + \frac{n}{2}} \ .$$

They are degenerate and the eigenvectors $|\psi^{\mu_1\cdots\mu_n}_{j,\nu}\rangle$ *form complete system*

$$\sum_{n=0}^{\infty} \mu(n) \int_{-\infty}^{+\infty} d\nu |\psi^{\mu_1\cdots\mu_n}_{j,\nu}\rangle\langle\psi^{\mu_1\cdots\mu_n}_{j,\nu}| = I \ , \qquad \mu(n) := \frac{2^{n-1}\Gamma(D/2+n)}{\pi^{D/2+1}\,n!} \ ,$$

We consider D-dimensional L-loop ladder integrals with arbitrary indices on the lines $\alpha_k, \beta_k, \gamma_k$:

$$I^{(L)}(x_1, x_2, x_3, x_4; \alpha_i, \beta_i \gamma_i) = \int d^D x_5 \ldots d^D x_{L+4} \times$$

$$\times \left. \frac{1}{(x_{1,5})^{2\gamma_0}} \prod_{i=1}^{L} \frac{1}{(x_{2,i+4})^{2\alpha_i}} \frac{1}{(x_{4,i+4})^{2\beta_i}} \frac{1}{(x_{i+4,i+5})^{2\gamma_i}} \right|_{x_{L+5} \equiv x_3} ,$$

and $I^{(0)} = x_{13}^{-2\gamma_0}$. These integrals are presented in the dual form in Fig.1, where integrations are over boldface vertices which are placed in the boxes.

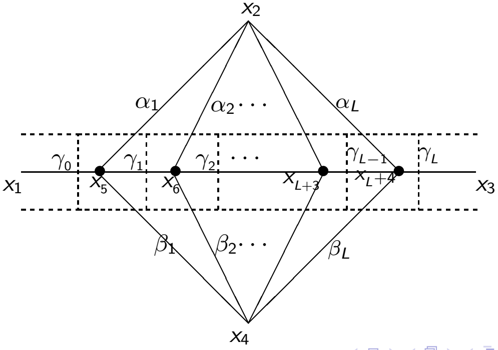


Fig.1

We consider the special case when all boldface vertices in Fig.1 are conformal

$$\gamma_{k-1} + \alpha_k + \beta_k + \gamma_k = D \qquad (\forall k) \; ,$$

and in addition we fix $\beta_k = \alpha_k = \beta$ $(\forall k)$, thus $\gamma_k = D/2 - \beta \equiv \beta'$. In this case

$$I^{(L)}(\tfrac{1}{x_i};\beta) = (x_1)^{2\beta'}(x_2)^{2L\beta}(x_3)^{2\beta'}(x_4)^{2L\beta} I^{(L)}(x_i;\beta) \; , \qquad (2)$$

and we have

$$(x_{24})^{2L\beta}(x_{13})^{2(D/2-\beta)} I^{(L)}(x_i;\beta) \equiv f_{(L)}(u,v;\beta) \; ,$$

where $f_{(L)}$ – is a conformal invariant function that depends only on two cross-ratios

$$u = \frac{x_{12}^2 x_{34}^2}{x_{24}^2 x_{13}^2} \; , \quad v = \frac{x_{14}^2 x_{23}^2}{x_{24}^2 x_{13}^2} \; .$$

Remark. For the choice $\beta_k = \alpha_k = \beta$, $\gamma_k = beta'$ $(\forall k)$, the functions (associated to the diagrams in Fig.1) give contributions to the 4-point amplitude in general D-dimensional bi-scalar fishnet CFT [V. Kazakov and E. Olivucci (2018)] with the Lagrangian

$$\mathcal{L} = \frac{1}{2}\mathrm{Tr}\Big(\phi_1\, \partial^{2\beta}\phi_1 + \phi_2\, \partial^{2\beta'}\phi_2 + g\phi_1\phi_2\phi_1\phi_2\Big).$$

Taking the limit $x_4 \to \infty$ in (2) we prove that all propagators to the point x_4 in Fig.1 disappeared and we deduce operator formula

$$f_{(L)}(u,v;\beta) = \frac{(\mathrm{x}-\mathrm{y})^{2\beta'}}{a(\beta)^{L+1}} \langle \mathrm{x}| \frac{1}{\hat{p}^{2\beta}} \left(\frac{1}{\hat{q}^{2\beta}} \frac{1}{\hat{p}^{2\beta}} \right)^L |\mathrm{y}\rangle$$

$$\mathrm{x} := \frac{1}{x_{12}} - \frac{1}{x_{42}}, \quad \mathrm{y} := \frac{1}{x_{32}} - \frac{1}{x_{42}}, \quad \frac{\mathrm{y}^2}{\mathrm{x}^2} = \frac{u}{v}, \quad \frac{(\mathrm{x}-\mathrm{y})^2}{\mathrm{x}^2} = \frac{1}{v}$$

$$f_{(L)}(u,v;\beta) = \frac{(\mathrm{x}-\mathrm{y})^{2\beta'}\mathrm{x}^{2\beta}}{a(\beta)^{L+1}} \langle \mathrm{x}| \left(\frac{1}{\hat{p}^{2\beta}\hat{q}^{2\beta}} \right)^{L+1} |\mathrm{y}\rangle = \boxed{\frac{(\mathrm{x}-\mathrm{y})^{2\beta'}\mathrm{x}^{2\beta}}{a(\beta)^{L+1}} \langle \mathrm{x}| \frac{1}{\mathcal{H}_\beta^{L+1}} |\mathrm{y}\rangle}$$

We need to use eigenvalues and eigenfunctions of graph building operators $\mathcal{H}_\beta$.

Remark. The generating function of the L-loop ladder conformal functions $f_{(L)}(u,v;\beta)$, related to the L-loop ladder integrals $I^{(L)}(x_i;\beta)$ is represented as a Green's function for D-dimensional conformal quantum mechanics

$$\frac{a(\beta)}{(\mathrm{x}-\mathrm{y})^{2(D/2-\beta)}} \cdot \sum_{L=0}^{\infty} \big(g\, a(\beta)\big)^L f_{(L)}(u,v;\beta) = \langle \mathrm{x}| \frac{1}{\hat{p}^{2\beta} - g\, \hat{q}^{-2\beta}} |\mathrm{y}\rangle ,$$

$$\frac{a(\beta)}{(x-y)^{2(D/2-\beta)}} = \langle x| \frac{1}{\hat{p}^{2\beta}} |y\rangle ,$$

It is convenient to replace the L-loop integrals $I^{(L)}$ and conformally invariant functions $f_{(L)}$ with the functions

$$\Phi_L^{(\beta)}(\mathrm{x},\mathrm{y}) = \frac{L!4^L a(\beta)^{L+1}}{(\mathrm{x}-\mathrm{y})^{2(D/2-\beta)}} f_{(L)} = L!4^L a(\beta)^{L+1} x_{12}^{2(D/2-\beta)} x_{23}^{2(D/2-\beta)} x_{24}^{2L\beta} I^{(L)} \,.$$

From operator representation by inserting the unity (completeness condition for eigenfunctions of operators $\mathcal{H}_\beta$) we find the integral representation

$$\Phi_L^{(\beta)}(\mathrm{x},\mathrm{y}) = \frac{\Gamma(\lambda)\, L!\, 4^L \mathrm{x}^{2\beta}}{2\pi^{\lambda+2}(\mathrm{x}^2\, \mathrm{y}^2)^{D/4}} \sum_{n=0}^{\infty} (n+\lambda)\, C_n^{(\lambda)} \left(\frac{z+\bar{z}}{2\sqrt{z\bar{z}}} \right) \int_{-\infty}^{+\infty} d\nu \frac{(z\bar{z})^{i\nu}}{\left[\tau_{n,\nu}(\beta;\lambda)\right]^{L+1}}$$

where $\lambda = \frac{D}{2} - 1$, parameters z and $\bar{z}$ are related to cross-ratios:

$$u = \frac{z\bar{z}}{(1-z)(1-\bar{z})} \,, \qquad v = \frac{1}{(1-z)(1-\bar{z})} \,. \tag{3}$$

and $\tau_{n,\nu}(\beta)$ are eigenvalues of $\mathcal{H}_\beta$.

Consider the simplest case $\beta = \alpha = 1$ $(\gamma = D/2 - 1)$. This choice of parameters in the case $D = 4$ leads to the usual ladder integrals, when all indices on the lines are equal to 1. For an arbitrary D, the formula for $\Phi_L(\mathrm{x}, \mathrm{y})$ gives the answer for the diagram Fig.1 with $D/2 - 1$ on the horizontal lines and 1 on the lines that go to x_2 and x_4. Using the definition of $\tau_{n,\nu}(\beta)$ with $\beta = 1$, we derive the formula

$$\Phi_L(\mathrm{x}, \mathrm{y}) = \frac{\Gamma(\lambda)\, L!\, \mathrm{x}^2}{8\pi^{\lambda+2}\, (\mathrm{x}^2\, \mathrm{y}^2)^{D/4}} \sum_{n=0}^{\infty} (n+\lambda)\, C_n^{(\lambda)}\, (\hat{\mathrm{x}}\hat{\mathrm{y}}) \;\cdot$$

$$\cdot \int_{-\infty}^{+\infty} \frac{d\nu \;\; (\mathrm{y}^2/\mathrm{x}^2)^{i\nu}}{\Big((\frac{D}{4} + \frac{n}{2} - i\nu)(\frac{D}{4} + \frac{n}{2} + i\nu - 1) \Big)^{L+1}}\,.$$

Integrating over ν gives

$$\Phi_L(\mathrm{x}, \mathrm{y}) = \frac{\Gamma(\lambda)}{4\pi^{\lambda+1}\mathrm{x}^{2\lambda}} \sum_{k=0}^{L} \frac{(-1)^k (2L-k)!}{k!(L-k)!}\, \mathrm{Log}^k(z\bar{z})\, \Sigma_s^{(\lambda)}(z, \bar{z}) \qquad (4)$$

where we introduce the function

$$\Sigma_s^{(\lambda)}(z, \bar{z}) = \sum_{n=0}^{\infty} C_n^{(\lambda)} \left(\frac{z + \bar{z}}{2\sqrt{z\bar{z}}} \right) (z\bar{z})^{n/2}\, \frac{1}{(\lambda+n)^s}\;,$$

and denote $s = 2L - k$.

Proposition 1. *For general* $D > 2$ *The function* $\Sigma_s^{(\lambda)}$ *is expressed in terms of known special functions:*

$$\Sigma_s^{(\lambda)}(z,\bar{z}) = P_\lambda(z\partial_z)\left(\frac{z^\lambda}{(z-\bar{z})^\lambda}\Phi(z,s,\lambda)\right) + P_\lambda(\bar{z}\partial_{\bar{z}})\left(\frac{\bar{z}^\lambda}{(\bar{z}-z)^\lambda}\Phi(\bar{z},s,\lambda)\right)$$

where

$$P_\lambda(z\partial_z) = \frac{1}{\Gamma(\lambda)}\frac{\Gamma(z\partial_z+\lambda)}{\Gamma(z\partial_z+1)} ,$$

and

$$\Phi(z,s,\lambda) = \sum_{n=0}^{\infty}\frac{z^n}{(\lambda+n)^s}$$

is a Lerch function, which generalizes the polylogarithms

$$\mathrm{Li}_s(z) = z\,\Phi(z,s,1) = \sum_{n=1}^{\infty}\frac{z^n}{n^s} .$$

If we restrict ourselves to the case $\lambda = D/2 - 1 \in \mathbb{Z}_{>0}$, which corresponds to even dimensions $D = 2(1+\lambda)$, then the operator $P_\lambda(z\partial_z)$ becomes a polynomial in $z\partial_z$:

$$P_\lambda(z\partial_z) = \frac{1}{\Gamma(\lambda)} \frac{\Gamma(z\partial_z + \lambda)}{\Gamma(z\partial_z + 1)} = \frac{1}{(\lambda-1)!}(z\partial_z + 1)\cdots(z\partial_z + \lambda - 1)\,, \qquad (5)$$

where $P_1(z\partial_z) = 1$, and the action of this operator gives explicit expressions for $\Sigma_s^{(\lambda)}(z,\bar{z})$.

Proposition 2. *In the case $\lambda = D/2 - 1 \in \mathbb{Z}_{>0}$, the function $\Sigma_s^{(\lambda)}(z,\bar{z})$ has explicit expression via polylogarithms*

$$\Sigma_s^{(\lambda)}(z,\bar{z}) = P_\lambda(z\partial_z)\frac{\mathrm{Li}_s(z)}{(z-\bar{z})^\lambda} + P_\lambda(\bar{z}\partial_{\bar{z}})\frac{\mathrm{Li}_s(\bar{z})}{(\bar{z}-z)^\lambda}$$

where operators $P_\lambda(z\partial_z)$ and $P_\lambda(\bar{z}\partial_{\bar{z}})$ were defined in (5).

Explicit answers for $\Sigma_s^{(1)}(z,\bar{z})$ in terms of polylogaritms and rational functions for several first values of λ.

- $\lambda = 1$:

$$\Sigma_s^{(1)}(z,\bar{z}) = \frac{\mathrm{Li}_s(z)}{z-\bar{z}} + (z \leftrightarrow \bar{z}).$$

- $\lambda = 2$:

$$\begin{aligned}\Sigma_s^{(2)}(z,\bar{z}) =& (z\partial_z + 1)\frac{\mathrm{Li}_s(z)}{(z-\bar{z})^2} + (z \leftrightarrow \bar{z}) \\ =& -\frac{z+\bar{z}}{(z-\bar{z})^3}\mathrm{Li}_s(z) + \frac{\mathrm{Li}_{s-1}(z)}{(z-\bar{z})^2} + (z \leftrightarrow \bar{z}).\end{aligned}$$

- $\lambda = 3$:

$$\begin{aligned}\Sigma_s^{(3)}(z,\bar{z}) =& \frac{1}{2}(z\partial_z + 2)(z\partial_z + 1)\frac{\mathrm{Li}_s(z)}{(z-\bar{z})^3} + (z \leftrightarrow \bar{z}) \\ =& \frac{z^2 + 4z\bar{z} + \bar{z}^2}{(z-\bar{z})^5}\mathrm{Li}_s(z) - \frac{3(z+\bar{z})}{2(z-\bar{z})^4}\mathrm{Li}_{s-1}(z) + \frac{\mathrm{Li}_{s-2}(z)}{2(z-\bar{z})^3} + (z \leftrightarrow\end{aligned}$$

- $\lambda = 4$:

$$\Sigma_s^{(4)}(z,\bar{z}) = \frac{1}{6}(z\partial_z + 3)(z\partial_z + 2)(z\partial_z + 1)\frac{\mathrm{Li}_s(z)}{(z-\bar{z})^4} + (z \leftrightarrow \bar{z})$$

$$= -\frac{(z+\bar{z})(z^2+8z\bar{z}+\bar{z}^2)}{(z-\bar{z})^7}\mathrm{Li}_s(z) + \frac{11z^2+38z\bar{z}+11\bar{z}^2}{6(z-\bar{z})^6}\mathrm{Li}_{s-1}(z)$$

$$-\frac{z+\bar{z}}{(z-\bar{z})^5}\mathrm{Li}_{s-2}(z) + \frac{\mathrm{Li}_{s-3}(z)}{6(z-\bar{z})^4} + (z \leftrightarrow \bar{z}).$$

- $\lambda = 5$:

$$\Sigma_s^{(5)}(z,\bar{z}) = \frac{1}{24}(z\partial_z + 4)(z\partial_z + 3)(z\partial_z + 2)(z\partial_z + 1)\frac{\mathrm{Li}_s(z)}{(z-\bar{z})^5} + (z \leftrightarrow \bar{z}) =$$

$$= \frac{z^4+16z^3\bar{z}+36z^2\bar{z}^2+16z\bar{z}^3+\bar{z}^4}{(z-\bar{z})^9}\mathrm{Li}_s(z) -$$

$$-\frac{5(z+\bar{z})(5z^2+32z\bar{z}+5\bar{z}^2)}{12(z-\bar{z})^8}\mathrm{Li}_{s-1}(z) + \frac{5(7z^2+22z\bar{z}+7\bar{z}^2)}{24(z-\bar{z})^7}\mathrm{Li}_{s-2}(z)$$

$$-\frac{5(z+\bar{z})}{12(z-\bar{z})^6}\mathrm{Li}_{s-3}(z) + \frac{\mathrm{Li}_{s-4}(z)}{24(z-\bar{z})^5} + (z \leftrightarrow \bar{z}).$$

At the end we give explicit expressions for of L-loop ladder integrals in $D = 4, 6, 8, 10$.

1. The case $D = 4$ ($\lambda = D/2 - 1 = 1$) and indices on the lines $\alpha = \beta = \gamma = 1$ [N.I. Usyukina and A.I. Davydychev (1993)]:

$$\Phi_L(\mathrm{x},\mathrm{y})|_{D=4} = \frac{(z-\bar{z})^{-1}}{4\pi^2\mathrm{x}^2}\sum_{k=0}^{L}\frac{(-1)^k(2L-k)!}{k!(L-k)!}\mathrm{Log}^k(z\bar{z})\Big(\mathrm{Li}_{2L-k}(z)-\mathrm{Li}_{2L-k}(\bar{z})\Big)\,. \tag{6}$$

2. The case $D = 6$ ($\lambda = 2$) and indices on the lines $\alpha = \beta = 1$, $\gamma = 2$:

$$\Phi_L(\mathrm{x},\mathrm{y})|_{D=6} = \frac{(z-\bar{z})^{-2}}{4\pi^3\mathrm{x}^4}\sum_{k=0}^{L}\frac{(-1)^k(2L-k)!}{k!(L-k)!}\mathrm{Log}^k(z\bar{z})\cdot \Big(-\frac{(z+\bar{z})}{(z-\bar{z})}(\mathrm{Li}_{2L-k}(z)-\mathrm{Li}_{2L-k}(\bar{z}))+\mathrm{Li}_{2L-k-1}(z)+\mathrm{Li}_{2L-k-1}(\bar{z})\Big)\,. \tag{7}$$

3. The case $D = 8$ ($\lambda = 3$) and indices on the lines $\alpha = \beta = 1$, $\gamma = 3$:

$$\Phi_L(\mathrm{x},\mathrm{y})|_{D=8} = \frac{(z-\bar{z})^{-3}}{4\pi^4\mathrm{x}^6}\sum_{k=0}^{L}\frac{(-1)^k(2L-k)!}{k!(L-k)!}\mathrm{Log}^k(z\bar{z})\cdot \Big(2\frac{(z^2+4z\bar{z}+\bar{z}^2)}{(z-\bar{z})^2}\mathrm{Li}_{2L-k}(z)+3\frac{(z+\bar{z})}{(z-\bar{z})}\mathrm{Li}_{2L-k-1}(z)+\mathrm{Li}_{2L-k-2}(z)+(z\leftrightarrow\bar{z})\Big)\,. \tag{8}$$

4. The case $D = 10$ ($\lambda = 4$) and indices on the lines $\alpha = \beta = 1$, $\gamma = 4$:

$$\begin{aligned}\Phi_L(\mathrm{x},\mathrm{y})|_{D=10} &= \frac{(z-\bar{z})^{-4}}{4\pi^5 \mathrm{x}^8} \sum_{k=0}^{L} \frac{(-1)^k (2L-k)!}{k!(L-k)!} \mathrm{Log}^k(z\bar{z}) \cdot \\ &\cdot \Big(-6 \frac{(z+\bar{z})(z^2+8z\bar{z}+\bar{z}^2)}{(z-\bar{z})^3} \mathrm{Li}_{2L-k}(z) + \\ &+ \frac{(11z^2+38z\bar{z}+11\bar{z}^2)}{(z-\bar{z})^2} \mathrm{Li}_{2L-k-1}(z) - 6\frac{(z+\bar{z})}{(z-\bar{z})} \mathrm{Li}_{2L-k-2}(z) + \\ &+ \mathrm{Li}_{2L-k-3}(z) + (z \leftrightarrow \bar{z}) \Big) . \end{aligned} \tag{9}$$

Remark 1. Define operator
[D. Simmons-Duffin (2014), F.Loebbert, S.F.Stawinski (2024)]

$$R_D = \frac{1}{z-\bar{z}} \left(z\partial_z - \bar{z}\partial_{\bar{z}} \right) . \tag{10}$$

This operator (for general D and β) is the shift operator in dimension of the space-time $D \to D+2$ for conformal (ladder) 4-point diagrams.
Indeed, the conformal functions

$$\widetilde{\Phi}_L^{(\beta)}(z,\bar{z};\lambda) = a(\beta)^{L+1} 2\pi^{\lambda+2} \frac{(z\bar{z})^{1/2}}{((1-z)(1-\bar{z}))^{\lambda-\beta+1}} \, f_L(z,\bar{z};\beta) \, ,$$

obeys differential equations (recursion in D)

$$R_D \, \widetilde{\Phi}_L^{(\beta)}(z,\bar{z};\lambda) = \widetilde{\Phi}_L^{(\beta)}(z,\bar{z};\lambda+1) \quad \Rightarrow \quad R_D \cdot \Sigma_s^{(\lambda)}(z,\bar{z}) = \lambda \, \Sigma_s^{(\lambda+1)}(z,\bar{z}) \, .$$

I.e., since $\lambda = D/2 - 1$, the operator R_D translate the expression for conformal L-loop 4-point ladder integral from dimension D to dimension $D+2$.

Remark 2. For all D and β, the functions $f_L(z,\bar{z};\beta)$ possess the symmetry

$$f_L(z,\bar{z};\, \beta) = f_L(1/z, 1/\bar{z};\, \beta) \, .$$

Remark 3. For $\beta = 1$ and arbitrary $\lambda = D/2 - 1$ the L-loop conformal functions

$$\tilde{\tilde{\Phi}}_L^{(1)}(z,\bar{z}) = \frac{8\pi^{\lambda+2}}{\Gamma(\lambda)} \left(x^2 y^2\right)^{\lambda/2} \Phi_L^{(1)}(x,y) = \frac{2L!\Gamma(\lambda)^L (z\bar{z})^{\lambda/2}}{\pi^{L\lambda+1}(1-z)^\lambda(1-\bar{z})^\lambda} f_{(L)}(z,\bar{z};1)$$

satisfy differential equations (recursion in L) [F.Loebbert, S.F.Stawinski (2024)]

$$R_L^{(1)}\, \tilde{\tilde{\Phi}}_L^{(1)}(z,\bar{z}) = \tilde{\tilde{\Phi}}_{L-1}^{(1)}(z,\bar{z})$$

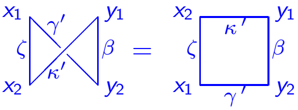
where

$$R_L^{(1)} = -\frac{1}{\log(z\bar{z})}\left(z\partial_z + \bar{z}\partial_{\bar{z}}\right),$$

The generalization of the graph building operator is

$$Q_{12}^{(\zeta,\kappa,\gamma)} := \frac{1}{a(\kappa)a(\gamma)} \mathcal{P}_{12}\, \hat{q}_{12}^{-2\zeta}\; \hat{p}_1^{-2\kappa}\; \hat{p}_2^{-2\gamma}\; \hat{q}_{12}^{-2\beta}\,, \qquad \zeta+\beta=\kappa+\gamma\,.$$

We depict the integral kernel of the D-dimensional operator $Q_{12}^{(\zeta,\kappa,\gamma)}$ as follows $((\kappa' := D/2-\kappa,\ \gamma' := D/2-\gamma))$

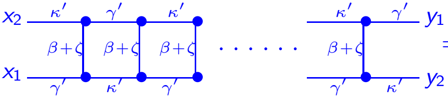


$$= \langle x_1, x_2 | Q_{12}^{(\zeta,\kappa,\gamma)} | y_1, y_2\rangle =$$

$$= \frac{1}{a(\kappa)a(\gamma)} \cdot \langle x_1, x_2 |\, \mathcal{P}_{12}\, \hat{q}_{12}^{-2\zeta}\; \hat{p}_1^{-2\kappa}\; \hat{p}_2^{-2\gamma}\; \hat{q}_{12}^{-2\beta}\, | y_1, y_2\rangle =$$

$$= \frac{1}{(x_1-x_2)^{2\zeta}(x_2-y_1)^{2\kappa'}(x_1-y_2)^{2\gamma'}(y_1-y_2)^{2\beta}}\,.$$

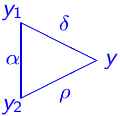
Thus, the operator $Q_{12}^{(\zeta,\kappa,\gamma)}$ is the GBO for the ladder diagrams



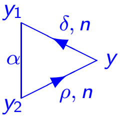
$$= (x_1-x_2)^{2\zeta}\,\langle x_1, x_2 | (\hat{Q}_{12}^{(\zeta,\kappa,\gamma)})^{2N} | y_1, y_2\rangle (y_1-y_2)^{2\beta},$$

Proposition 2. The eigenfunction for the operator $Q_{12}^{(\zeta,\kappa,\gamma)}$ is given by 3-point correlation function (conformal triangle)

$$\langle y_1, y_2 | \Psi_{\delta,\rho}^{(n,u)}(y)\rangle \; = $$



$$\cdot \left(\frac{(u, y-y_1)}{(y-y_1)^2} - \frac{(u, y-y_2)}{(y-y_2)^2} \right)^n \; \equiv$$



where we depict the nontrivial rank-n tensor numerator as arrows on the lines (the rank is fixed by indices on the lines: $\rho \to (\rho, n)$, etc) and denote

$$2\alpha = \Delta_1 + \Delta_2 - (\Delta - n) \, , \quad 2\delta = \Delta_1 - \Delta_2 + (\Delta - n) \, , \quad 2\rho = \Delta_2 - \Delta_1 + (\Delta - n) \, ,$$

i.e., conformal dimensions $\Delta, \Delta_1, \Delta_2$ are arbitrary parameters in this case. Thus, we have

$$Q_{12}^{(\zeta,\kappa,\gamma)} \, |\Psi_{\delta,\rho}^{(n,u)}(y)\rangle = \bar{\tau}(\kappa, \gamma; \delta, \alpha; n) \, |\Psi_{\delta,\rho}^{(n,u)}(y)\rangle \; .$$

where $\alpha + \rho = \kappa'$, $\alpha + \delta = \gamma'$ and $\bar{\tau}(\kappa, \gamma; \delta, \alpha; n)$ is an eigenvalue

$$\bar{\tau}(\kappa, \gamma; \delta, \alpha; n) = (-1)^n \cdot \tau(\delta', \kappa, n) \cdot \tau(\alpha, \gamma, n) \, ,$$

$$\tau(\alpha, \beta, n) = (-1)^n \, \frac{\pi^{D/2} \Gamma(\beta) \Gamma(\alpha) \Gamma(\alpha' - \beta + n)}{\Gamma(\beta') \Gamma(\alpha' + n) \Gamma(\alpha + \beta)}$$

Remark 1. We introduce new notation $\beta + \zeta = -2u$ – spectral parameter, and express parameters α, δ, ρ via conf. dimensions $\Delta_{1,2}$:

$$\beta - \zeta = D - \Delta_1 - \Delta_2 \, , \quad \gamma - \zeta = D/2 - \Delta_1 \, , \quad \kappa - \zeta = D/2 - \Delta_2 \, .$$

In this case the general GBO dependes on u, Δ_1, Δ_2 and is equal (up to a normalization factor) to the R-operator [D. Chicherin, S. Derkachov, A. P. Isaev (2013)]

$$R_{\Delta_1 \Delta_2}(u) = a(\kappa) a(\gamma) Q_{12}^{(\zeta,\kappa,\gamma)} =$$

$$= \mathcal{P}_{12} \, \hat{q}_{12}^{2(u+\frac{D-\Delta_1-\Delta_2}{2})} \hat{p}_1^{2(u+\frac{\Delta_2-\Delta_1}{2})} \, \hat{p}_2^{2(u+\frac{\Delta_1-\Delta_2}{2})} \, \hat{q}_{12}^{2(u+\frac{\Delta_1+\Delta_2-D}{2})}$$

which is conformal invariant solution of the Yang-Baxter equation

$$R_{\Delta_1 \Delta_2}(u-v) \, R_{\Delta_1 \Delta_3}(u) R_{\Delta_2 \Delta_3}(v) = R_{\Delta_2 \Delta_3}(v) \, R_{\Delta_1 \Delta_3}(u) R_{\Delta_1 \Delta_2}(u-v) \, .$$

The operator $R_{\Delta_1\Delta_2}(u)$ intertwines two spaces $V_{\Delta_1} \otimes V_{\Delta_2} \to V_{\Delta_2} \otimes V_{\Delta_1}$, where V_{Δ_i} is the space of scalar conf. fields with dimensions Δ_i. Let we have $V_{\Delta_1} \otimes V_{\Delta_2} = \sum_{\Delta,n} V_{\Delta}^{(n)}$, where $V_{\Delta}^{(n)}$ – is the space of tensor fields of rank n. Formally we have

$$R_{\Delta_1\Delta_2}(u) = \int d\mu(y) \sum_{n,\Delta} C(n,\Delta)|\Psi^{(n,u)}_{\Delta_1,\Delta_2,\Delta}(y)\rangle\langle\Psi^{(n,u)}_{\Delta_1,\Delta_2,\Delta}(y)|$$

Thus, eigenfunctions of $R_{\Delta_1\Delta_2}(u) = a(\kappa)a(\gamma)Q_{12}^{(\zeta,\kappa,\gamma)}$ should describe the fusion of two scalar conformal fields with dimensions Δ_1, Δ_2 into the composite tensor field with dimension Δ. Thus, conformal triangles are Clebsch-Gordan coefficients which produce this fusion.

Remark 2. The special case (for $D = 1$ and $\Delta_1 = \Delta_2 \equiv \frac{D}{2} - \xi$) of this R-operator $R_{\Delta_1 \Delta_2}(u)$ underlies Lipatov's integrable model of the high-energy asymptotics of multicolor QCD. Indeed, we have
(see D.Chicherin, S.Derkachov, API, JHEP 04 (2013) 020)

$$\mathcal{P}_{12} R_{12}^{(\kappa,\xi)} = \hat{q}_{12}^{2(u+\xi)} \; \hat{p}_1^{2u} \; \hat{p}_2^{2u} \; \hat{q}_{12}^{2(u-\xi)} \quad \stackrel{u \to 0}{\to} \quad 1 + u \, h_{12}^{(\xi)} + \ldots ,$$

$$h_{12}^{(\xi)} = 2 \ln q_{12}^2 + \hat{q}_{12}^{2\xi} \ln(\hat{p}_1^2 \; \hat{p}_2^2) \; \hat{q}_{12}^{-2\xi} \; ,$$

where $h_{12}^{(\xi)}$ is a local density of the Lipatov's Hamiltonian (holomorphic part of)

$$\mathcal{H} = \sum_{i=1}^{N-1} h_{i,i+1}^{(\xi)} \; .$$

Conclusion.

1.) In this report, we found new explicit formulas for L-loop ladder conformal integrals in even dimensions $D = 6, 8, 10$ and reproduce known result [N.I. Usyukina and A.I. Davydychev (1993)] for $D = 4$.

2.) The method of evaluating of the L-loop ladder conformal integrals is based on the use of graph building operators $\mathcal{H}_\beta$ which form the commutative family in view of STR [A.P.I. NPB (2003)].

3.) We also wonder if it is possible to apply our D-dimensional generalizations to evaluation similar 4-points functions (with fermions) that arise in the generalized "fishnet" model, in double scaling limit of γ-deformed $N = 4$ SYM theory.

4.) Very recent paper M. Kade, M. Staudacher, Supersymmetric brick wall diagrams and the dynamical fishnet, arXiv:2408.05805 [hep-th]. Supersymmetric generalizations of Basso-Dixon fishnet and brick wall diagrams.

S.E.Derkachov, A.P.I., L.A.Shumilov, Conformal triangles and zig-zag diagrams, Phys.Lett.B 830 (2022) 137150, 2201.12232 [hep-th];

S.E.Derkachov, A.P.I., L.A.Shumilov, Ladder and zig-zag Feynman diagrams, operator formalism and conformal triangles, JHEP 06 (2023) 059, 2302.11238 [hep-th]

S.E.Derkachov, A.P.I., L.A.Shumilov, Conformal four-point ladder integrals in diverse dimensions and polylogarithms, in preparation